2 M3-4-5 A16 Assessed Problems # 2 Due 2pm 1 Dec 2011

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 2.1 ***$\mathbb{R}^3$-bracket for Maxwell-Bloch equations***

The real-valued Maxwell-Bloch system for $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}3$ is given by

$$\dot{x}_1 = x_2 \,, \quad \dot{x}_2 = x_1 x_3 \,, \quad \dot{x}_3 = -x_1 x_2 \,.$$

(a) *Write this system in three-dimensional vector $\mathbb{R}3$-bracket notation as*

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2 \,,$$

*where H_1 and H_2 are two conserved functions, one of whose level sets (let it be H_1) may be taken as **circular cylinders** oriented along the x_1-direction and the other (let it be H_2) whose level sets may be taken as **parabolic cylinders** oriented along the x_2-direction.*

(b) *Restrict the equations and their $\mathbb{R}3$ Poisson bracket to a level set of H_2. Show that the Poisson bracket on the parabolic cylinder $H_2 = const$ is symplectic.*

(c) *Derive the equation of motion on a level set of H_2 and express them in the form of Newton's Law. Do they reduce to something familiar?*

(d) *Identify steady solutions and determine which are unstable (saddle points) and which are stable (centers).*

Exercise 2.2 ***The fish: quadratically nonlinear oscillator***

Consider the Hamiltonian dynamics on a symplectic manifold of a system comprising two real degrees of freedom, with real phase space variables (x, y, θ, z), symplectic form

$$\omega = dx \wedge dy + d\theta \wedge dz$$

and Hamiltonian

$$H = \tfrac{1}{2}y^2 + x\left(\tfrac{1}{3}x^2 - z\right) - \tfrac{2}{3}z^{3/2}$$

(a) Write the canonical Poisson bracket for this system.

(b) Write Hamilton's canonical equations for this system. Explain how to keep $z \geq 0$, so that H and θ remain real.

(c) At what values of x, y and H does the system have stationary points in the (x, y) plane?

(d) Propose a strategy for solving these equations. In what order should they be solved?

(e) Identify the constants of motion of this system and explain why they are conserved.

(f) Compute the associated Hamiltonian vector field X_H and show that it satisfies

$$X_H \lrcorner\, \omega = dH$$

(g) Write the Poisson bracket that expresses the Hamiltonian vector field X_H as a divergenceless vector field in $\mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. Explain why this Poisson bracket satisfies the Jacobi identity.

(h) Identify the Casimir function for this $\mathbb{R}^3$ bracket. Show explicitly that it satisfies the definition of a Casimir function.

(i) Sketch a graph of the intersections of the level surfaces in $\mathbb{R}^3$ of the Hamiltonian and Casimir function. Show the directions of flow along these intersections. Identify the locations and types of any relative equilibria at the tangent points of these surfaces.

(j) Linearise around the relative equilibria on a level set of the Casimir (z) and compute its eigenvalues.

(k) If you found a hyperbolic equilibrium point in the previous part connected to itself by a homoclinic orbit, then reduce the equation for the homoclinic orbit to an indefinite integral expression.

Exercise 2.3 *2D coupled oscillators*

Consider the 2D oscillator Hamiltonian $H : \mathbb{C}^2 \to \mathbb{R}$, with complex 2-vector $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2$ and constant frequencies ω_j,

$$H = \frac{1}{2}\sum_{j=1}^{2} \omega_j |a_j|^2 = \frac{1}{4}(\omega_1+\omega_2)\big(|a_1|^2+|a_2|^2\big) + \frac{1}{4}(\omega_1-\omega_2)\big(|a_1|^2-|a_2|^2\big)\,.$$

(a) Compute its canonical Hamiltonian dynamics with

$$\{a_j, a_k^*\} = -2i\delta_{jk}.$$

Explain why this is the sum of a $1:1$ resonant oscillator and a $1:-1$ oscillator.

(b) Find the transformations generated by X, Y, Z, R on a_1, a_2, where

$$\begin{aligned} R &= |a_1|^2 + |a_2|^2\,, \\ Z &= |a_1|^2 - |a_2|^2\,, \\ X - iY &= 2a_1 a_2^*\,. \end{aligned}$$

Express these infinitesimal transformations as matrix operations and identify their corresponding finite transformations.

(c) For the starting Hamiltonian,

$$\begin{aligned} H &= \frac{\omega_1}{2}\,(R+Z) + \frac{\omega_2}{2}\,(R-Z) \\ &= \frac{1}{2}(\omega_1+\omega_2)R + \frac{1}{2}(\omega_1-\omega_2)Z\,, \end{aligned}$$

write the equations $\dot{X}, \dot{Y}, \dot{Z}, \dot{R}$ for the S^1 invariants X, Y, Z, R of the $1:1$ resonance.

Write these equations in vector form, with $\mathbf{X} = (X, Y, Z)^T$, and describe this motion in terms of level sets of the Poincaré sphere and the Hamiltonian H.

Exercise 2.4 ***Matrix rigid body equations & cotangent lift momentum maps***

(a) Let the Lie group $SO(n)$ act on itself with infinitesimal transformation

$$\Phi_\Xi(Q) = Q\Xi \quad \text{for} \quad Q \in SO(n) \quad \text{and} \quad \Xi = -\,\Xi^T \in \mathfrak{so}(n)$$

Compute the cotangent lift (CL) momentum map for this action and its CL infinitesimal action on $T^*SO(n)$.

