2 M3-4-5 A16 Assessed Problems # 2 Due 2pm 1 Dec 2011

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 2.1 ***$\mathbb{R}^3$-bracket for Maxwell-Bloch equations***

The real-valued Maxwell-Bloch system for $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ *is given by*

$$\dot{x}_1 = kx_2\,, \quad \dot{x}_2 = x_1x_3\,, \quad \dot{x}_3 = -x_1x_2\,,$$

where k *is a constant with the same units as that of* (x_1, x_2, x_3) *and time is dimensionless.*

(a) Write this system in three-dimensional vector $\mathbb{R}^3$*-bracket notation as*

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 *and* H_2 *are two conserved functions, one of whose level sets (let it be* H_1*) may be taken as* ***circular cylinders*** *oriented along the* x_1*-direction and the other (let it be* H_2*) whose level sets may be taken as* ***parabolic cylinders*** *oriented along the* x_2*-direction.*

Answer

The real-valued Maxwell-Bloch system is expressible in three-dimensional vector notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where H_1 and H_2 are the two conserved functions

$$H_1 = \frac{1}{2}(x_2^2 + x_3^2) \quad \text{and} \quad H_2 = kx_3 + \frac{1}{2}x_1^2\,.$$

▲

(b) Restrict the equations and their $\mathbb{R}^3$ *Poisson bracket to a level set of* H_2*. Show that the Poisson bracket on the parabolic cylinder* $H_2 = const$ *is symplectic.*

Answer

A level set of $H_2 = kx_3 + \frac{1}{2}x_1^2$ is a ***parabolic cylinder*** oriented along the x_2-direction. On a level set of H_2, one has

$$H_1 = \frac{1}{2}x_2^2 + \frac{1}{2k^2}\left(H_2 - \frac{1}{2}x_1^2\right)^2 =: \frac{1}{2}x_2^2 + V(x_1)\,,$$

so that

$$d^3x = dx_1 \wedge dx_2 \wedge dx_3 = dx_1 \wedge dx_2 \wedge dH_2\,.$$

The $\mathbb{R}^3$ bracket restricts to such a level set as

$$\{F,\, H\}d^3x = dH_2 \wedge \{F,\, H_1\}_{p-cyl} dx_1 \wedge dx_2\,,$$

where the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic,

$$\{F,\, H_1\}_{p-cyl} = \frac{\partial F}{\partial x_1}\frac{\partial H_1}{\partial x_2} - \frac{\partial H_1}{\partial x_1}\frac{\partial F}{\partial x_2}\,.$$

▲

(c) Derive the equation of motion on a level set of H_2 *and express them in the form of Newton's Law.*

Answer

Hence, the equations of motion on the parabolic cylinder $H_2 = const$ are

$$\begin{aligned}\dot{x}_1 &= \frac{\partial H_1}{\partial x_2} = x_2\,, \\ \dot{x}_2 &= -\frac{\partial H_1}{\partial x_1} = -\frac{x_1}{k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)\,.\end{aligned}$$

Therefore, an equation of motion for x_1 emerges, which may be expressed in the form of Newton's Law for the ***Duffing oscillator***,

$$\ddot{x}_1 = -\frac{x_1}{k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)\,.$$

▲

(d) *Identify steady solutions and determine which are unstable (saddle points) and which are stable (centers).*

Answer

The Duffing oscillator has critical points at

$$(x_1,\, x_2) = \big(0,\, 0\big) \quad \text{and} \quad \big(\pm\sqrt{2H_2}, 0\big)\,.$$

The first of these critical points is unstable (a saddle point) and the other two are stable (centers).

▲

(e) *Determine the geometric and dynamic phases of a closed orbit on a level set of H_2.*

Answer

The ***geometric phase*** for any closed orbit on the level set of H_2 is the integral

$$\Delta\phi_{geom} = \frac{1}{H_2}\int_A dx_1 \wedge dx_2 = -\frac{1}{H_2}\oint_{\partial A} x_2 dx_1\,,$$

the latter by Stokes theorem. Here A is the area enclosed by the solution orbit ∂A on a level set of H_2. Then

$$\begin{aligned}\Delta\phi_{geom} &= -\frac{1}{H_2}\oint_{\partial A} x_2\dot{x}_1\,dt = -\frac{1}{H_2}\oint_{\partial A} x_2\frac{\partial H}{\partial x_2}\,dt \\ &= = -\frac{1}{H_2}\oint_{\partial A} x_2^2\,dt = -\frac{2T}{H_2}\Big(H - \big\langle V\big\rangle\Big)\,,\end{aligned}$$

where

$$\big\langle V\big\rangle = \frac{1}{T}\oint_{\partial A}\frac{1}{2k^2}\Big(H_2 - \frac{1}{2}x_1^2\Big)^2 dt$$

is the average of the potential energy over the orbit.

