1 Solutions of M3-4A16 Assessed Problems # 1

Exercise 1.1

1. *Define the sphere S^{n-1} and its tangent space TS^{n-1} in $\mathbb{R}^n$. What is the dimension of TS^{n-1}?*

Answer

$$TS^{n-1} = \Big\{ (\mathbf{x}, \dot{\mathbf{x}}) \in \mathbb{R}^n \times \mathbb{R}^n \Big| \; |\mathbf{x}|^2 = 1 \text{ and } \mathbf{x} \cdot \dot{\mathbf{x}} = 0 \Big\}$$

The dimension of TS^{n-1} is $2n - 2$, since two relations are imposed in $\mathbb{R}^n \times \mathbb{R}^n$, whose dimension is $2n$. ▲

2. *Prove that the two sets of planar coordinates arising from the stereographic projections of the sphere $S^2 = \{(x, y, z) | \, x^2 + y^2 + z^2 = 1\}$ from its North and South poles $z = \pm 1$*
(1) (valid everywhere except $z = 1$): $\xi_N = \frac{-x}{1-z}$, $\quad \eta_N = \frac{y}{1-z}$,
(2) (valid everywhere except $z = -1$): $\xi_S = \frac{x}{1+z}$, $\quad \eta_S = \frac{y}{1+z}$.
are diffeomorphic. That is, construct the mapping from $(\xi_N, \eta_N) \to (\xi_S, \eta_S)$ and verify that it is a diffeomorphism. Hint: $(1 + z)(1 - z) = 1 - z^2 = x^2 + y^2$.

Answer

$$(\xi_S, \eta_S) = \frac{1 - z}{1 + z} (\xi_N, \eta_N) = \frac{1}{\xi_N^2 + \eta_N^2} (\xi_N, \eta_N)$$

Hence, the map $(\xi_N, \eta_N) \to (\xi_S, \eta_S)$ is smooth and invertible everywhere except at $(\xi_N, \eta_N) = (0, 0)$ (the North Pole). ▲

3. *If θ is co-latitude, ϕ is azimuth (longitude), of the stereographic projection of the sphere S^2 from its North pole, show that each latitude on the sphere projects to a circle given in the complex plane by $\zeta = \cot \frac{\theta}{2} e^{i\phi}$.*

Answer At a fixed azimuth, e.g., $\phi = 0$ (in the $\xi, z-$plane) a point on the sphere at co-latitude θ from the North Pole has coordinates

$$\xi = \sin\theta \quad z = \cos\theta$$

Its projection strikes the $\xi\eta-$plane at radius r and angle ψ, given by

$$\cot\psi = r = \frac{r - \sin\theta}{\cos\theta}$$

Thus $\psi = \theta/2$, since,

$$\cot\psi = r = \frac{\sin\theta}{1 - \cos\theta} = \cot(\theta/2)$$

The stereographic projection of the circle at polar angle θ thus describes a circle in the complex plane at $\zeta = \xi + i\eta = \cot \frac{\theta}{2} e^{i\phi}$. ▲

4. *Let $\phi_t : S^2 \to S^2$ rotate points on S^2 about a fixed axis through an angle $\psi(t)$. Show that ϕ_t is the flow of a certain vector field on S^2.*

Answer When the axis of rotation is taken as the diameter $\hat{\mathbf{z}}$ connecting the North and South poles, then in $\mathbb{R}^3$ the corresponding vector field is

$$X = (\hat{\mathbf{z}} \times \mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{x}} \, .$$

In spherical polar coordinates (θ, ϕ) this is

$$X = \frac{\partial}{\partial \phi} \, .$$

The points on the sphere then follow latitudes. ▲

5. *Let $f : S^2 \to \mathbb{R}$ be defined by $f(x, y, z) = z$, for $(x, y, z) \in \mathbb{R}^3$. Compute the differential df using spherical coordinates (θ, ϕ).*

Answer The map to spherical coordinates is $(x + iy) = \sin\theta e^{i\phi}$, $z = \cos\theta$. The differential of f is

$$\begin{aligned} df &= f_x dx + f_y dy + f_z dz \\ &= f_\theta d\theta + f_\phi d\phi \end{aligned}$$

For the case $f = z = \cos\theta$, we have $df = dz = d(\cos\theta) = -\sin\theta d\theta$.

