M3/4A16 Solutions to Assessed Coursework 2 Darryl Holm

1. ***Exercise.*** **Planar Isotropic Simple Harmonic Oscillator (PISHO)**
The canonical bracket operation on $T^*\mathbb{R}^2$ given by

$$\{F,\, H\} := \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}}\,, \tag{1}$$

satisfies the conditions to be a Poisson bracket. (It's bilinear, skew-symmetric, Leibnitz and satisfies the Jacobi identity.)

(a) *Determine whether any of the following transformations of coordinates are* **canonical**.
   i. $T^*\mathbb{R}^2\ \{0\} \to T^*\mathbb{R}_+ \times T^*S^1$, *given by* $q_1 + iq_2 = re^{i\theta}\,, \quad p_1 + ip_2 = (p_r + ip_\theta/r)e^{i\theta}$.
   ii. $T^*\mathbb{R}^2 \to \mathbb{C}^2$, *given by* $a_k = q_k + ip_k\,, \quad a_k^* = q_k - ip_k$, *with* $k = 1, 2$.
   iii. $T^*\mathbb{R}^2\ \{0\} \to \mathbb{R}^2_+ \times T^2$, *given by* $I_k = \frac{1}{2}(q_k^2 + p_k^2)\,, \quad \phi_k = \tan^{-1}(p_k/q_k)$, *with* $k = 1, 2$.

$\boxed{\textit{Answer}}$

i. Evaluating $\mathrm{Re}\big((p_x + ip_y)(dx - idy)\big) = \mathrm{Re}\big((p_r + ip_\theta/r)(dr - ird\theta)\big)$ yields

$$p_x dx + p_y dy = p_r dr + p_\theta d\theta.$$

Taking exterior derivative gives

$$dp_x \wedge dx + dp_y \wedge dy = dp_r \wedge dr + dp_\theta \wedge d\theta.$$

ii. $da \wedge da^* = (dq_k + idp_k) \wedge (dq_k - idp_k) = -2idq \wedge dp$

iii. For each k (no sum) define

$$a_k = q_k + ip_k\,, \quad r_k = |a_k| = (q_k^2 + p_k^2)^{1/2}\,, \quad \phi_k = \tan(p_k/q_k)$$

Then for each k (no sum) with $I_k = \frac{1}{2}r_k^2$ and

$$\begin{aligned} dI_k \wedge d\phi_k = \tfrac{1}{2}dr_k^2 \wedge d\phi_k &= (q_k dq_k + p_k dp_k) \wedge \frac{q_k dp_k - p_k dq_k}{q_k^2 + p_k^2} \\ &= \frac{q_k^2 dq_k \wedge dp_k - p_k^2 dp_k \wedge dq_k}{q_k^2 + p_k^2} \\ &= dq_k \wedge dp_k \end{aligned}$$

▲

(b) *Write the Hamiltonian equations for the Planar Isotropic Simple Harmonic Oscillator (PISHO)*

$$\ddot{\mathbf{q}} = -\,\mathbf{q} \quad \textit{with} \quad \mathbf{q} \in \mathbb{R}^2,$$

using each of the coordinate systems of the previous exercise.
(If any of the solutions are obvious to you, go ahead and write them.)

$\boxed{\textit{Answer}}$

i. $H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + r^2\right)$,

$$\begin{aligned}\dot{r} &= \partial H/\partial p_r = p_r\,, \quad \dot{p}_r = -\partial H/\partial r = -r + \frac{p_\theta^2}{r^3}\\ \dot{\theta} &= \partial H/\partial p_\theta = p_\theta/r^2\,, \quad \dot{p}_\theta = -\partial H/\partial \theta = 0\,.\end{aligned}$$

ii. $H = \frac{1}{2}(|a_1|^2 + |a_2|^2)$ and $\{a,\, a^*\} = -2i$, so

$$\dot{a}_k = \{a_k,\, H\} = -ia_k \quad \text{and} \quad a_k(t) = a_k(0)e^{-it}\,.$$

PISHO dynamics is just a linear phase shift at frequency -1.

iii. $H = I_1 + I_2$ and $\{I_k,\, \phi_l\} = \delta_{kl}$, so

$$\begin{aligned}\dot{I}_k &= \{I_k,\, H\} = \frac{\partial H}{\partial \phi_k} = 0\,,\\ \dot{\phi}_k &= \{\phi_k,\, H\} = -\frac{\partial H}{\partial I_k} = -1 \quad \text{and} \quad \phi(t) = \phi(0) - t\,.\end{aligned}$$

▲

(c) *Write the Hamiltonian forms of the PISHO equations in terms of S^1-invariant quantities, for the following two cases:*

i. $(X_1, X_2, X_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q}\cdot\mathbf{p})$ *and* $p_\theta^2 = |\mathbf{p}\times\mathbf{q}|^2$ *with* $(\mathbf{q},\mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2$.

ii. $R = |a_1|^2 + |a_2|^2$, $Y_1 + iY_2 = 2a_1^*a_2$ *and* $Y_3 = |a_1|^2 - |a_2|^2$, *with* $a_k := q_k + ip_k \in \mathbb{C}^1$ *for* $k = 1, 2$.

Answer

i. Note $p_\theta^2 = |\mathbf{p}\times\mathbf{q}|^2 = |\mathbf{q}|^2|\mathbf{p}|^2 - (\mathbf{p}\cdot\mathbf{q})^2 = X_1X_2 - X_3^2 =: S^2(\mathbf{X})$ and write the flow on $\mathbf{X} \in \mathbb{R}^3$ with $H(\mathbf{X}) = \frac{1}{2}(X_1 + X_2)$ for PISHO as *rotation* about the $(\frac{1}{2}, \frac{1}{2}, 0)$-axis

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\end{bmatrix} = \dot{\mathbf{X}} = \nabla S^2(\mathbf{X}) \times \nabla H(\mathbf{X}) = \begin{bmatrix}2X_3\\ -2X_3\\ -X_1 + X_2\end{bmatrix}$$

ii. In the second case, $H = R/2$ for PISHO and $R^2 = |\mathbf{Y}|^2 = Y_1^2 + Y_2^2 + Y_3^2$

$$\begin{bmatrix}\dot{Y}_1\\ \dot{Y}_2\\ \dot{Y}_3\end{bmatrix} = \dot{\mathbf{Y}} = 2\nabla|\mathbf{Y}|^2 \times \nabla H(\mathbf{Y}) = \nabla R^2 \times \nabla R = \mathbf{0} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

Substituting $a_k(t) = a_k(0)e^{-it}$ shows that each of the S^1-invariant quantities in the vector $\mathbf{Y}$ are indeed invariant under the dynamics of PISHO.

▲

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2. ***Exercise.*** **Hamiltonian flows**

(a) *Use the canonical Poisson brackets in (1) to compute ($\{X_1, X_2\}$, etc.) among the three rotationally invariant quadratic phase space functions*

$$(X_1, X_2, X_3) = (|\mathbf{q}|^2, |\mathbf{p}|^2, \mathbf{q}\cdot\mathbf{p}) \,, \qquad (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2 \,. \tag{2}$$

Answer

$$\{X_i,\, X_j\} = -2\epsilon_{ijk}\frac{\partial S^2}{\partial X^k} \quad \text{with} \quad S^2(\mathbf{X}) = X_1X_2 - X_3^2 \,.$$

▲

(b) *Show that these Poisson brackets may be expressed as a closed system*

$$\{X_i, X_j\} = c_{ij}^k X_k, \qquad i, j, k = 1, 2, 3, \tag{3}$$

in terms of these invariants, by computing the coefficients c_{ij}^k.

