M3-4A16 Assessed Problems # 3 Do all four problems

[#1] Exercises in exterior calculus operations

Vector notation for differential basis elements:
One denotes differential basis elements dx^i *and* $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, *for* $i, j, k = 1, 2, 3$, *in vector notation as*

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\
d\mathbf{S} &= (dS_1, dS_2, dS_3) \\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,, \\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,, \\
d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}
$$

(1a) ***Vector algebra operations***

(i) *Show that contraction with the vector field* $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ *recovers the following familiar operations among vectors*

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\
(\textit{or,} \quad X \,\lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k) \\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\
X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,, \\
Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,, \\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot\mathbf{Z}\,.
\end{aligned}
$$

(ii) *Show that these are consistent with*

$$
X \,\lrcorner\, (\alpha\wedge\beta) = (X \,\lrcorner\, \alpha)\wedge\beta + (-1)^k\alpha\wedge(X \,\lrcorner\, \beta)\,,
$$

for a k*-form* α.

(iii) *Use (ii) to compute* $Y \,\lrcorner\, X \,\lrcorner\, (\alpha\wedge\beta)$ *and* $Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, (\alpha\wedge\beta)$.

(1b) ***Exterior derivative examples in vector notation***
Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$
\begin{aligned}
df &= f_{,j}\,dx^j =: \nabla f\cdot d\mathbf{x} \\
0 = d^2 f &= f_{,jk}\,dx^k\wedge dx^j \\
df\wedge dg &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k =: (\nabla f\times\nabla g)\cdot d\mathbf{S} \\
df\wedge dg\wedge dh &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k\wedge h_{,l}\,dx^l =: (\nabla f\cdot\nabla g\times\nabla h)\,d^{\,3}x
\end{aligned}
$$

Likewise, show that

$$
\begin{aligned}
d(\mathbf{v}\cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S} \\
d(\mathbf{A}\cdot d\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\,d^{\,3}x\,.
\end{aligned}
$$

Verify the compatibility condition $d^2 = 0$ for these forms as

$$
\begin{aligned}
0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathrm{grad}\, f) \cdot d\mathbf{S}\,, \\
0 = d^2(\mathbf{v} \cdot d\mathbf{x}) = d\big((\mathrm{curl}\,\mathbf{v}) \cdot d\mathbf{S}\big) &= (\mathrm{div}\,\mathrm{curl}\,\mathbf{v})\, d^{\,3}x\,.
\end{aligned}
$$

Verify the exterior derivatives of these contraction formulas for $X = \mathbf{X} \cdot \nabla$

(i) $d(X \,\lrcorner\, \mathbf{v} \cdot d\mathbf{x}) = d(\mathbf{X} \cdot \mathbf{v}) = \nabla(\mathbf{X} \cdot \mathbf{v}) \cdot d\mathbf{x}$

(ii) $d(X \,\lrcorner\, \boldsymbol{\omega} \cdot d\mathbf{S}) = d(\boldsymbol{\omega} \times \mathbf{X} \cdot d\mathbf{x}) = \mathrm{curl}\,(\boldsymbol{\omega} \times \mathbf{X}) \cdot d\mathbf{S}$

(iii) $d(X \,\lrcorner\, f\, d^{\,3}x) = d(f\mathbf{X} \cdot d\mathbf{S}) = \mathrm{div}\,(f\mathbf{X})\, d^{\,3}x$

(1c) *Use Cartan's formula,*

$$
\pounds_X \alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)
$$

for a $k-$form α, $k = 0, 1, 2, 3$ in $\mathbb{R}^3$ to verify the Lie derivative formulas:

(i) $\pounds_X f = X \,\lrcorner\, df = \mathbf{X} \cdot \nabla f$

(ii) $\pounds_X\,(\mathbf{v} \cdot d\mathbf{x}) = \big(-\ \mathbf{X} \times \mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{X} \cdot \mathbf{v})\big) \cdot d\mathbf{x}$

(iii) $\pounds_X(\boldsymbol{\omega} \cdot d\mathbf{S}) = \big(\mathrm{curl}\,(\boldsymbol{\omega} \times \mathbf{X}) + \mathbf{X}\,\mathrm{div}\,\boldsymbol{\omega}\big) \cdot d\mathbf{S}$
$= \big(-\boldsymbol{\omega} \cdot \nabla\mathbf{X} + \mathbf{X} \cdot \nabla\boldsymbol{\omega} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{X}\big) \cdot d\mathbf{S}$

(iv) $\pounds_X(f\, d^{\,3}x) = (\mathrm{div}\, f\mathbf{X})\, d^{\,3}x$

(v) *Derive these formulas from the dynamical definition of Lie derivative.*

(1d) *Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:*

(i) $\pounds_{fX}\alpha = f\pounds_X\alpha + df \wedge (X \,\lrcorner\, \alpha)$

(ii) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

(iii) $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$

(iv) $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$

(v) $\pounds_X(\alpha \wedge \beta) = (\pounds_X\alpha) \wedge \beta + \alpha \wedge \pounds_X\beta$

[#2] Operations among vector fields

The Lie derivative of one vector field by another is called the ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y \cdot X - \nabla X \cdot Y = -\,\pounds_Y X$$

In components, the Jacobi-Lie bracket is

$$[X\,,\,Y] = \Big[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\Big] = \Big(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\Big)\frac{\partial}{\partial x^l}$$

The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

Verify the following formulas

(2a) $X \,\lrcorner\, (Y \,\lrcorner\, \alpha) = -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha)$

(2b) $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha)$*, for zero-forms (functions) and one-forms.*

(2c) $\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$*, as a result of* **(b)***. Use* **2(c)** *to verify the Jacobi identity.*

(2d) *Verify formula 2(b) for $k-$forms using the dynamical definition of the Lie derivative.*

[#3] A steady Euler fluid flow

A steady Euler fluid flow in a rotating frame satisfies

$$\pounds_u(\mathbf{v}\cdot d\mathbf{x}) = -\,d(p + \tfrac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v})\,,$$

where $\pounds_u$ is Lie derivative with respect to the divergenceless vector field $u = \mathbf{u}\cdot\nabla$, with $\nabla\cdot\mathbf{u} = 0$, and $\mathbf{v} = \mathbf{u} + \mathbf{R}$, with Coriolis parameter $\mathrm{curl}\,\mathbf{R} = 2\boldsymbol{\Omega}$.