▲

6. *Compute the tangent lifts for the two stereographic projections of $S^2 \to \mathbb{R}^2$ above. That is, assuming (x, y, z) depend smoothly on t, find*

   (a) *How $(\dot\xi_N, \dot\eta_N)$ depend on $(\dot x, \dot y, \dot z)$. Likewise, for $(\dot\xi_S, \dot\eta_S)$.*

   Answer Write $\xi^i_N = \psi^i_N(x^1, x^2, x^3)$ for $i = 1, 2$. Then

$$\dot\xi^i_N = \frac{\partial\psi^i_N}{\partial x^A}\dot x^A$$

   Likewise, when $N \to S$. ▲

   (b) *How $(\dot\xi_N, \dot\eta_N)$ depend on $(\dot\xi_S, \dot\eta_S)$.*

   Hint: Recall $(1 + z)(1 - z) = 1 - z^2 = x^2 + y^2$ and use $x\dot x + y\dot y + z\dot z = 0$ when $(\dot x, \dot y, \dot z)$ is tangent to S^2 at (x, y, z).

   Answer

$$\dot\xi^i_N = \frac{\partial\xi^i_N}{\partial\xi^j_S}\dot\xi^j_S$$

▲

7. *Consider two sets of local coordinates q^i and s^i on a manifold Q, related by $s^i = \psi^i\left(q^1, \dots, q^n\right)$. Check that tangent lifted coordinates $\dot q^i$ and $\dot s^i$ are related by*

$$\dot s^i = \frac{\partial\psi^i}{\partial q^A}\dot q^A =: (D\psi)^i_A\,\dot q^A,$$

and thus show that the corresponding

$$\frac{\partial}{\partial s^i} = \left((D\psi)^{-1}\right)^A_i \frac{\partial}{\partial q^A}\,.$$

Answer Vector field components will be related by

$$Y^i\frac{\partial}{\partial\psi^i} = X^A\frac{\partial}{\partial q^A}\,, \quad \text{so} \quad Y^i = \frac{\partial\psi^i}{\partial q^A}X^A\,,$$

in which the quantity called the ***tangent lift***

$$Tf = \frac{\partial\psi}{\partial q}\,,$$

of the function f arises from the chain rule and is equal to the Jacobian matrix for the transformation $Tf : TQ \mapsto TQ$. ▲

8. *Perform the corresponding calculations on the cotangent bundle side.*

Answer The ***cotangent lift*** of the function f,

$$T^*f = \frac{\partial q}{\partial \psi}\,,$$

arises from

$$\beta_i d\psi^i = \alpha_A dq^A\,, \quad \text{so} \quad \beta_i = \alpha_A \frac{\partial q^A}{\partial \psi^i}\,,$$

and $T^*f : T^*Q \mapsto T^*Q$. ▲

Exercise 1.2

1. *Explain why one can conclude that the zero locus map for*

$$S = \{U \in GL(n,\mathbb{R}) |\, UKU^T - K = 0\}$$

is a submersion for $K = K^T \in GL(n,\mathbb{R})$. (Pay close attention to establishing the constant rank condition for the linearization of this map.)

Answer
First, S is the zero locus of the mapping

$$U \to U^T KU - K\,, \qquad \text{(locus map)}$$

Let $U \in S$, and let δU be an arbitrary element of $R^{n\times n}$. Then linearize to find

$$(U+\delta U)^T K(U+\delta U) - K = U^T KU - K + \delta U^T KU + U^T K\delta U + O(\delta U)^2\,.$$

We may conclude that S is a submanifold of $R^{n\times n}$ if we can show that the linearization of the locus map, namely the linear mapping defined by

$$L \equiv \delta U \to \delta U^T KU + U^T K\delta U\,, \qquad R^{n\times n} \to R^{n\times n}$$

has constant rank for all $U \in S$.

Lemma 1.3 *The linearization map L is onto the space of $n\times n$ of symmetric matrices. Hence, it has constant rank and the original map is a submersion.*

Proof that L is onto.

- Both the original locus map and the image of L lie in the subspace of $n\times n$ symmetric matrices.
- Indeed, given U and any symmetric matrix S we can find δU such that

$$\delta U^T KU + U^T K\delta U = S\,.$$

  Namely

$$\delta U = K^{-1}U^{-T}S/2\,.$$

- Thus, the linearization map L is onto the space of $n\times n$ of symmetric matrices. That is, it has constant rank. This means the original locus map $U \to UKU^T - K$ to the space of symmetric matrices is a submersion.