Answer

The non-vanishing entries among the skew-symmetric coefficients $c_{ji}^k = -\, c_{ij}^k$ are:

$$c_{12}^3 = 4\,, \quad c_{31}^1 = -\,2\,, \quad c_{23}^2 = -\,2$$

▲

(c) *Write the Poisson brackets among the invariants $\{X_i, X_j\}$ as a 3×3 skew-symmetric table.*

Answer

The table of Poisson brackets $\{X_i,\, X_j\}$ is given as

$$\{X_i,\, X_j\} = \begin{array}{|c|cccc|}\hline \{\,\cdot\,,\,\cdot\,\} & X_1 & X_2 & X_3 & p_\theta \\ \hline X_1 & 0 & 4X_3 & 2X_1 & 0 \\ X_2 & -4X_3 & 0 & -2X_2 & 0 \\ X_3 & -2X_1 & 2X_2 & 0 & 0 \\ p_\theta & 0 & 0 & 0 & 0 \\ \hline \end{array} \tag{4}$$

▲

(d) *Write the Poisson brackets for functions of these three invariants (X_1, X_2, X_3) as a vector cross product of gradients of functions of $\mathbf{X} \in \mathbb{R}^3$.*

Answer

$$\{F,\, H\} = -\,2\nabla S^2 \cdot \nabla F \times \nabla H$$

▲

(e) *Take the Poisson brackets of the three invariants* (X_1, X_2, X_3) *with the function,*

$$S^2 = X_1X_2 - X_3^2 \,. \tag{5}$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$.

Answer

$$\{S^2,\, H\} = -\,2\nabla S^2 \cdot \nabla S^2 \times \nabla H = 0 \quad \text{for all} \quad H(X_1, X_2, X_3)$$

The Poisson bracket vanishes, because the cross product of parallel vectors in $\mathbb{R}^3$ vanishes. ▲

(f) *Write the results of applying the Poisson brackets in the form*

$$\{\,\mathbf{z}\,,\, X_k\} = m_k\mathbf{z}\,,$$

for $\mathbf{z} = (\mathbf{q}, \mathbf{p})^T \in \mathbb{R}^2 \times \mathbb{R}^2$ *and* 2×2 *matrices* m_k, *with* $k = 1, 2, 3$. *Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a* 3×3 *skew-symmetric table of their matrix commutation relations* $[m_i,\, m_j]$, *etc. Compare it with the table in Part 2c.*

Answer

These Poisson brackets result in traceless constant matrices

$$m_1 = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \; m_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \; m_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

which satisfy a matrix commutator table that corresponds to the Poisson bracket table (4). In particular,

$$[m_1, m_2] = 4m_3, \quad [m_2, m_3] = -2m_2, \quad [m_3, m_1] = -2m_1\,.$$

▲

(g) *Show that the flows*

$$\phi_k : \mathbf{z}(t) = e^{m_k t}\mathbf{z}(0) = \sum_{n=0}^{\infty} \frac{1}{n!}(m_k t)^n \mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,,\, X_k\}$ *arising from the three rotationally invariant quadratic phase space functions in (2) acting on the phase space vector* $\mathbf{z} = (\mathbf{q}, \mathbf{p})^T$ *may be written as symplectic matrix transformations* $\mathbf{z}(t) = S(t)\mathbf{z}(0)$, *with* $S^TJS = J$.

Answer

Taking the derivative of the definition of symplectic matrix transformations $\mathbf{z}(t) = S(t)\mathbf{z}(0)$, with $S^T(t)JS(t) = J$, yields

$$Jm_i + m_i^TJ = 0\,, \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

That is, if $(Jm_i) = (Jm_i)^T$ is a symmetric matrix, then its flow

$$\phi_k : \mathbf{z}(t) = e^{m_k t}\mathbf{z}(0)$$

is a symplectic matrix transformation. This is easily verified for the matrices m_k, $k = 1, 2, 3$, in the previous part, as

$$Jm_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad Jm_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad Jm_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

are all symmetric. ▲

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3. ***Exercise.*** **Poisson brackets for 1:1 invariants**

(a) *Use the canonical Poisson brackets in (1) to compute ($\{Y_1, Y_2\}$, etc.) among the three S^1-invariant quadratic phase space functions for a 1:1 resonance*

$$Y_1 + iY_2 = 2a_1^* a_2 \quad \textit{and} \quad Y_3 = |a_1|^2 - |a_2|^2 \,, \tag{8}$$

with $a_k := q_k + ip_k \in \mathbb{C}^1$ for $k = 1, 2$.

Answer

$$\{Y_i,\, Y_j\} = 2\epsilon_{ijk}\frac{\partial |\mathbf{Y}|^2}{\partial Y^k} \quad \text{with} \quad |\mathbf{Y}|^2 = Y_1^2 + Y_2^2 + Y_3^2 \,.$$

▲

(b) *Show that these Poisson brackets may be expressed as a closed system*

$$\{Y_i, Y_j\} = c_{ij}^k Y_k, \qquad i, j, k = 1, 2, 3, \tag{9}$$

in terms of these invariants, by computing the coefficients c_{ij}^k.

Answer

The non-vanishing entries among the skew-symmetric coefficients $c_{ji}^k = -\,c_{ij}^k$ are:

$$c_{12}^3 = 4\,, \quad c_{31}^2 = 4\,, \quad c_{23}^1 = 4$$

That is, $c_{ij}^k = 4\,\epsilon_{ijk}$. ▲

(c) *Write the Poisson brackets $\{Y_i, Y_j\}$ among these invariants as a 3×3 skew-symmetric table.*

Answer

The table of Poisson brackets $\{Y_i,\, Y_j\}$ is given as

$$\{Y_i,\, Y_j\} = \begin{array}{|c|cccc|} \hline \{\,\cdot\,,\,\cdot\,\} & Y_1 & Y_2 & Y_3 & R \\ \hline Y_1 & 0 & 4Y_3 & -4Y_2 & 0 \\ Y_2 & -4Y_3 & 0 & 4Y_1 & 0 \\ Y_3 & 4Y_2 & -4Y_1 & 0 & 0 \\ R & 0 & 0 & 0 & 0 \\ \hline \end{array} \tag{10}$$

▲

(d) *Write the Poisson brackets for functions of these three invariants (Y_1, Y_2, Y_3) as a vector cross product of gradients of functions of $\mathbf{Y} \in \mathbb{R}^3$.*

Answer

$$\{F,\, H\} = 2\nabla|\mathbf{Y}|^2 \cdot \nabla F \times \nabla H = 0 \text{ for smooth functions } F,\, H:\ \mathbb{R}^3 \to \mathbb{R}$$

▲

(e) *Take the Poisson brackets of the three invariants (Y_1, Y_2, Y_3) with the function,*

$$R = |a_1|^2 + |a_2|^2\,. \tag{11}$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$.

Answer

$$\{|\mathbf{Y}|^2,\, H\} = 2\nabla|\mathbf{Y}|^2 \cdot \nabla|\mathbf{Y}|^2 \times \nabla H = 0 \quad \text{for all} \quad H(Y_1, Y_2, Y_3)$$

The Poisson bracket vanishes, because the cross product of parallel vectors in $\mathbb{R}^3$ vanishes.

As an alternative explanation, $|\mathbf{Y}|^2 = R^2$ and $\{a_k,\, R\} = -\,2ia_k$, so R generates an S^1 phase shift, but the quantities (Y_1, Y_2, Y_3) are all invariant under this phase shift; so they Poisson commute with smooth functions of R. ▲

(f) *Write the results of applying the Poisson brackets in the form*

$$\{\,\mathbf{a}\,,\, Y_k\} = c_k\mathbf{a} \quad k = 1, 2, 3\,,$$

for $\mathbf{a} = (a_1, a_2)^T \in \mathbb{C}^2$ and 2×2 matrices c_k, with $k = 1, 2, 3$. Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a 3×3 skew-symmetric table of their matrix commutation relations $[c_i,\, c_j]$, etc. Compare it with the table in Part 3c.

