# 3 M3-4-5 A16 Assessed Problems # 3 Due 2pm 15 Dec 2011

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

**Exercise 3.1** ***A cyclically symmetric problem with three oscillators on*** $\mathbb{C}^3$

Consider the (1,2,3) cyclically symmetric dynamical system,

$$\frac{da_1^*}{dt} = a_2 a_3 \,, \qquad \frac{da_2^*}{dt} = a_3 a_1 \,, \qquad \frac{da_3^*}{dt} = a_1 a_2 \,,$$

where $a_1, a_2, a_3 \in \mathbb{C}^3$ and $a_k^*$ denotes the complex conjugate of $a_k$.

***Problem statement:***

(a) Show that this system is Hamiltonian for the canonical Poisson bracket $\{a_j \,,\, a_k^*\} = -2i\delta_{jk}$.

Note: a dynamical system $\dot{x} = F(x)$ is called Hamiltonian, if it can be expressed as $\dot{x} = \{x, H\}$ for a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ and Hamiltonian $H(x)$. Here you are given the Poisson bracket and the phase space, $T^*\mathbb{C}^3$. Just find the Hamiltonian.

(b) Find two other constants of motion that generate $S^1$ symmetries of the Hamiltonian.

(c) Is this dynamical system completely integrable? That is, are there enough symmetries and constants of motion in involution to reduce it to a Hamiltonian system on the phase plane, and reconstruct the 'angles' canonically conjugate to the constants of motion?

(i) Begin your answer by showing that the following transformation of variables is canonical,

$$a_1 = ze^{-i(\phi_2+\phi_3)} \,, \qquad a_2 = |a_2|e^{i\phi_2} \,, \qquad a_3 = |a_3|e^{i\phi_3} \,, \quad \text{with} \quad z = |z|e^{i\zeta} \in \mathbb{C}.$$

(ii) Show that the Hamiltonian may be written solely in terms of $z, z^*, I_2, I_3$.

(iii) Are there any limitations to the range of variables? If so, what are they?

(iv) Draw a figure showing the level sets of $H$ in the $(z_1, z_2)$ phase plane with $z = z_1 + iz_2$.

(v) Write a closed equation in $z$ that provides the dynamics of both the amplitude and phase of $z = |z|e^{i\zeta}$. Show that solving it would determine the magnitudes, $|a_2|$ and $|a_3|$.

(vi) Extra credit: Write the phase equations and solve for the phases $\zeta$, $\phi_2$ and $\phi_3$.

(d) Will the analysis here generalise to $n$ degrees of freedom? Is the corresponding system on $\mathbb{C}^n$ completely integrable? Write this system explicitly and justify your answer.

**Exercise 3.2** ***Spherical pendulum in a constant vertical magnetic field***

Explain the effects that an external constant vertical magnetic field $B_0\hat{\mathbf{e}}_3$ can have on a spherical pendulum with unit charge on its mass.

Draw your conclusions analytically by taking the following steps.

***Problem statement:***

Begin with the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 + B_0\hat{\mathbf{e}}_3 \cdot (\mathbf{x} \times \dot{\mathbf{x}}) - g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \tfrac{1}{2}\mu(1 - |\mathbf{x}|^2),$$

in which the Lagrange multiplier $\mu$ constrains the motion to remain on the sphere $S^2$ by enforcing $(1 - |\mathbf{x}|^2) = 0$ when it is varied in Hamilton's principle.

(a) Derive the constrained Euler–Lagrange equation from Hamilton's principle for this Lagrangian.

(b) Use the $S^1$ symmetry of the Lagrangian under rotations about the vertical to obtain its Noether conservation laws.

(c) Transform to the Hamiltonian side and write the Hamiltonian in terms of $S^1$-invariant quantities that reduce to those used in class for the spherical pendulum in the absence of a magnetic field.

(d) Write the Hamiltonian equations of motion for the $S^1$ invariants in $\mathbb{R}^3$-bracket form and reduce the dynamics to a phase plane.

(e) Draw its phase portraits and discuss the types of motion that are available to this system, as the magnitude $B_0$ of its external magnetic field is varied.

**Exercise 3.3** *Complex Maxwell-Bloch equations*

Begin by reading Section 6.1 of GM Part 1, which derives the complex Maxwell-Bloch (CMB) equations

$$\dot{x} = y, \quad \dot{y} = xz, \quad \dot{z} = \frac{1}{2}\left(x^* y + x y^*\right), \quad \text{for } (x, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R},$$

by averaging the Lagrangian for the Maxwell-Schrödinger equations. Section 6.1 of GM Part 1 points out that the 5D CMB system conserves three quantities,

$$H = \frac{i}{2}\left(x^* y - x y^*\right), \quad K = z + \frac{1}{2}|x|^2 \quad \text{and} \quad C = |y|^2 + z^2 .$$

This exercise is about understanding the relationships among the conserved quantities $H, K, C$.

Let's first introduce notation that will conveniently make all factors real in the subsequent calculations.

$$U_0 = \begin{bmatrix} w_0 & q_0 \\ r_0 & -w_0 \end{bmatrix} = \begin{bmatrix} -4z & -4iy \\ 4iy^* & 4z \end{bmatrix} \qquad U_1 = \begin{bmatrix} w_1 & q_1 \\ r_1 & -w_1 \end{bmatrix} = \begin{bmatrix} Q & 2ix \\ 2ix^* & -Q \end{bmatrix}$$

where the variable $Q$ is a space-holder for the $6^{th}$ dimension (!) that will later be set to zero.

***Problem statement:***

(a) Show that the CMB equations arise from the Lax pair $(L, M)$ given by the $2 \times 2$ matrix commutator relation

$$\frac{dL}{dt} = [M, L]\,, \quad \text{with} \quad L = A\lambda^2 + U_1\lambda + U_0 \quad \text{and} \quad M = A\lambda + U_1$$

where $A$ is a $2 \times 2$ constant matrix to be determined.

(b) Explain why the conservation laws for this system $h_1, h_2, h_3, w_1$ are obtained from $\mathrm{tr}L^2$, the trace of the square of the Lax matrix and how they are related to ones we already know, $H, K, C$.

(c) Find these conservation laws.

(d) Compute the three $6 \times 6$ Hamiltonian matrix operators of the forms

$$D_0 = -\begin{bmatrix} J_1 & J_2 \\ J_2 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} J_0 & 0 \\ 0 & -J_2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & J_0 \\ J_0 & J_1 \end{bmatrix}$$

by identifying the $3 \times 3$ matrices $J_0, J_1, J_2$, that recover the CMB equations in vector form $U = (q_0, r_0, w_0, q_1, r_1, w_1)^T$ via the tri-Hamiltonian ladder relations,

$$\dot{U} = D_0 \nabla h_3 = D_1 \nabla h_2 = D_2 \nabla h_1$$

That is, obtain the CMB equations in this vector tri-Hamiltonian ladder form from the Hamiltonians $h_1, h_2, h_3$ in the equation above, after setting $w_1 = 0$, by determining the submatrices $J_0, J_1, J_2$ that produce the Hamiltonian matrices $D_0, D_1, D_2$.

(e) Find the Casimir functions for the the tri-Hamiltonian operators, $D_0, D_1, D_2$.

(f) Does this system possess enough constants of motion to be completely solved analytically? Prove it.