The ***dynamic phase*** is given by the formula,

$$\begin{aligned}\Delta\phi_{dyn} &= \frac{1}{H_2}\oint_{\partial A}\Big(x_2\dot{x}_1 + H_2\dot{\phi}\Big)dt \\ &= \frac{1}{H_2}\oint_{\partial A}\Big(x_2\frac{\partial H}{\partial x_2} + H_2\frac{\partial H}{\partial H_2}\Big)dt \\ &= \frac{1}{H_2}\oint_{\partial A} x_2^2 dt + \oint_{\partial A}\tfrac{1}{k^2}(H_2 - \tfrac{1}{2}x_1^2)\,dt \\ &= -\Delta\phi_{geom} + \frac{T}{k}\Big\langle\sqrt{2V}\Big\rangle \\ &= \frac{2T}{H_2}\Big(H - \big\langle V\big\rangle + \frac{H_2}{2k}\Big\langle\sqrt{2V}\,\Big\rangle\Big)\end{aligned}$$

where ϕ is the angle conjugate to H_2 and T is the period of the orbit around which the integration is performed. Thus, the total phase change around the orbit is

$$\Delta\phi_{tot} = \Delta\phi_{dyn} + \Delta\phi_{geom} = \frac{T}{k}\Big\langle \sqrt{2V} \Big\rangle .$$

▲

Exercise 2.2 ***The fish: quadratically nonlinear oscillator***

Consider the Hamiltonian dynamics on a symplectic manifold of a system comprising two real degrees of freedom, with real phase space variables (x, y, θ, z), symplectic form

$$\omega = dx \wedge dy + d\theta \wedge dz$$

and Hamiltonian

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2}$$

(a) Write the canonical Poisson bracket for this system.

Answer

$$\{F\,,\,H\} = H_y F_x - H_x F_y + H_z F_\theta - H_\theta F_z$$

▲

(b) Write Hamilton's canonical equations for this system. Explain how to keep $z \ge 0$, so that H and θ remain real.

Answer

Hamilton's canonical equations for this system are

$$\dot{x} = \{x, H\} = H_y = y\,,$$
$$\dot{y} = \{y, H\} = -\,H_x = -(x^2 - z)\,,$$

and

$$\dot{\theta} = \{\theta, H\} = H_z = -(x + \sqrt{z}\,)\,,$$
$$\dot{z} = \{z, H\} = -\,H_\theta = 0\,.$$

For H and θ to remain real, one need only choose the initial value of the constant of motion $z \ge 0$. ▲

(c) At what values of x, y and H does the system have stationary points in the (x, y) plane?

Answer

The system has (x, y) stationary points when its time derivatives vanish: at $y = 0$, $x = \pm\sqrt{z}$ and $H = -\frac{4}{3}z^{3/2}$. ▲

(d) Propose a strategy for solving these equations. In what order should they be solved?

Answer

Since z is a constant of motion, the equation for its conjugate variable $\theta(t)$ decouples from the others and may be solved as a quadrature *after* first solving for $x(t)$ and $y(t)$ on a level set of z. ▲

(e) Identify the constants of motion of this system and explain why they are conserved.

Answer

There are two constants of motion:
(i) The Hamiltonian H for the canonical equations is conserved, because the Poisson bracket in $\dot{H} = \{H, H\}$ is antisymmetric.
(ii) The momentum z conjugate to θ is conserved, because $H_\theta = 0$. ▲

(f) *Compute the associated Hamiltonian vector field X_H and show that it satisfies*

$$X_H \lrcorner\, \omega = dH$$

Answer

$$\begin{aligned} X_H = \{\,\cdot\,,\, H\} &= H_y\partial_x - H_x\partial_y + H_z\partial_\theta - H_\theta\partial_z \\ &= y\partial_x - (x^2 - z)\partial_y - (x + \sqrt{z})\partial_\theta , \end{aligned}$$

so that

$$X_H \lrcorner\, \omega = ydy + (x^2 - z)dx - (x + \sqrt{z}\,)dz = dH$$

▲

(g) *Write the Poisson bracket that expresses the Hamiltonian vector field X_H as a divergenceless vector field in $\mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Explain why this Poisson bracket satisfies the Jacobi identity.*

Answer

Write the evolution equations for $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ as

$$\begin{aligned} \dot{\mathbf{x}} = \{\mathbf{x}, H\} = \nabla H \times \nabla z &= (H_y, -H_x, 0)^T \\ &= (y, z - x^2, 0)^T \\ &= (\dot{x}, \dot{y}, \dot{z})^T . \end{aligned}$$

Hence, for any smooth function $F(\mathbf{x})$,

$$\{F, H\} = \nabla z \cdot \nabla F \times \nabla H = F_x H_y - H_x F_y \,.$$

This is the canonical Poisson bracket for one degree of freedom, which is known to satisfy the Jacobi identity. ▲

(h) *Identify the Casimir function for this $\mathbb{R}^3$ bracket. Show explicitly that it satisfies the definition of a Casimir function.*