Answer

One writes

$$\mathbf{Y} = a_k \boldsymbol{\sigma}_{kl} a_l^*\,.$$

where $\boldsymbol{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$ is the 'vector' of Pauli Spin Matrices, given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{12}$$

Then $\{a_k, a_l^*\} = -2i\delta_{kl}$ implies

$$\{\,\mathbf{a}\,,\, Y_k\} = -2i\sigma_k\mathbf{a} = c_k\mathbf{a} \quad k = 1, 2, 3. \tag{13}$$

These satisfy a matrix commutator table that corresponds exactly to the Poisson bracket table (10). In particular,

$$[c_1, c_2] = 4c_3, \quad [c_2, c_3] = 4c_1, \quad [c_3, c_1] = 4c_2 \,.$$

▲

(g) *Show that the flows*

$$\phi_k : \mathbf{z}(t) = e^{c_k t}\mathbf{z}(0) = \sum_{n=0}^{\infty} \frac{1}{n!}(c_k t)^n \mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,, Y_k\}$ *arising from the three* S^1 *phase invariant quadratic phase space functions in (8) acting on the phase space vector* $\mathbf{a} = (a_1, a_2)^T \in \mathbb{C}^2$ *may be written as* $SU(2)$ *matrix transformations* $\mathbf{a}(t) = U(t)\mathbf{a}(0)$, *with* $U^\dagger U = \mathrm{Id}$.

Answer

Taking the derivative of the definition of unitary matrix transformations $\mathbf{a}(t) = U(t)\mathbf{a}(0)$, with $U(t)^\dagger U(t) = \mathrm{Id}$, yields

$$c_k + c_k^\dagger = 0$$

where superscript dagger (†) denotes Hermitian conjugate. Thus, the matrices $c_k = -2i\sigma_k$ in (13) satisfy $c_k^\dagger = -c_k$. The flows of such matrices, $\phi_k : \mathbf{a}(t) = e^{c_k t}\mathbf{a}(0)$ are unitary matrix transformations. ▲

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4. ***Exercise.*** **Poisson brackets for angular momentum**

(a) *Compute the canonical Poisson brackets* ($\{L_1, L_2\}$, *etc.) among the three components of the angular momentum vector* $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ *for* $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$ *so that* $\mathbf{L} \in \mathbb{R}^3$ *also.*

Answer

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2 \,.$$

▲

(b) *Show that these Poisson brackets may be expressed as a closed system*

$$\{L_i, L_j\} = c_{ij}^k L_k, \qquad i, j, k = 1, 2, 3, \tag{14}$$

in terms of these invariants, by computing the coefficients c_{ij}^k.

Answer

$$\{L_i, L_j\} = \epsilon_{ijk} L_k, \qquad i, j, k = 1, 2, 3.$$

▲

(c) *Write the Poisson brackets $\{L_i, L_j\}$ among the (L_1, L_2, L_3) as a 3×3 skew-symmetric table.*

Answer

$$
\{L_i,\,L_j\} = \begin{array}{|c|ccc|}
\hline
\{\cdot\,,\,\cdot\} & L_1 & L_2 & L_3 \\
\hline
L_1 & 0 & L_3 & -L_2 \\
L_2 & -L_3 & 0 & L_1 \\
L_3 & L_2 & -L_1 & 0 \\
\hline
\end{array}
$$

▲

(d) *Write the Poisson brackets for functions of the three components (L_1, L_2, L_3) as a vector cross product of gradients of functions of $\mathbf{L}\in\mathbb{R}^3$.*

Answer

$$
\{F\,,\,H\}(\mathbf{L}) = \mathbf{L}\cdot\frac{\partial F}{\partial \mathbf{L}}\times\frac{\partial H}{\partial \mathbf{L}} = \frac{1}{2}\frac{\partial L^2}{\partial \mathbf{L}}\cdot\frac{\partial F}{\partial \mathbf{L}}\times\frac{\partial H}{\partial \mathbf{L}}\,.
$$

▲

(e) *Take the Poisson brackets of the three components (L_1, L_2, L_3) with the function,*

$$
L^2 = L_1^2 + L_2^2 + L_3^2\,. \tag{15}
$$

Explain your answers geometrically in terms of vectors in $\mathbb{R}^3$.

Answer

$$
\{\mathbf{L}\,,\,L^2\} = -\mathbf{L}\times\mathbf{L} = 0\,.
$$

▲

(f) *Write the results of applying the Poisson brackets in the form*

$$
\{\,\mathbf{z}\,,\,L_k\} = r_k\mathbf{z}\,,
$$

for $\mathbf{z} = (\mathbf{q},\mathbf{p})^T\in\mathbb{R}^3\times\mathbb{R}^3$ and 3×3 matrices r_k, with $k = 1,2,3$. Identify the type of matrix that results (symmetric, skew symmetric, etc.) Write a 3×3 skew-symmetric table of their matrix commutation relations $[r_i,\,r_j]$, etc. Compare it with the table in Part 4c.

Answer

$$
\{\,\mathbf{z}\,,\,\boldsymbol{\xi}\cdot\mathbf{L}\} = \boldsymbol{\xi}\times\mathbf{z} = \widehat{\boldsymbol{\xi}}\,\mathbf{z}\,,
$$

so that

$$
\{\,z_i\,,\,\boldsymbol{\xi}\cdot\mathbf{L}\} = \epsilon_{ijk}\xi_j z_k = -\epsilon_{ikj}\xi_j z_k = \widehat{\xi}_{ik}\,z_k\,,
$$

with

$$
\widehat{\boldsymbol{\xi}} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} = \xi_k r_k
$$

where

$$r_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

whose matrix commutators satisfy

$$[r_i, r_j] = \epsilon_{ijk} r_k, \qquad i, j, k = 1, 2, 3.$$

just as in the table in Part 4c. ▲

(g) Show that the flows

$$\phi_k : \mathbf{z}(t) = e^{r_k t}\mathbf{z}(0) = \sum_{n=0}^{\infty} \frac{1}{n!}(r_k t)^n \mathbf{z}(0)$$

of the Hamiltonian vector fields $\{\,\cdot\,,\,L_k\}$ *arising from the three components of the vector* $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ *acting on the phase space vector* $\mathbf{z} = (\mathbf{q}, \mathbf{p})^T$ *may be written as* $SO(3)$ *orthogonal matrix transformations* $\mathbf{z}(t) = O(t)\mathbf{z}(0)$*, with* $O^T O = \mathrm{Id}$.

Answer

The matrices r_k in (16) are skew symmetric; that is, they satisfy $r_k^T = -r_k$. The flows of such matrices, $\phi_k : \mathbf{z}(t) = e^{r_k t}\mathbf{z}(0)$ are orthogonal matrix transformations, as one may see by taking the time derivative of the relation $O^T(t)O(t) = \mathrm{Id}$ with $O(t) = e^{r_k t}$ to find

$$0 = \dot{O}O^T + O\dot{O}^T = \dot{O}O^T + (\dot{O}O^T)^T = r_k + r_k^T\,.$$

▲

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5. ***Exercise.*** **Reduced Kepler problem**

Newton's equation for the ***reduced Kepler problem*** *for planetary motion is*

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0\,, \tag{17}$$

in which μ *is a constant and* $r = |\mathbf{r}|$ *with* $\mathbf{r} \in \mathbb{R}^3$.