(b) Compute the variations in Hamilton's principle $\delta S = 0$ with Clebsch-constrained action integral

$$S(\Omega, Q, P) = \int_a^b l(\Omega) + \mathrm{tr}\Big(P^T\,\big(\dot{Q} - Q\Omega\big)\Big)\,dt\,.$$

Discuss the relation between these variational equations and the equations for the infinitesimal Lie algebra actions associated with CL momentum maps.

(c) Show that the Clebsch-constrained Hamilton's principle implies that $M = \partial l/\partial\Omega$ satisfies the Euler-Poincaré equation

$$\frac{dM}{dt} = \mathrm{ad}^*_\Omega M = -\,\big[\,\Omega,\,M\,\big]\,.$$

Exercise 2.5 $\boxed{2\!:\!1 \textit{ resonance}}$

The Hamiltonian $\mathbb{C}^2 \to \mathbb{R}$ *for a certain* $2\!:\!1$ *resonance is given by*

$$H = \tfrac{1}{2}|a_1|^2 - |a_2|^2 + \tfrac{1}{2}\mathrm{Im}(a_1^{*2}a_2)\,,$$

in terms of canonical variables $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$ *whose Poisson bracket relation*

$$\{a_j, a_k^*\} = -2i\delta_{jk}, \quad \textit{for} \quad j,k = 1,2\,,$$

is invariant under the $2\!:\!1$ *resonance* S^1 *transformation*

$$a_1 \to e^{i\phi} \quad \textit{and} \quad a_2 \to e^{2i\phi}.$$

(a) Write the motion equations in terms of the canonical variables $(a_1, a_1^*, a_2, a_2^*) \in \mathbb{C}^2$.

(b) Introduce the orbit map $\mathbb{C}^2 \to \mathbb{R}^4$

$$\pi :\ (a_1, a_1^*, a_2, a_2^*) \to \{X, Y, Z, R)\}$$

and transform the Hamiltonian H *on* $\mathbb{C}^2$ *to new variables* $X, Y, Z, R \in \mathbb{R}^4$ *given by*

$$\begin{aligned} R &= \tfrac{1}{2}|a_1|^2 + |a_2|^2\,, \\ Z &= \tfrac{1}{2}|a_1|^2 - |a_2|^2\,, \\ X - iY &= 2a_1^{*2}a_2\,, \end{aligned}$$

that are invariant under the $2\!:\!1$ *resonance* S^1 *transformation.*

(c) Show that these variables are functionally dependent, because they satisfy a cubic algebraic relation $C(X, Y, Z, R) = 0$.

(d) Use the orbit map $\pi : \mathbb{C}^2 \to \mathbb{R}^4$ *to make a table of Poisson brackets among the four quadratic* $2\!:\!1$ *resonance* S^1*-invariant variables* $X, Y, Z, R \in \mathbb{R}^4$.

(e) Show that both R *and the cubic algebraic relation* $C(X, Y, Z, R)$ $= 0$ *are Casimirs for these Poisson brackets.*

(f) Write the Hamiltonian, Poisson bracket and equations of motion in terms of the remaining variables $\mathbf{X} = (X, Y, Z)^T \in \mathbb{R}^3$.

(g) Describe this motion in terms of level sets of the Hamiltonian H *and the orbit manifold for the* $2\!:\!1$ *resonance, given by* $C(X, Y, Z, R) = 0$.

(h) Restrict the dynamics to a level set of the Hamiltonian and show that it reduces there to the equation of motion for a point particle in a cubic potential. Explain its geometrical meaning.

(i) Compute the geometric and dynamic phases for any closed orbit on a level set of H.

Exercise 2.6 ***Three-wave equations***

The three-wave equations of motion take the symmetric form

$$i\dot{A} = B^*C\,, \quad i\dot{B} = CA^*\,, \quad i\dot{C} = AB\,, \quad \textit{for} \quad (A,B,C) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C} \simeq \mathbb{C}^3\,.$$

(a) Write these equations as a Hamiltonian system. How many degrees of freedom does it have?

(b) Find two additional constants of motion for it, besides the Hamiltonian.

(c) Use the Poisson bracket to identify the symmetries of the Hamiltonian associated with the two additional constants of motion, by computing their Hamiltonian vector fields and integrating their characteristic equations.

(d) Set:

$$A = |A|\exp(i\xi)\,, \quad B = |B|\exp(i\eta)\,, \quad C = Z\exp(i(\xi+\eta))\,.$$

Determine whether this transformation is canonical.

(e) Express the three-wave problem entirely in terms of the variable $Z = |Z|e^{i\zeta}$, reduce the motion to a single equation for $|Z|$ then reconstruct the full solution as,

$$A = |A|\exp(i\xi)\,, \quad B = |B|\exp(i\eta)\,, \quad C = |Z|\exp\big(i(\xi+\eta+\zeta)\big)\,.$$

That is, reduce the motion to a single equation for $|Z|$ then write the various differential equations for $|A|$, ξ, $|B|$, η and η.