Answer

Substituting $F = \Phi(z)$ for a smooth function Φ into the bracket expression yields

$$\{\Phi(z), H\} = \nabla z \cdot \nabla\Phi(z) \times \nabla H = \nabla H \cdot \nabla z \times \nabla\Phi(z) = 0,$$

for all H. This proves that $F = \Phi(z)$ is a Casimir function for any smooth Φ. ▲

(i) *Sketch a graph of the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function. Show the directions of flow along these intersections. Identify the locations and types of any relative equilibria at the tangent points of these surfaces.*

Answer

The sketch should show a saddle-node fish shape pointing rightward in the (x, y) plane with elliptic equilibrium at $(x, y) = (\sqrt{z}, 0)$, hyperbolic equilibrium at $(x, y) = (-\sqrt{z}, 0)$ and directions of flow with sign($\dot{x}$)=sign(y). The fish shape is sketched in Figure 1 for $z = 1$. ▲

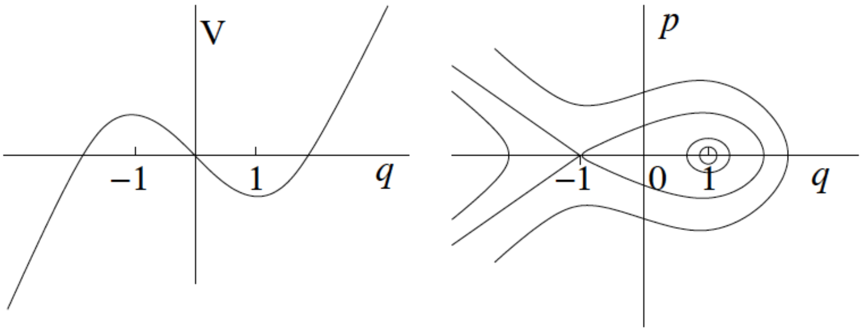

Figure 1: Phase plane for the saddle-node fish shape arising from the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function.

(j) *Linearise around the relative equilibria on a level set of the Casimir (z) and compute its eigenvalues.*

Answer

On a level surface of z the (x, y) coordinates satisfy $\dot{x} = y$ and $\dot{y} = z - x^2$. Linearising around $(x_e, y_e) = (\pm\sqrt{z}, 0)$ yields with $(x, y) = (x_e + \phi_1(t), y_e + \phi_2(t))$

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2x_e & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

Its characteristic equation,

$$\det \begin{bmatrix} \lambda & -1 \\ 2x_e & \lambda \end{bmatrix} = \lambda^2 + 2x_e = 0,$$

yields $\lambda^2 = -2x_e = \mp 2\sqrt{z}$.
Hence, the eigenvalues are,
$\lambda = \pm i\sqrt{2}z^{1/4}$ at the elliptic equilibrium $(x_e, y_e) = (\sqrt{z}, 0)$, and
$\lambda = \pm\sqrt{2}z^{1/4}$ at the hyperbolic equilibrium $(x_e, y_e) = (-\sqrt{z}, 0)$. ▲

(k) *If you found a hyperbolic equilibrium point in the previous part connected to itself by a homoclinic orbit, then reduce the equation for the homoclinic orbit to an indefinite integral expression.*

Answer

On the homoclinic orbit the Hamiltonian vanishes, so that

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2} = 0.$$

Using $y = \dot{x}$, rearranging and integrating implies the indefinite integral expression, or "quadrature",

$$\int \frac{dx}{\sqrt{2z^{3/2} - x^3 + 3zx}} = \sqrt{\tfrac{2}{3}} \int dt\,.$$

After some work this integrates to

$$\frac{x(t) + \sqrt{z}}{3\sqrt{z}} = \operatorname{sech}^2\left(\frac{z^{1/4}t}{\sqrt{2}}\right).$$

From this equation, one may also compute the evolution of $\theta(t)$ on the homoclinic orbit by integrating the θ-equation,

$$\frac{d\theta}{dt} = -(x(t) + \sqrt{z}\,).$$

▲

Exercise 2.3 ***2D coupled oscillators***

Consider the 2D oscillator Hamiltonian $H : \mathbb{C}^2 \to \mathbb{R}$, with complex 2-vector $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2$ and constant frequencies ω_j,

$$H = \frac{1}{2}\sum_{j=1}^{2} \omega_j |a_j|^2 = \frac{1}{4}(\omega_1 + \omega_2)\big(|a_1|^2 + |a_2|^2\big) + \frac{1}{4}(\omega_1 - \omega_2)\big(|a_1|^2 - |a_2|^2\big)\,.$$

(a) Compute its canonical Hamiltonian dynamics with

$$\{a_j, a_k^*\} = -2i\delta_{jk}.$$

Explain why this is the sum of a $1:1$ resonant oscillator and a $1:-1$ oscillator.