(a) Show directly that Newton's equation (17) conserves the quantities,

$$\begin{aligned} E &= \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{r} \quad \text{(energy)}\,, \\ \mathbf{L} &= \mathbf{r} \times \dot{\mathbf{r}} \quad \text{(specific angular momentum)}\,. \end{aligned}$$

Since, $\mathbf{r} \cdot \mathbf{L} = 0$*, the planetary motion in* $\mathbb{R}^3$ *takes place in a plane to which vector* $\mathbf{L}$ *is perpendicular. This is the orbital plane.*

Answer

$\dot{E} = 0$ and $\dot{\mathbf{L}} = 0$ both follow easily by direct verification using Newton's equation for Kepler's problem. ▲

(b) Show that Newton's equation (17) also conserves the vector quantities,

$$\begin{aligned}
\mathbf{K} &= \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad \text{(Hamilton's vector)} \\
\mathbf{J} &= \dot{\mathbf{r}} \times \mathbf{L} - \mu\mathbf{r}/r \quad \text{(Laplace-Runge-Lenz vector, or LRL vector)} \\
&= \mathbf{K} \times \mathbf{L}\,,
\end{aligned}$$

with

$$\hat{\mathbf{r}} = (\cos\theta, \sin\theta) \quad \textit{and} \quad \hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta).$$

These vectors lie in the orbital plane, since $\mathbf{K}\cdot\mathbf{L} = 0 = \mathbf{J}\cdot\mathbf{L}$. *These conservation laws, particularly constancy of the LRL vector, mean that certain attributes of the orbit, particularly, its orientation and eccentricty, are fixed in the orbital plane.*

Answer

Conservation of Hamilton's vector. Conservation of $\mathbf{K}$ follows from Newton's equation and its conservation of $\mathbf{L}$, with some additional geometry.

Begin with the relations,

$$L = |\mathbf{L}|\,, \quad \frac{\dot\theta}{L} = \frac{1}{r^2}\,, \quad \frac{dL}{dt} = 0 \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot\theta\,\hat{\mathbf{r}}\,,$$

the last of which follows from the geometric definitions

$$\hat{\mathbf{r}} = (\cos\theta, \sin\theta) \quad \text{and} \quad \hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta).$$

Newton's equation of motion (17) for the Kepler problem may then be rewritten equivalently as

$$0 = \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = \ddot{\mathbf{r}} + \frac{\mu}{L}\dot\theta\,\hat{\mathbf{r}} = \frac{d}{dt}\Big(\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big)\,.$$

This equation implies conservation of the following vector in the plane of motion

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad (\textit{\textbf{Hamilton's vector}})\,.$$

The other vector $\mathbf{J}$ in the plane is given by the cross-product of the two conserved vectors $\mathbf{K}$ and $\mathbf{L}$,

$$\mathbf{J} = \mathbf{K}\times\mathbf{L} = \dot{\mathbf{r}}\times\mathbf{L} - \mu\hat{\mathbf{r}} \quad (\textit{\textbf{Laplace-Runge-Lenz vector}})\,,$$

so it is also conserved. The vectors $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{L}$ are mutually orthogonal, with $\mathbf{L}$ normal to the orbital plane. ▲

(c) From their definitions, show that these conserved quantities are related by

$$L^2 + \frac{J^2}{(-2E)} = \frac{\mu^2}{(-2E)}\,, \tag{18}$$

where $J := |\mathbf{J}|$ *is the magnitude of the LRL vector and* $-2E > 0$ *for bounded orbits.*

Answer

From their definitions, these conserved quantities are related by

$$K^2 = 2E + \frac{\mu^2}{L^2} = \frac{J^2}{L^2}\,.$$

Hence, $J^2 = 2E\,L^2 + \mu^2$ and (18) follows. ▲

(d) *Choose the conserved LRL vector* $\mathbf{J}$ *in the orbital plane to point along the reference line for the measurement of the polar angle* θ*, say from the center of the orbit (Sun) to the perihelion (point of nearest approach, at mid-summer's day), so that*

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L} - \mu\,\mathbf{r}/r)\,.$$

Use this relation to write the Kepler orbit $r(\theta)$ *in plane polar coordinates, as*

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1 + e\cos\theta}\,,$$

with eccentricity $e = J/\mu$ *and semi latus rectum* $l_\perp = L^2/\mu$*. The expression* $r(\theta)$ *for the Kepler orbit is the formula for a conic section. This is* ***Kepler's First Law****. How is the value of the eccentricity associated to the types of orbits?*

Answer

The scalar product of $\mathbf{r}$ and $\mathbf{J}$ yields an elegant result for the Kepler orbit in plane polar coordinates. We are given

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L} - \mu\,\mathbf{r}/r) = L^2 - \mu r\,,$$

which implies an expression for $r(\theta)$

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1 + e\cos\theta} \quad \text{with eccentricity} \quad e := J/\mu\,.$$

As expected, the orbit is a ***conic section***. The eccentricity e takes values $0 < e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola. For bounded periodic orbits (for which $-2E > 0$) the formula for $r(\theta)$ describes an ellipse in polar coordinates that are centred at one of the two foci. This verifies ***Kepler's First Law***. The eccentricity of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$e^2 = J^2/\mu^2 = 1 - b^2/a^2$$

and its half-width at a focus (the semi latus rectum) is given by

$$l_\perp = L^2/\mu = (1 - e^2)a = b^2/a$$

Hence, $J^2 = 2E\,L^2 + \mu^2$ implies $(1 - J^2/\mu^2) = -2E(L^2/\mu^2)$, so that

$$1 = -2E(a/\mu) \quad \text{or} \quad -2E = \frac{\mu}{a}$$

for elliptical orbits. (The sign changes to $+2E = \mu/a$ for hyperbolic orbits.) The eccentricity vanishes ($e = 0$) for a circle and correspondingly $K = 0$ implies that $\dot{\mathbf{r}} = \mu\,\hat{\boldsymbol{\theta}}/L$.

▲

(e) *Use conservation of the magnitude $L = |\mathbf{L}|$ to show that the orbit sweeps out equal areas in equal times. This is* ***Kepler's Second Law****. Express the period in terms of $L = |\mathbf{L}|$ and the spatial orbital parameters. For an elliptical orbit, write the period first in terms of angular momentum and the area. Then write it in terms of angular momentum, semi-major axis and eccentricity of the elliipse. Hint: use $L = r^2\dot{\theta}$.*

Answer

We use $L = r^2\dot{\theta}$ in computing the area swept out during time t_1 to time t_2, as

$$A = \int_{\theta(t_1)}^{\theta(t_2)} \tfrac{1}{2} r \cdot r d\theta = \int_{t_1}^{t_2} \tfrac{1}{2} r \cdot (r\dot{\theta}) dt = \int_{t_1}^{t_2} \tfrac{1}{2} L dt = \tfrac{1}{2} L (t_2 - t_1)$$

So the area swept out is linear in the duration $t_2 - t_1$. This is Kepler's second law. For an elliptic orbit with semi-axes a and b, the area (πab) and period (T) are related by

$$\pi ab = \tfrac{1}{2} LT$$

So the period of the orbit satisfies

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^2b^2}{L^2} = \frac{a^4(1-e^2)}{L^2}$$

with eccentricity $e \geq 0$ defined by $b^2/a^2 = (1 - e^2)$.