(3a) *Write out this Lie-derivative relation in Cartesian coordinates.*

(3b) *By taking the exterior derivative, show that this relation implies that the exact two-form*

$$\mathrm{curl}v \,\lrcorner\, d^{\,3}x = \mathrm{curl}\mathbf{v}\cdot\nabla \,\lrcorner\, d^{\,3}x = \mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x}) =: d\Xi\wedge d\Pi$$

is invariant under the flow of the divergenceless vector field u.

(3c) *Show that Cartan's formula for the Lie derivative in the steady Euler flow condition implies that*

$$u \,\lrcorner\, \Big(\mathrm{curl}v \,\lrcorner\, d^{\,3}x\Big) = dH(\Xi,\Pi)$$

and identify the function H.

(3d) *Use the result of (3c) to write $\pounds_u\Xi = \mathbf{u}\cdot\nabla\Xi$ and $\pounds_u\Pi = \mathbf{u}\cdot\nabla\Xi$ in terms of the partial derivatives of H.*

(3e) *What do the results of (3d) mean geometrically? Hint: Is a symplectic form involved?*

[#4] Ertel's theorem for stratified fluids

The Euler–Boussinesq equations for the incompressible motion of an ideal flow of a stratified fluid and velocity $\mathbf{u}$ *satisfying* $\operatorname{div}\mathbf{u} = 0$ *in a rotating frame with Coriolis parameter* $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ *are given by*

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{acceleration} = \underbrace{-\,gb\nabla z}_{buoyancy} + \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{Coriolis} - \underbrace{\nabla p}_{pressure} \tag{1}$$

where $-g\nabla z$ *is the constant downward acceleration of gravity and* b *is the buoyancy, which satisfies the* ***advection relation****,*

$$\partial_t\, b + \mathbf{u}\cdot\nabla b = 0\,. \tag{2}$$

(4a) *Explain why the divergence-free condition on the fluid velocity,* $\nabla\cdot\mathbf{u} = 0$*, preserves volume*

(4b) *Derive the Poisson equation for pressure* p *by requiring that the fluid velocity remain divergence-free (volume-preserving,* $\nabla\cdot\mathbf{u} = 0$*), and identify the boundary condition on the pressure.*

(4c) *Rearrange the Newton's law form of the Euler–Boussinesq equations in (1) as*

$$\partial_t\,\mathbf{v} - \ \mathbf{u}\times\operatorname{curl}\mathbf{v} + gb\nabla z + \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = 0\,, \tag{3}$$

where $\mathbf{v} \equiv \mathbf{u} + \mathbf{R}$ *and* $\nabla\cdot\mathbf{u} = 0$*. Verify that equations (2), (3) and the divergence-free condition* $\nabla\cdot\mathbf{u} = 0$ *may be written geometrically, as*

$$\big(\partial_t + \pounds_u\big)b = 0\,, \qquad \big(\partial_t + \pounds_u\big)v + gbdz + d\varpi = 0\,, \quad \text{and} \quad \pounds_u\, d^{\,3}x = 0\,, \tag{4}$$

and identify the quantities v *and* ϖ*.*

(4d) *Denote the fluid velocity vector field as* $u = \mathbf{u}\cdot\nabla$ *and the circulation one-form as* $v = \mathbf{v}\cdot d\mathbf{x}$*. Write the exterior derivatives of the two equations in (4) and determine from the product rule for Lie derivatives and the antisymmetry of the wedge product that*

$$\big(\partial_t + \pounds_u\big)(dv\wedge db) = 0 \quad \text{or} \quad \partial_t\, q + \mathbf{u}\cdot\nabla q = 0\,, \tag{5}$$

in which the quantity q *is defined by the projection*

$$q = \nabla b\cdot\operatorname{curl}\mathbf{v} \tag{6}$$

is conserved on fluid particles.

(4e) ***Ertel theorem for the vorticity vector field***
Write the vorticity vector field $\omega = \boldsymbol{\omega}\cdot\nabla$ *and show that*

$$\big(\partial_t + \pounds_u\big)\omega = \partial_t\omega + [u,\,\omega] = -g\nabla b\times\nabla z\cdot\nabla \tag{7}$$

Use this expression to show that conservation of the potential vorticity may be proved by the product rule for Lie derivatives.

(4f) ***Conservation laws***
Show that the constancy of the scalar quantities b *and* q *on fluid parcels implies conservation of the spatially integrated quantity,*

$$C_\Phi = \int_D \Phi(b,q)\,d^{\,3}x\,,$$

for any smooth function Φ for which the integral exists.

Show that in addition to C_Φ the Euler–Boussinesq fluid equations (3) also conserve the total energy

$$E = \int_D \frac{1}{2}|\mathbf{u}|^2 + gbz \;d^{\,3}x\,,$$

which is the sum of the kinetic and potential energies.