Answer

The flow generated by the Hamiltonian vector field of $R = |a_1|^2 + |a_2|^2$ given by

$$\frac{da_j}{dr} := \{a_j, R\} = -2i\,\frac{\partial R}{\partial a_j^*} = -2i\,a_j$$

whose solution is the $1:1$ resonant oscillator motion

$$R : (a_1, a_2) \to (e^{-2ir} a_1, e^{-2ir} a_2)$$

Likewise, with $Z = |a_1|^2 - |a_2|^2$ we have

$$\frac{da_j}{dz} := \{a_j, Z\} = -2i\,\frac{\partial Z}{\partial a_j^*}\,, \qquad \frac{da_1}{dz} = -2i\,a_1 \qquad \frac{da_2}{dz} = +2i\,a_2$$

which has the $1:-1$ resonant oscillator solution

$$Z : (a_1, a_2) \to (e^{-2iz} a_1, e^{+2iz} a_2)$$

▲

(b) Find the transformations generated by X, Y, Z, R on a_1, a_2, where

$$\begin{aligned} R &= |a_1|^2 + |a_2|^2\,, \\ Z &= |a_1|^2 - |a_2|^2\,, \\ X - iY &= 2a_1 a_2^*\,. \end{aligned}$$

Express these infinitesimal transformations as matrix operations and identify their corresponding finite transformations.

Answer See GM1 page 375.
From the definition of the Hamiltonian vector field

$$\{\,\cdot\,,\, H\,\} = -2i\left(\frac{\partial H}{\partial a_j^*}\frac{\partial}{\partial a_j} - \frac{\partial H}{\partial a_j}\frac{\partial}{\partial a_j^*}\right)$$

one finds the following linear transformations for the quadratic quantities, X, Y, Z, R,

$$\frac{d}{dr}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -2i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad \frac{d}{dz}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

and

$$\frac{d}{dx}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -2i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad \frac{d}{dy}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -2i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

These linear transformations summon the four 2×2 Pauli spin matrices $(\sigma_R, \sigma_X,\ \sigma_Y,\ \sigma_Z)$ given, respectively, by

$$\sigma_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The corresponding finite transformations are found by solving the differential equations in the previous part,

$$\begin{bmatrix} a_1(r) \\ a_2(r) \end{bmatrix} = \begin{bmatrix} e^{-2ir}a_1(0) \\ e^{-2ir}a_2(0) \end{bmatrix}, \qquad \begin{bmatrix} a_1(z) \\ a_2(z) \end{bmatrix} = \begin{bmatrix} e^{-2iz}a_1(0) \\ e^{+2iz}a_2(0) \end{bmatrix}$$
$$\begin{bmatrix} a_1(x) \\ a_2(x) \end{bmatrix} = \begin{bmatrix} \cos(2x) & -\sin(2x) \\ \sin(2x) & \cos(2x) \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix}, \qquad \begin{bmatrix} a_1(y) \\ a_2(y) \end{bmatrix} = \begin{bmatrix} \cos(2y) & -\sin(2y) \\ \sin(2y) & \cos(2y) \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix}$$

▲

(c) For the starting Hamiltonian,

$$\begin{aligned} H &= \frac{\omega_1}{2}(R+Z) + \frac{\omega_2}{2}(R-Z) \\ &= \frac{1}{2}(\omega_1+\omega_2)R + \frac{1}{2}(\omega_1-\omega_2)Z\,, \end{aligned}$$

write the equations $\dot X, \dot Y, \dot Z, \dot R$ for the S^1 invariants X, Y, Z, R of the $1:1$ resonance.

Write these equations in vector form, with $\mathbf{X} = (X, Y, Z)^T$, and describe this motion in terms of level sets of the Poincaré sphere and the Hamiltonian H.

Answer See GM1 page 377.
The dynamics is given by the Poisson bracket relation

$$\begin{aligned} \dot F = \{F, H\} &= -\nabla\frac{R^2}{2}\cdot\nabla F\times\nabla H(X,Y,Z) \\ &= -\frac{1}{2}(\omega_1-\omega_2)\,\nabla\frac{R^2}{2}\cdot\nabla F\times\nabla Z\,. \end{aligned}$$

Then $\dot R = 0 = \dot Z$ and

$$\dot X = \frac{1}{2}(\omega_1-\omega_2)Y\,, \qquad \dot Y = -\frac{1}{2}(\omega_1-\omega_2)X\,.$$

In vector form, with $\mathbf{X} = (X, Y, Z)^T$, this is

$$\dot{\mathbf{X}} = \frac{1}{2}(\omega_1-\omega_2)\mathbf{X}\times\widehat{\mathbf{Z}}\,,$$

where $\widehat{\mathbf{Z}}$ is the unit vector in the Z-direction ($\cos\theta = 0$). This motion is uniform rotation in the positive direction along a latitude of the Poincaré sphere $R = const$. This azimuthal rotation on a latitude at fixed polar angle on the sphere occurs along the intersections of level sets of the Poincaré sphere $R = const$ and the planes $Z = const$, which are level sets of the Hamiltonian for a fixed value of R.