▲

(f) *Use the result of Part 5e and the geometric properties of ellipses to show that the period of the orbit is given by*

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}$$

Hint: use the relationships between the orbital parameters and the conservaton laws. The relation T^2/a^3 = constant is ***Kepler's Third Law****. The constant was found by Newton.*

Answer

From previous parts of the problem, we know that

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^4(1-e^2)}{L^2}$$

with $b^2/a^2 = (1 - e^2)$ for elliptic orbits and

$$L^2 = \frac{\mu^2 - J^2}{-2E} = \frac{\mu^2(1-e^2)}{-2E} = \mu(1-e^2)a$$

after using $e^2 := J^2/\mu^2$ and $-2E = \mu/a$ for elliptic orbits. Therefore,

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}$$

The first equation is Kepler's Third Law. The second equation relates the period of the orbit to its energy. ▲

(g) *Write the Kepler motion equation (17) in Hamiltonian form*

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}\,, \qquad \dot{\mathbf{p}} = -\,\frac{\partial H}{\partial \mathbf{q}}\,.$$

Identify the position $\mathbf{q}$ *and the canonical momentum* $\mathbf{p}$ *in terms of* $\mathbf{r}$ *and* $\dot{\mathbf{r}}$. *Write the Hamiltonian* $H(\mathbf{q},\mathbf{p})$ *explicitly.*

$\boxed{\textit{Answer}}$

$$\begin{aligned}
\dot{\mathbf{q}} = \{\,\mathbf{q}\,,\,H\} &= \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p}\,,\\
\dot{\mathbf{p}} = \{\,\mathbf{p}\,,\,H\} &= -\,\frac{\partial H}{\partial \mathbf{q}} = -\mu\,\frac{\mathbf{q}}{|\mathbf{q}|^3}\,,\\
H &= \tfrac{1}{2}|\mathbf{p}|^2 - \frac{\mu}{|\mathbf{q}|}\,.
\end{aligned}$$

▲

(h) *List the Poisson brackets amongst the various components of the vectors* $\mathbf{L}$ *and* $\mathbf{J}$. *Show that they close amongst themselves. You may wish to recall that* $\{\mathbf{q},\mathbf{L}\cdot\boldsymbol{\xi}\} = \boldsymbol{\xi}\times\mathbf{q}$ *and* $\{\mathbf{p},\mathbf{L}\cdot\boldsymbol{\xi}\} = \boldsymbol{\xi}\times\mathbf{p}$ *for a given constant vector* $\boldsymbol{\xi}\in\mathbb{R}^3$.

$\boxed{\textit{Answer}}$

To recall, Poisson brackets with angular momentum components produces rotations of vectors:

$$\begin{aligned}
\{\,q_i\,,\,L_j\} &= \frac{\partial L_j}{\partial p_i} = \epsilon_{ijk}q_k,\\
\{\,p_i\,,\,L_j\} &= -\,\frac{\partial L_j}{\partial q_i} = \epsilon_{ijk}p_k,
\end{aligned}$$

so the vector field $X_{(\mathbf{L}\cdot\boldsymbol{\xi})} = \{\,\cdot\,,\,(\mathbf{L}\cdot\boldsymbol{\xi})\}$ infinitesimally rotates vectors in phase space around the constant direction $\boldsymbol{\xi}$ as $X_{(\mathbf{L}\cdot\boldsymbol{\xi})}\mathbf{J} = \boldsymbol{\xi}\times\mathbf{J}$.
Poisson brackets amongst the components of the vectors $\mathbf{L}$ and $\mathbf{J}$ are, as follows.

$$\begin{aligned}
\{L_i,\,L_j\} &= \epsilon_{ijk}L_k\,,\\
\{J_i,\,L_j\} &= \epsilon_{ijk}J_k\,,\\
\{J_i,\,J_j\} &= -\,2H\,\epsilon_{ijk}L_k\,.
\end{aligned}$$

Note that sign of last term; $-2H > 0$ for bounded orbits. ▲

(i) *Show that the Poisson brackets in the previous part imply*

$$\{\mathbf{J},\, J^2\} = -4H\mathbf{J}\times\mathbf{L} = -2H\{L^2,\, \mathbf{J}\}\,.$$

Answer

These formulas follow algebraically from the previous part. In particular,

$$\begin{aligned}
\{L_i,\, L^2\} &= 2\epsilon_{ijk}L_jL_k = 0\,,\\
\{J_i,\, L^2\} &= 2\epsilon_{ijk}L_jJ_k = 2(\mathbf{L}\times\mathbf{J})_i\,,\\
\{J_i,\, J^2\} &= -\,4H\,\epsilon_{ijk}J_jL_k = -\,4H\,(\mathbf{J}\times\mathbf{L})_i\,.
\end{aligned}$$

▲

(j) *The conservation laws* $\{L^2, H\} = 0$ *and* $\{\mathbf{J}, H\} = 0$ *allow the use of formula (18) to check consistency of the previous Poisson bracket relations, since that formula implies*

$$\big\{\mathbf{J},\big(J^2 - 2H\,L^2\big)\big\} = \big\{J_i, \mu^2\big\} = 0\,.$$

Upon referring to the relationships between the orbital parameters and the conservation laws derived in Part 5d, explain how the canonical transformations generated by $\mathbf{J}$ *affect the (i) energy, (ii) eccentricity and (iii) width of the orbit.*

Answer

- Poisson brackets with the LRL vector $\mathbf{J}$ affect the shape of the ellipse, *both* in its eccentricity (J^2) and its latus rectum (L^2), while preserving energy.
- Constancy of the LRL vector $\mathbf{J}$ and angular momentum vector $\mathbf{L}$ implies that the shape and orientation of the planar orbit remain constant.

▲

(k) *Suppose Newton had postulated the Yukawa potential with length scale* $\alpha > 0$

$$V(r) = -\,\mu\frac{e^{-r/\alpha}}{r}$$

as the potential energy for planetary motion. How much difference would that have made in the comparison with Kepler's Laws?

Answer

- In polar coordinates (r, θ) the modified conserved energy for a particle of unit mass would read

$$E = KE + PE = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}r^2\dot{\theta}^2 - \mu\frac{e^{-r/\alpha}}{r}$$

- Since the Yukawa potential still produces a central force, the angular momentum vector normal to the orbital plane with magnitude $L = r^2\dot{\theta}$ will still be conserved. This that means Kepler's Law of equal areas in equal times with still hold. However, the other two Kepler Laws will be altered.