▲

Exercise 2.4 *Matrix rigid body equations & cotangent lift momentum maps*

(a) Let the Lie group $SO(n)$ act on itself with infinitesimal transformation

$$\Phi_\Xi(Q) = Q\Xi \quad \text{for} \quad Q \in SO(n) \quad \text{and} \quad \Xi = -\,\Xi^T \in \mathfrak{so}(n)$$

Compute the cotangent lift (CL) momentum map for this action and its CL infinitesimal action on $T^*SO(n)$.

Answer

From the definition of CL momentum map, $M(P,Q) : T^*SO(n) \to \mathfrak{so}(n)^*$, we have

$$M^\Xi = \langle M(P,Q)\,,\,\Xi\rangle = \langle P_Q\,,\,\Phi_\Xi(Q)\rangle = \operatorname{tr}\left(P^TQ\Xi\right) = \operatorname{tr}\left(\frac{1}{2}\left(P^TQ - Q^TP\right)\Xi\right)$$

So

$$M(P,Q) = -\,\frac{1}{2}\left(P^TQ - Q^TP\right)$$

The corresponding infinitesimal action on $(Q,P) \in T^*SO(n)$ by CL are given by the canonical equations for $M^\Xi(Q,P)$,

$$Q' = \{Q, M^\Xi\} = \frac{\partial M^\Xi}{\partial P} = \Phi_\Xi(Q) = Q\Xi$$

and

$$P' = \{P, M^\Xi\} = -\,\frac{\partial M^\Xi}{\partial Q} = P\Xi$$

▲

(b) Compute the variations in Hamilton's principle $\delta S = 0$ with Clebsch-constrained action integral

$$S(\Omega, Q, P) = \int_a^b l(\Omega) + \operatorname{tr}\Big(P^T\left(\dot{Q} - Q\Omega\right)\Big)\,dt\,.$$

Discuss the relation between these variational equations and the equations for the infinitesimal Lie algebra actions associated with CL momentum maps.

Answer

The variational equations are:

$$M := \frac{\partial l}{\partial \Omega} = \frac{1}{2}(P^TQ - Q^TP)$$

and

$$\dot{Q} = Q\Omega \quad \text{and} \quad \dot{P} = P\Omega\,,$$

as a result of the constraints.

These take exactly the same form as the equations for the infinitesimal Lie algebra actions associated with CL momentum maps. ▲

(c) Show that the Clebsch-constrained Hamilton's principle implies that $M = \partial l/\partial\Omega$ satisfies the Euler-Poincaré equation

$$\frac{dM}{dt} = \operatorname{ad}^*_\Omega M = -\left[\,\Omega,\, M\,\right].$$

Answer

This is a direct calculation that uses the Jacobi identity. It also follows because CL momentum maps are infinitesimally equivariant, so they satisfy the EP equation. ▲

Exercise 2.5 1:2 ***resonance***

The Hamiltonian $\mathbb{C}^2 \to \mathbb{R}$ *for a certain* 1:2 *resonance is given by*

$$H = \tfrac{1}{2}|a_1|^2 - |a_2|^2 + \tfrac{1}{2}\mathrm{Im}(a_1^{*2}a_2)\,,$$

in terms of canonical variables $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$ *whose Poisson bracket relation*

$$\{a_j, a_k^*\} = -2i\delta_{jk}, \quad \textit{for} \quad j,k = 1,2\,,$$

is invariant under the 1:2 *resonance* S^1 *transformation*

$$a_1 \to e^{i\phi} \quad \textit{and} \quad a_2 \to e^{2i\phi}.$$

(a) *Write the motion equations in terms of the canonical variables* $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$.

Answer

The canonical Poisson bracket relations, $\{a_j, a_k^*\} = -2i\delta_{jk}$ for $j,k = 1,2$ imply

$$\dot{a}_1 = \{a_1, H\} = -2i\,\frac{\partial H}{\partial a_1^*} = -\,ia_1 - a_1^* a_2 \quad \text{and} \quad \dot{a}_2 = \{a_2, H\} = -2i\,\frac{\partial H}{\partial a_2^*} = 2ia_2 + \frac{1}{2}a_1^2$$

▲

(b) *Introduce the orbit map* $\mathbb{C}^2 \to \mathbb{R}^4$

$$\pi:\ (a_1, a_1^*, a_2, a_2^*) \to \{X, Y, Z, R)\}$$

and transform the Hamiltonian H *on* $\mathbb{C}^2$ *to new variables* $X, Y, Z, R \in \mathbb{R}^4$ *given by*

$$\begin{aligned} R &= \tfrac{1}{2}|a_1|^2 + |a_2|^2\,, \\ Z &= \tfrac{1}{2}|a_1|^2 - |a_2|^2\,, \\ X - iY &= 2a_1^{*2}a_2\,, \end{aligned}$$

that are invariant under the 1:2 *resonance* S^1 *transformation.*

Answer

Substitution of the definitions of X, Y, Z, R above yields

$$H = \tfrac{1}{2}|a_1|^2 - |a_2|^2 + \tfrac{1}{2}\mathrm{Im}(a_1^{*2}a_2) = Z - \tfrac{1}{4}Y$$

▲

(c) *Show that these variables are functionally dependent, because they satisfy a cubic algebraic relation* $C(X, Y, Z, R) = 0$.

Answer

One shows that these variables are not independent by verifying that the cubic equation,

$$C(X, Y, Z, R) = X^2 + Y^2 - 2(R - Z)(R + Z)^2 = 0$$

so that $C(X, Y, Z, R)$ vanishes identically.

▲

(d) *Use the orbit map* $\pi : \mathbb{C}^2 \to \mathbb{R}^4$ *to make a table of Poisson brackets among the four quadratic* 1:2 *resonance* S^1*-invariant variables* $X, Y, Z, R \in \mathbb{R}^4$.

Answer

We have $\nabla C = 2\big(X,\,Y,\,(R^2 - 2ZR - 3Z^2)\big)$. Denoting $(X_1, X_2, X_3) = (X, Y, Z)$ gives

$$\{X_i, X_j\} = -\,\epsilon_{ijk}\frac{\partial C}{\partial X_k} \quad \text{and} \quad \{X_i, R\} = 0 \quad \text{so} \quad \dot{\mathbf{X}} = \nabla C \times \nabla H$$

▲

(e) *Show that both R and the cubic algebraic relation $C(X, Y, Z, R) = 0$ are Casimirs for these Poisson brackets.*

Answer

The Poisson brackets $\{R, a_1\} = ia_1$ and $\{R, a_2\} = 2ia_2$, $\{R, \cdot\}$ show that the quantity R generates the $1{:}2$ resonance S^1 transformation. This implies that

$$\{R, X\} = \{R, Y\} = \{R, Z\} = 0\,.$$

because X, Y, Z, R are invariant under the $1{:}2$ resonance S^1 phase shift. Likewise, the definition of the $\mathbb{R}^3$ Nambu bracket

$$\{F, H\} = -\,\nabla C \cdot \nabla F \times \nabla H$$

implies that C is its Casimir. That is,

$$\{C, H\} = -\,\nabla C \cdot \nabla C \times \nabla H = 0$$

for any Hamiltonian $H(X, Y, Z)$. ▲

(f) *Write the Hamiltonian, Poisson bracket and equations of motion in terms of the remaining variables $\mathbf{X} = (X, Y, Z)^T \in \mathbb{R}^3$.*

Answer

Hamiltonian: $H = Z - Y/4$,
Poisson bracket: $\{F, H\} = -\,\nabla C \cdot \nabla F \times \nabla H$,
Equations of motion: $\dot{\mathbf{X}} = \nabla C \times \nabla H$

$$\begin{aligned}
\dot{X} &= \{X,\, H\} = -2Y - \frac{1}{2}(R^2 - 2ZR - 3Z^2) \\
\dot{Y} &= \{Y,\, H\} = -2X \\
\dot{Z} &= \{Z,\, H\} = -X/2
\end{aligned}$$

▲

(g) *Describe this motion in terms of level sets of the Hamiltonian H and the orbit manifold for the $1{:}2$ resonance, given by $C(X, Y, Z, R) = 0$.*

Answer

The motion takes place along intersections of the level sets of C (which are cubic surfaces of revolution indexed by the value of R) and H (which are X-invariant planes of positive slope $(dZ/dY = 1/4)$. ▲

(h) *Restrict the dynamics to a level set of the Hamiltonian and show that it reduces there to the equation of motion for a point particle in a cubic potential. Explain its geometrical meaning.*

Answer

As in Part (f), the equations of motion: $\dot{\mathbf{X}} = \nabla C \times \nabla H$ in components $(X_1, X_2, X_3) = (X, Y, Z)$ are

$$\begin{aligned}\dot{X} &= \{X,\, H\} = -2Y - \frac{1}{2}(R^2 - 2ZR - 3Z^2)\\ \dot{Y} &= \{Y,\, H\} = -2X\\ \dot{Z} &= \{Z,\, H\} = -X/2\end{aligned}$$

Inserting $Z = H + Y/4$, taking a time derivative to obtain $\ddot{Y}$ and eliminating X and Z yields Newton's law for Y with a cubic potential,

$$\ddot{Y} = -V'(Y) \quad \text{with} \quad V(Y) = -\frac{1}{32}Y^3 + \frac{1}{8}(8 - R - 3H)Y^2 - \frac{1}{2}(3H^2 + 2RH - R^2)Y$$

The solution behaviour of this equation depends on the values of R and H. In particular, it undergoes nonlinear oscillations when the discriminant of the quadratic equilibrium condition $V'(Y) = 0$ is positive. In this case, the phase plane has a homoclinic orbit in the shape of a fish heading *leftward*, i.e., in the opposite sense from Exercise **2.2**.

At the hyperbolic point the Hamiltonian plane intersects the reduced orbit manifold $C(X, Y, Z, R) = 0$ at its corner singularity. ▲

(i) Compute the geometric and dynamic phases for any closed orbit on a level set of H.