– First, the motion equation now contains a fixed length-scale. This means the equation of motion has no symmetry under scaling of length and time, so Kepler's scaling law involving orbital sizes and orbital periods cannot hold.
– Second, on writing the energy as the Hamiltonian in polar coordinates,
$$H = \tfrac{1}{2}p^2 + \frac{L^2}{2r^2} - \mu\frac{e^{-r/\alpha}}{r}\,,$$
with canonical variables (r, p) and (θ, L), we find Hamilton's equations
$$\dot{r} = \frac{\partial H}{\partial p} = p \quad\text{and}\quad \dot{p} = -\frac{\partial H}{\partial r} = \frac{L^2}{r^3} - \mu\frac{e^{-r/\alpha}}{r^2}\Big(\frac{r}{\alpha}+1\Big)\,.$$
The corresponding equilibrium for a given constant magnitude of angular momentum L satisfies $p = 0$ and in the limit of $\alpha \to 0$
$$\lim_{\alpha\to 0}\frac{L^2}{\mu\, r} = \lim_{\alpha\to 0} e^{-r/\alpha}\Big(\frac{r}{\alpha}+1\Big) = 0$$
Hence, for any finite angular momentum L^2 and small enough α, the centre cannot hold and the equilibrium radius r_e will move to an asymptotically largevalue. On the other hand, for $\alpha \to \infty$ one recovers the usual equilibrium condition, $r_e = L^2/\mu$.
– Third, the orbit equation
$$\ddot{r} - \frac{L^2}{r^3} = -\mu\frac{e^{-r/\alpha}}{r^2}\Big(1+\frac{r}{\alpha}\Big)$$
transforms under $\dot{\theta} = L/r^2$, $r = 1/u$, $\dot{r} = -Ldu/d\theta$, $\ddot{r} = -(1/u^2)d^2u/d\theta^2$ into
$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2}e^{-1/u\alpha}\Big(1+\frac{1}{u\alpha}\Big)\,,$$
whose solutions are no longer simple trigonometric functions, except in the limit $\alpha \to \infty$, for which the right-hand side takes the constant value μ/L^2. This means the orbits for finite α will not be ellipses. For example, when $1/(u\alpha) = r/\alpha \ll 1$, the orbit becomes
$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2}\Big(1-\frac{1}{u^2\alpha^2}\Big) + O\Big(\frac{1}{u^4\alpha^4}\Big)\,,\quad\text{for}\quad \frac{1}{u^2\alpha^2} \ll 1,$$
whose solution is nearly an ellipse, but the perturbation terms cause it to precess instead of closing.
In an attempt to see the precession, we may use (17) to compute the evolution of Hamilton's vector
$$\begin{aligned}
\frac{d}{dt}\mathbf{K} = \frac{d}{dt}\Big(\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big) &= \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3}\\
&= \frac{\partial}{\partial\mathbf{r}}\Big(\mu\frac{e^{-r/\alpha}}{r}\Big) + \frac{\mu\mathbf{r}}{r^3}\\
&= -\frac{\mu\mathbf{r}}{r^3}\Big(e^{-r/\alpha}\Big(1+\frac{r}{\alpha}\Big) - 1\Big)\\
&= \frac{\mu\mathbf{r}}{\alpha^2 r}\Big(1 - O\Big(\frac{r^2}{\alpha^2}\Big)\Big)\quad\text{for}\quad \frac{r^2}{\alpha^2} \ll 1\,.
\end{aligned}$$

So the rate of change of Hamilton's vector is in the radial direction in the orbital plane. The Laplace-Runge-Lenz vector $\mathbf{J} = \mathbf{K} \times \mathbf{L}$

$$\mathbf{J} = \mathbf{K} \times \mathbf{L} = \dot{\mathbf{r}} \times \mathbf{L} - \mu\hat{\mathbf{r}}\,,$$

then satisfies the equation

$$\frac{d}{dt}\mathbf{J} = \frac{\mu\mathbf{r}}{\alpha^2 r}\left(1 - O\left(\frac{r^2}{\alpha^2}\right)\right) \times \mathbf{L} \quad \text{for} \quad \frac{r^2}{\alpha^2} \ll 1,$$

So the rate of change of the Laplace-Runge-Lenz vector is in the angular direction in the orbital plane.
The magnitude of $\mathbf{J}$ is *not* preserved for the Yukawa potential, but its projection in the $\hat{\mathbf{r}}$ direction varies slowly at a nearly constant rate for large α, as

$$\frac{d}{dt}(\hat{\mathbf{r}} \cdot \mathbf{J}) = -\frac{\mu L}{\alpha^2}\left(1 - O\left(\frac{r^2}{\alpha^2}\right)\right) \quad \text{for} \quad \frac{r^2}{\alpha^2} \ll 1.$$

▲

★

6. ***Exercise.*** **$\mathbb{R}^3$-bracket equations**

Part 1: Maxwell-Bloch equations

The real-valued Maxwell-Bloch system for $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is given by

$$\dot{x}_1 = kx_2\,, \quad \dot{x}_2 = x_1x_3\,, \quad \dot{x}_3 = -x_1x_2\,,$$

where k is a constant with the same units as that of (x_1, x_2, x_3) and time is dimensionless.

(a) Write this system in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

*where H_1 and H_2 are two conserved functions, one of whose level sets (let it be H_1) may be taken as **circular cylinders** oriented along the x_1-direction and the other (let it be H_2) whose level sets may be taken as **parabolic cylinders** oriented along the x_2-direction.*

$\boxed{\textit{Answer}}$

The real-valued Maxwell-Bloch system is expressible in three-dimensional vector notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are the two conserved functions

$$H_1 = \frac{1}{2}(x_2^2 + x_3^2) \quad \text{and} \quad H_2 = kx_3 + \frac{1}{2}x_1^2\,.$$

▲

(b) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of H_2. Show that the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic.

$\boxed{\textit{Answer}}$

A level set of $H_2 = kx_3 + \frac{1}{2}x_1^2$ is a ***parabolic cylinder*** oriented along the x_2-direction. On a level set of H_2, one has

$$H_1 = \frac{1}{2}x_2^2 + \frac{1}{2k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)^2 =: \frac{1}{2}x_2^2 + V(x_1)\,,$$

so that

$$d^{\,3}x = dx_1 \wedge dx_2 \wedge dx_3 = dx_1 \wedge dx_2 \wedge dH_2\,.$$

The $\mathbb{R}^3$ bracket restricts to such a level set as

$$\big\{F,\,H\big\}d^{\,3}x = dH_2 \wedge \big\{F,\,H_1\big\}_{p-cyl}dx_1 \wedge dx_2\,,$$

where the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic,

$$\big\{F,\,H_1\big\}_{p-cyl} = \frac{\partial F}{\partial x_1}\frac{\partial H_1}{\partial x_2} - \frac{\partial H_1}{\partial x_1}\frac{\partial F}{\partial x_2}\,.$$

▲

(c) *Derive the equation of motion on a level set of H_2 and express them in the form of Newton's Law.*

$\boxed{\textit{Answer}}$

Hence, the equations of motion on the parabolic cylinder $H_2 = const$ are

$$\begin{aligned}
\dot{x}_1 &= \frac{\partial H_1}{\partial x_2} = x_2\,,\\
\dot{x}_2 &= -\frac{\partial H_1}{\partial x_1} = -\frac{x_1}{k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)\,.
\end{aligned}$$

Therefore, an equation of motion for x_1 emerges, which may be expressed in the form of Newton's Law for the ***Duffing oscillator***,

$$\ddot{x}_1 = -\frac{x_1}{k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)\,.$$

▲

(d) *Identify steady solutions and determine which are unstable (saddle points) and which are stable (centers).*

$\boxed{\textit{Answer}}$

The Duffing oscillator has critical points at

$$(x_1,\,x_2) = \big(0,\,0\big) \quad \text{and} \quad \big(\pm\sqrt{2H_2},0\big)\,.$$

The first of these critical points is unstable (a saddle point) and the other two are stable (centers).

▲

(e) Determine the geometric and dynamic phases of a closed orbit on a level set of H_2.

Answer

The ***geometric phase*** for any closed orbit on the level set of H_2 is the integral

$$\Delta\phi_{geom} = \frac{1}{H_2}\int_A dx_1 \wedge dx_2 = -\frac{1}{H_2}\oint_{\partial A} x_2 dx_1\,,$$

the latter by Stokes theorem. Here A is the area enclosed by the solution orbit ∂A on a level set of H_2. Then

$$\begin{aligned}\Delta\phi_{geom} &= -\frac{1}{H_2}\oint_{\partial A} x_2\dot{x}_1\,dt = -\frac{1}{H_2}\oint_{\partial A} x_2\frac{\partial H}{\partial x_2}\,dt\\ &= = -\frac{1}{H_2}\oint_{\partial A} x_2^2\,dt = -\frac{2T}{H_2}\Big(H - \Big\langle V\Big\rangle\Big)\,,\end{aligned}$$

where

$$\Big\langle V\Big\rangle = \frac{1}{T}\oint_{\partial A}\frac{1}{2k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)^2\,dt$$

is the average of the potential energy over the orbit.

The ***dynamic phase*** is given by the formula,

$$\begin{aligned}\Delta\phi_{dyn} &= \frac{1}{H_2}\oint_{\partial A}\Big(x_2\dot{x}_1 + H_2\dot{\phi}\Big)dt\\ &= \frac{1}{H_2}\oint_{\partial A}\Big(x_2\frac{\partial H}{\partial x_2} + H_2\frac{\partial H}{\partial H_2}\Big)dt\\ &= \frac{1}{H_2}\oint_{\partial A} x_2^2 dt + \oint_{\partial A}\tfrac{1}{k^2}(H_2 - \tfrac{1}{2}x_1^2)\,dt\\ &= -\Delta\phi_{geom} + \frac{T}{k}\Big\langle\sqrt{2V}\Big\rangle\\ &= \frac{2T}{H_2}\Big(H - \Big\langle V\Big\rangle + \frac{H_2}{2k}\Big\langle\sqrt{2V}\,\Big\rangle\Big)\end{aligned}$$

where ϕ is the angle conjugate to H_2 and T is the period of the orbit around which the integration is performed. Thus, the total phase change around the orbit is

$$\Delta\phi_{tot} = \Delta\phi_{dyn} + \Delta\phi_{geom} = \frac{T}{k}\Big\langle\sqrt{2V}\Big\rangle\,.$$

▲

★

Part 2: Modulation equations – only one difference from Maxwell-Bloch!

The real 3-wave modulation equations on $\mathbb{R}^3$ differ in only one term from the Maxwell-Bloch equations in the previous part,

$$\dot{X}_1 = X_2X_3\,,\quad \dot{X}_2 = X_1X_3\,,\quad \dot{X}_3 = -X_1X_2\,.$$

Many of the calculations in the Maxwell-Bloch solution still go through after this change, but some may not. Of course, the method is still the same. Investigate the differences in the results. Time is still dimensionless.

(a) *Write these equations using an $\mathbb{R}^3$ bracket of the form,*

$$\dot{\mathbf{X}} = \nabla H \times \nabla C\,,$$

where level sets of H and C are both circular cylinders.

$\boxed{\textit{Answer}}$

These equations may be written in the $\mathbb{R}^3$ bracket form as,

$$\dot{\mathbf{X}} = \nabla H \times \nabla C = \nabla\frac{1}{2}(X_2^2 + X_3^2) \times \nabla\frac{1}{2}(X_1^2 + X_3^2)\,,$$

where level sets of H and C are circular cylinders, with H oriented along the X_1 and C oriented along the X_2 axis. ▲

(b) *Characterise the equilibrium points geometrically in terms of the gradients of H and C. How many are there? Which are stable?*

$\boxed{\textit{Answer}}$

Equilibria occur at points where the cross product of gradients $\nabla H \times \nabla C$ vanishes. In the orthogonal intersection of two circular cylinders as above, this may occur at points where the circular cylinders are tangent, and at points where the axis of one cylinder punctures normally through the surface of the other.
The elliptic cylinders are tangent at one $\mathbb{Z}_2$-symmetric pair of points along the X_3 axis, and the elliptic cylinders have normal axial punctures at two other $\mathbb{Z}_2$-symmetric pairs of points along the X_1 and X_2 axes. There is a total of 6 equilibrium points. 4 are stable and 2 are unstable. ▲

(c) *Choose cylindrical polar coordinates along the axis of the circular cylinder that represents the level set of C and restrict the $\mathbb{R}^3$ Poisson bracket to that level set. Show that the Poisson bracket on the circular cylinder C is symplectic.*

$\boxed{\textit{Answer}}$

Cylindrical polar coordinates are chosen along the axis of the circular cylinder level set of C by writing $(X_1, X_3) = (-r\cos\theta,\, r\sin\theta)$. Thus,

$$X_1^2 + X_3^2 = r^2(\cos^2\theta + \sin^2\theta) = 2C$$

Thus,

$$d^3X = -dX_2 \wedge dX_1 \wedge dX_3 = dX_2 \wedge d\frac{r^2}{2} \wedge d\theta = dC \wedge d\theta \wedge dX_2.$$

Restricting the $\mathbb{R}^3$ Poisson bracket to a level set of C yields

$$\{F,\, H\}d^3X = dC \wedge \{F,\, H\}_C\, d\theta \wedge dX_2$$

where on a level set of C,

$$\{F,\, H\}_C = \frac{\partial F}{\partial\theta}\frac{\partial H}{\partial X_2} - \frac{\partial H}{\partial\theta}\frac{\partial F}{\partial X_2} \quad \text{so that} \quad \{\theta,\, X_2\}_C = 1\,.$$

▲

(d) *Write the equations of motion on that level set.*

Answer

The Hamiltonian H on a level set of C is given by

$$H = \frac{1}{2}X_2^2 + 2C\sin^2\theta\,.$$

The equations of motion on a level set of C are given by

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial X_2} = X_2 \qquad \frac{dX_2}{dt} = -\,\frac{\partial H}{\partial \theta} = -\,2C\sin\theta\cos\theta$$

These reduce to the ***pendulum equation***,

$$\frac{d^2\theta}{dt^2} = -\,C\sin 2\theta\,.$$

▲

(e) *Determine the geometric and dynamic phases of a closed orbit on a level set of C.*

Answer

The ***geometric phase*** for any closed orbit on the level set of C is the integral

$$\Delta\phi_{geom} = \frac{1}{C}\int_A d\theta\wedge dX_2 = -\frac{1}{C}\oint_{\partial A} X_2 d\theta\,,$$

by Stokes theorem. Here A is the area enclosed by the solution orbit ∂A on a level set of C. Then

$$\begin{aligned}
\Delta\phi_{geom} &= -\frac{1}{C}\oint_{\partial A} X_2\,\dot\theta(t)\,dt = -\frac{1}{C}\oint_{\partial A} X_2\frac{\partial H}{\partial X_2}\,dt \\
&= = -\frac{1}{C}\oint_{\partial A} 2H - 4C\sin^2\theta\,dt = -\frac{2T}{C}\Big(H - 2\Big\langle V\Big\rangle\Big)\,,
\end{aligned}$$

where

$$\Big\langle V\Big\rangle = \frac{1}{T}\oint_{\partial A} 2C\sin^2\theta(t)\,dt$$

is the average of the potential energy over the orbit.

The ***dynamic phase*** is given by the formula,

$$\begin{aligned}
\Delta\phi_{dyn} &= \frac{1}{C}\oint_{\partial A}\Big(X_2\dot\theta + C\dot\phi\Big)dt \\
&= \frac{1}{C}\oint_{\partial A}\Big(X_2\frac{\partial H}{\partial X_2} + C\frac{\partial H}{\partial C}\Big)dt \\
&= \frac{1}{C}\oint_{\partial A} X_2^2 + 2C\sin^2\theta\,dt \\
&= \frac{1}{C}\oint_{\partial A} 2H - 2C\sin^2\theta\,dt \\
&= \frac{2T}{C}\Big(H - \Big\langle V\Big\rangle\Big)
\end{aligned}$$

where ϕ is the angle conjugate to C and T is the period of the orbit around which the integration is performed. Thus, the total phase change around the orbit is

$$\Delta\phi_{tot} = \Delta\phi_{dyn} + \Delta\phi_{geom} = \frac{2T}{C}\Big\langle V \Big\rangle .$$

▲

★

7. ***Exercise.* The fish: quadratically nonlinear oscillator**

Consider the Hamiltonian dynamics on a symplectic manifold of a system comprising two real degrees of freedom, with real phase space variables (x, y, θ, z), symplectic form

$$\omega = dx \wedge dy + d\theta \wedge dz$$

and Hamiltonian

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2}$$

(a) Write the canonical Poisson bracket for this system.