Answer

On a level set of H the motion is canonical in terms of Y and its canonical momentum $P = \dot{Y} = -2X$ with the Hamiltonian

$$h(P, Y, H, R) = \frac{1}{2}P^2 + V(Y, H, R)$$

Thus,

$$H d\phi = -PdY + p_j dq_j$$

The geometric phase is given by the area of the orbit

$$H\Delta\phi_{geom} = H \oint d\phi = -\oint PdY = -\iint dP \wedge dY = 2\iint dX \wedge dY$$

and the dynamic phase for orbits of period T is given by the sum,

$$\begin{aligned}H\Delta\phi_{dyn} = \oint p_j \dot{q}_j dt &= \int_0^T (P\dot{Y} + H\dot{\phi})\, dt = \int_0^T \left(P\frac{\partial h}{\partial P} + H\frac{\partial h}{\partial H} \right) dt\\ &= \int_0^T \left(2(h - V) + H\frac{\partial V}{\partial H} \right) dt\end{aligned}$$

Since V is not a monomial in H, we as may well leave the expression for $H\Delta\phi_{dyn}$ as it is. The final result may be expressed in terms of time averages as

$$H\Delta\phi_{dyn} = 2Th - T\left\langle V - H\frac{\partial V}{\partial H} \right\rangle.$$

▲

Exercise 2.6 ***Three-wave equations***

The three-wave equations of motion take the symmetric form

$$i\dot{A} = B^*C\,, \quad i\dot{B} = CA^*\,, \quad i\dot{C} = AB\,, \quad \textit{for} \quad (A,B,C) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C} \simeq \mathbb{C}^3\,. \tag{1}$$

(a) Write these equations as a Hamiltonian system. How many degrees of freedom does it have?

Answer a

The three-wave interaction equations (1) may be written in canonical form with Hamiltonian $H = \Re(ABC^*)$ and Poisson brackets

$$\{A, A^*\} = \{B, B^*\} = \{C, C^*\} = -2i\,.$$

There are 3 complex-canonical degrees of freedom. ▲

(b) Find two additional constants of motion for it, besides the Hamiltonian.

Answer b

The three-wave equations conserve the following three quantities:

$$H = \frac{1}{2}(ABC^* + A^*B^*C) = \Re(ABC^*)\,, \tag{2}$$

$$J = |A|^2 - |B|^2\,, \tag{3}$$

$$N = |A|^2 + |B|^2 + 2|C|^2\,. \tag{4}$$

▲

(c) Use the Poisson bracket to identify the symmetries of the Hamiltonian associated with the two additional constants of motion, by computing their Hamiltonian vector fields and integrating their characteristic equations.

Answer c

The Hamiltonian vector field $X_H = \{\,\cdot\,, H\}$ generates the motion, while $X_J = \{\,\cdot\,, J\}$ and $X_N = \{\,\cdot\,, N\}$ generate S^1 symmetries $S^1 \times \mathbb{C}^3 \mapsto \mathbb{C}^3$ of the Hamiltonian H. The S^1 symmetries associated to J and N are the following:

$$J : \begin{pmatrix} A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} e^{-2i\phi}A \\ e^{2i\phi}B \\ C \end{pmatrix} \qquad N : \begin{pmatrix} A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} e^{-2i\psi}A \\ e^{-2i\psi}B \\ e^{-4i\psi}C \end{pmatrix}$$

The constant of motion J represents the angular momentum about the vertical in the new variables, while N is the new conserved quantity arising from phase-averaging in the Lagrangian L to obtain $\langle L\rangle$.

The following positive-definite combinations of N and J are physically significant:

$$N_1 \equiv \frac{1}{2}(N + J) = |A|^2 + |C|^2\,, \qquad N_2 \equiv \frac{1}{2}(N - J) = |B|^2 + |C|^2\,.$$

These combinations are known as the ***Manley-Rowe invariants*** in the extensive literature about three-wave interactions. The quantities H, N_1 and N_2 provide three independent constants of the motion. The S^1 symmetries associated to N_1 and N_2 are the following:

$$N_1 : \begin{pmatrix} A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} e^{-2i\phi_1}A \\ B \\ e^{-2i\phi_1}C \end{pmatrix} \qquad N_2 : \begin{pmatrix} A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} A \\ e^{-2i\phi_2}B \\ e^{-2i\phi_2}C \end{pmatrix}$$

▲

(d) Set:

$$A = |A| \exp(i\phi_1)\,, \quad B = |B| \exp(i\phi_2)\,, \quad C = Z \exp(i(\phi_1 + \phi_2))\,.$$

Determine whether this transformation is canonical.