Answer

$$\{F\,,\,H\} = H_y F_x - H_x F_y + H_z F_\theta - H_\theta F_z$$

▲

(b) Write Hamilton's canonical equations for this system. Explain how to keep $z \ge 0$, so that H and θ remain real.

Answer

Hamilton's canonical equations for this system are

$$\dot{x} = \{x, H\} = H_y = y\,,$$
$$\dot{y} = \{y, H\} = -\,H_x = -(x^2 - z)\,,$$

and

$$\dot{\theta} = \{\theta, H\} = H_z = -(x + \sqrt{z}\,)\,,$$
$$\dot{z} = \{z, H\} = -\,H_\theta = 0\,.$$

For H and θ to remain real, one need only choose the initial value of the constant of motion $z \ge 0$. ▲

(c) At what values of x, y and H does the system have stationary points in the (x, y) plane?

Answer

The system has (x, y) stationary points when its time derivatives vanish: at $y = 0$, $x = \pm\sqrt{z}$ and $H = -\frac{4}{3}z^{3/2}$. ▲

(d) *Propose a strategy for solving these equations. In what order should they be solved?*

Answer

Since z is a constant of motion, the equation for its conjugate variable $\theta(t)$ decouples from the others and may be solved as a quadrature *after* first solving for $x(t)$ and $y(t)$ on a level set of z. ▲

(e) *Identify the constants of motion of this system and explain why they are conserved.*

Answer

There are two constants of motion:
(i) The Hamiltonian H for the canonical equations is conserved, because the Poisson bracket in $\dot{H} = \{H, H\}$ is antisymmetric.
(ii) The momentum z conjugate to θ is conserved, because $H_\theta = 0$. ▲

(f) *Compute the associated Hamiltonian vector field X_H and show that it satisfies*

$$X_H \,\lrcorner\, \omega = dH$$

Answer

$$\begin{aligned} X_H = \{\,\cdot\,,\, H\} &= H_y\partial_x - H_x\partial_y + H_z\partial_\theta - H_\theta\partial_z \\ &= y\partial_x - (x^2 - z)\partial_y - (x + \sqrt{z})\partial_\theta , \end{aligned}$$

so that

$$X_H \,\lrcorner\, \omega = ydy + (x^2 - z)dx - (x + \sqrt{z}\,)dz = dH$$

▲

(g) *Write the Poisson bracket that expresses the Hamiltonian vector field X_H as a divergenceless vector field in $\mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Explain why this Poisson bracket satisfies the Jacobi identity.*

Answer

Write the evolution equations for $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ as

$$\begin{aligned} \dot{\mathbf{x}} = \{\mathbf{x}, H\} = \nabla H \times \nabla z &= (H_y, -H_x, 0)^T \\ &= (y, z - x^2, 0)^T \\ &= (\dot{x}, \dot{y}, \dot{z})^T . \end{aligned}$$

Hence, for any smooth function $F(\mathbf{x})$,

$$\{F, H\} = \nabla z \cdot \nabla F \times \nabla H = F_x H_y - H_x F_y \,.$$

This is the canonical Poisson bracket for one degree of freedom, which is known to satisfy the Jacobi identity. ▲

(h) Identify the Casimir function for this $\mathbb{R}^3$ bracket. Show explicitly that it satisfies the definition of a Casimir function.

Answer

Substituting $F = \Phi(z)$ for a smooth function Φ into the bracket expression yields

$$\{\Phi(z), H\} = \nabla z \cdot \nabla\Phi(z) \times \nabla H = \nabla H \cdot \nabla z \times \nabla\Phi(z) = 0,$$

for all H. This proves that $F = \Phi(z)$ is a Casimir function for any smooth Φ. ▲

(i) Sketch a graph of the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function. Show the directions of flow along these intersections. Identify the locations and types of any relative equilibria at the tangent points of these surfaces.

Answer

The sketch should show a saddle-node fish shape pointing rightward in the (x, y) plane with elliptic equilibrium at $(x, y) = (\sqrt{z}, 0)$, hyperbolic equilibrium at $(x, y) = (-\sqrt{z}, 0)$ and directions of flow with sign($\dot{x}$)=sign(y). The fish shape is sketched in Figure 1 for $z = 1$. ▲

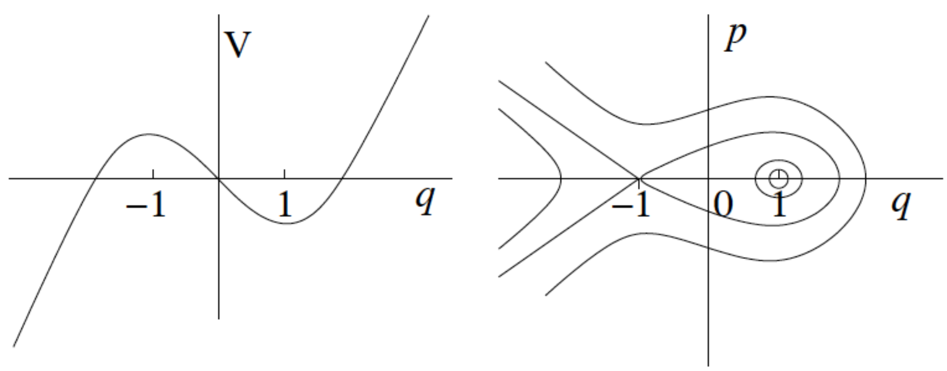

Figure 1: Phase plane for the saddle-node fish shape arising from the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function.

(j) Linearise around the relative equilibria on a level set of the Casimir (z) and compute its eigenvalues.

Answer

On a level surface of z the (x, y) coordinates satisfy $\dot{x} = y$ and $\dot{y} = z - x^2$. Linearising around $(x_e, y_e) = (\pm\sqrt{z}, 0)$ yields with $(x, y) = (x_e + \xi(t), y_e + \eta(t))$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2x_e & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Its characteristic equation,

$$\det \begin{bmatrix} \lambda & -1 \\ 2x_e & \lambda \end{bmatrix} = \lambda^2 + 2x_e = 0,$$

yields $\lambda^2 = -2x_e = \mp 2\sqrt{z}$.
Hence, the eigenvalues are,
$\lambda = \pm i\sqrt{2}z^{1/4}$ at the elliptic equilibrium $(x_e, y_e) = (\sqrt{z}, 0)$, and
$\lambda = \pm\sqrt{2}z^{1/4}$ at the hyperbolic equilibrium $(x_e, y_e) = (-\sqrt{z}, 0)$. ▲

(k) *If you found a hyperbolic equilibrium point in the previous part connected to itself by a homoclinic orbit, then reduce the equation for the homoclinic orbit to an indefinite integral expression.*

Answer

On the homoclinic orbit the Hamiltonian vanishes, so that

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2} = 0.$$

Using $y = \dot{x}$, rearranging and integrating implies the indefinite integral expression, or "quadrature",

$$\int \frac{dx}{\sqrt{2z^{3/2} - x^3 + 3zx}} = \sqrt{\tfrac{2}{3}} \int dt\,.$$

After some work this integrates to

$$\frac{x(t) + \sqrt{z}}{3\sqrt{z}} = \operatorname{sech}^2\left(\frac{z^{1/4}t}{\sqrt{2}}\right).$$

From this equation, one may also compute the evolution of $\theta(t)$ on the homoclinic orbit by integrating the θ-equation,

$$\frac{d\theta}{dt} = -(x(t) + \sqrt{z}\,).$$

▲

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