Answer d

The transformation

$$\begin{aligned} A &= |A| \exp(i\phi_1)\,, \\ B &= |B| \exp(i\phi_2)\,, \\ C &= Z \exp(i(\phi_1 + \phi_2))\,. \end{aligned} \tag{5}$$

is canonical, since it preserves the symplectic form. Namely, as one may compute directly,

$$dA \wedge dA^* + dB \wedge dB^* + dC \wedge dC^* = dZ \wedge dZ^* - i(dN_1 \wedge d\phi_1 + dN_2 \wedge d\phi_2)\,.$$

In these variables, the Hamiltonian is independent of the phases ϕ_1 and ϕ_2,

$$H = \frac{1}{2}(Z + Z^*)\sqrt{N_1 - |Z|^2}\sqrt{N_2 - |Z|^2}\,.$$

The Poisson bracket is $\{Z, Z^*\} = -2i$ and the canonical equations reduce to

$$\begin{aligned} &i\dot{Z} = i\{Z, H\} = 2\frac{\partial H}{\partial Z^*}\,, \\ &\dot{N}_k = -\frac{\partial H}{\partial \phi_k} \quad \text{and} \quad \dot{\phi}_k = \frac{\partial H}{\partial N_k} \quad \text{for} \quad k = 1, 2. \end{aligned}$$

As we shall see, these equations eventually provide the dynamics of both the amplitude and phase of $Z = |Z|e^{i\zeta}$.

▲

(e) Express the three-wave problem entirely in terms of the variable $Z = |Z|e^{i\zeta}$, reduce the motion to a single equation for $|Z|$ then reconstruct the full solution as,

$$A = |A| \exp(i\phi_1)\,, \quad B = |B| \exp(i\phi_2)\,, \quad C = |Z| \exp\big(i(\phi_1 + \phi_2 + \zeta)\big)\,.$$

That is, reduce the motion to a single equation for $|Z|$ then write the various differential equations for $|A|$, ϕ_1, $|B|$, ϕ_2 and ϕ_2.

Answer e In these variables, the Hamiltonian $H = |Z|\cos(\zeta)|A||B|$ is

$$H = |Z|\cos(\zeta)\sqrt{N_1 - |Z|^2}\sqrt{N_2 - |Z|^2}\,.$$

Changing to polar variables $Z = |Z|e^{i\zeta}$ will allow us to obtain an implicit solution for $Q = |Z|^2$ as an integral (quadrature). Since

$$dZ \wedge dZ^* = -i\, dQ \wedge d\zeta = -2i\, dq \wedge dp \quad \text{for} \quad Z = q + ip \quad \text{with} \quad \{q, p\} = 1$$

we acquire a factor of $1/2$ in the Poisson bracket, $\{Q, \zeta\} = -1/2$, so that

$$\frac{dQ}{dt} = \{Q, H\} = -\frac{1}{2}\frac{\partial H}{\partial \zeta} = \frac{1}{2}\sqrt{Q}\sin(\zeta)\sqrt{N_1 - Q}\sqrt{N_2 - Q}\,.$$

Then

$$
\begin{aligned}
\left(\frac{dQ}{dt}\right)^2 &= \frac{1}{4}Q(1-\cos^2(\zeta))(N_1-Q)(N_2-Q) \\
&= \frac{1}{4}Q\left(1-\frac{H^2}{(N_1-Q)(N_2-Q)}\right)(N_1-Q)(N_2-Q) \\
&= \frac{1}{4}Q\Big((N_1-Q)(N_2-Q)-H^2\Big)
\end{aligned}
$$

Consequently, the amplitude $Q=|Z|^2=|C|^2$ is obtained in closed form in terms of Jacobi elliptic functions as the solution of the quadrature,

$$
\frac{2d|Z|^2}{\sqrt{|Z|^2\Big((N_1-|Z|^2)(N_2-|Z|^2)-H^2\Big)}} = \pm\, dt\,.
$$

Once $|Z|$ is known, $|A|$ and $|B|$ follow immediately from the Manley-Rowe relations,

$$
|A| = \sqrt{N_1-|Z|^2}\,, \qquad |B| = \sqrt{N_2-|Z|^2}\,.
$$

The phases ϕ_1 and ϕ_2 may now be determined from

$$
\begin{aligned}
\dot{\phi}_1 = \{\phi_1, H\} = \frac{1}{2}\frac{\partial H}{\partial N_1} = \frac{1}{4}\frac{H}{|A|^2}\,, \\
\dot{\phi}_2 = \{\phi_2, H\} = \frac{1}{2}\frac{\partial H}{\partial N_2} = \frac{1}{4}\frac{H}{|B|^2}\,,\,,
\end{aligned}
$$

so that ϕ_1 and ϕ_2 can be integrated by quadratures once $|A|(t)$ and $|B|(t)$ are known. Finally, the phase ζ of Z is determined unambiguously by

$$
\frac{d|Z|^2}{dt} = \{|Z|^2, H\} = -\frac{\partial H}{\partial \zeta} = -\,2H\tan\zeta \qquad \text{and} \qquad H = |A||B||Z|\cos\zeta\,. \tag{6}
$$

Hence, we can now reconstruct the full solution as,

$$
A = |A|\exp(i\phi_1)\,, \quad B = |B|\exp(i\phi_2)\,, \quad C = |Z|\exp\big(i(\phi_1+\phi_2+\zeta)\big)\,.
$$

▲