▲

2. *Write the defining relation for the tangent space to S at the identity, $T_I S$.*

Answer The tangent space $T_I S$ at the identity of the matrix Lie group S defined by $S = \{U \in GL(n,\mathbb{R}) |\, UKU^T - K = 0\}$ is the linear space of matrices A satisfying

$$A^T K + KA = 0$$

Proof. Near the identity the defining condition for S expands to

$$(I + \epsilon A^T + O(\epsilon^2))K(I + \epsilon A + O(\epsilon^2)) = K\,, \quad \text{for} \quad \epsilon \ll 1\,.$$

At linear order $O(\epsilon)$ one finds,

$$A^T K + KA = 0\,.$$

This relation defines the linear space of matrices $A \in T_I S$. ▲

3. *Show that for any pair of matrices $A, B \in T_I S$, the matrix commutator satisfies*

$$[A, B] \equiv AB - BA \in T_I S.$$

Answer Using $[A, B]^T = [B^T, A^T]$, we check closure by a direct computation,

$$\begin{aligned}
[B^T, A^T]K + K[A, B] &= B^T A^T K - A^T B^T K + KAB - KBA \\
&= B^T A^T K - A^T B^T K - A^T KB + B^T KA \\
&= B^T(A^T K + KA) - A^T(B^T K + KB) = 0\,.
\end{aligned}$$

Hence, the tangent space of S at the identity $T_I S$ is closed under the matrix commutator $[\cdot\,,\,\cdot]$.

▲

4. *Suppose the $n \times n$ matrices A and M satisfy*

$$AM + MA^T = 0\,.$$

Show that $\exp(At)M\exp(A^T t) = M$ *for all* t.

Answer

$$\frac{d}{dt}\Big(\exp(At)M\exp(A^T t)\Big) = \exp(At)(AM + MA^T)\exp(A^T t) = 0$$

▲

Exercise 1.4

1. ***Gauge invariance*** *Show that the Euler-Lagrange equations are unchanged under*

$$L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \to L' = L + \frac{d}{dt}\gamma(\mathbf{q}(t), \dot{\mathbf{q}}(t))\,,$$

for any function $\gamma : \mathbb{R}^{6N} = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q}, \dot{\mathbf{q}} \in \mathbb{R}^{3N}\} \to \mathbb{R}$.

Answer Hamilton's principle for the difference is

$$0 = \delta \int_{t_1}^{t_2} \Big(L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) - L'(\mathbf{q}(t), \dot{\mathbf{q}}(t))\Big)\, dt = \delta\Big[\gamma(\mathbf{q}(t), \dot{\mathbf{q}}(t))\Big]_{t_1}^{t_2}$$

However, this vanishes for variations $\delta\mathbf{q}(t)$ that vanish at the endpoints in time. ▲

2. ***Generalized coordinate theorem*** *Show that the Euler-Lagrange equations are* ***unchanged in form*** *under any smooth invertible mapping* $f : \{\mathbf{q} \mapsto \mathbf{s}\}$. *That is, with*

$$L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = \tilde{L}(\mathbf{s}(t), \dot{\mathbf{s}}(t))\,,$$

show that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \quad \Longleftrightarrow \quad \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\mathbf{s}}}\right) - \frac{\partial \tilde{L}}{\partial \mathbf{s}} = 0\,.$$

Answer This amounts just to a change of notation, so it clearly holds. ▲

3. *How do the Euler-Lagrange equations transform under $\mathbf{q}(t) = \mathbf{r}(t) + \mathbf{s}(t)$, when $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are independent of each other?*

Answer A sum of two separate Euler-Lagrange equations is obtained. ▲

4. *State and prove Noether's theorem that each smooth symmetry of Hamilton's principle implies a conservation law for the corresponding Euler-Lagrange equations on the tangent space TM of a smooth manifold M.*

Answer In the family of smoothly deformed curves $q_s(t) = Q(q,t,s)$ with $q_0 = Q(q,t,0) = q(t)$ during the time interval $t \in [t_1,\, t_2]$, the action $S = \int_{t_1}^{t_2} L(q,\dot q,t)\,dt$ transforms to

$$S = \int_{t_1}^{t_2} L\left(Q(q,t,s), \frac{dQ(q,t,s)}{d\tau(t,s)}, \tau(t,s)\right)\, d\tau(t,s)\,.$$

We denote

$$\delta q(t) = \frac{d}{ds}\bigg|_{s=0} Q(q,t,s) = \xi(q(t),t)\,, \qquad \delta t = \frac{d}{ds}\bigg|_{s=0} \tau(t,s) = \theta(t)\,,$$

so that at *linear order* in s we have

$$Q(q,t,s) = q(t) + s\xi(q,t)\,, \qquad \tau(t,s) = t + s\theta(t)\,, \qquad \frac{dQ(q,t,s)}{d\tau(t,s)} = \frac{dq}{dt} + s\Big(\dot\xi(q,t) - \dot q\dot\theta\Big)\,.$$

Here the dot-notation as in $\dot\xi(q,t) = \partial_t\xi + \dot q\partial_q\xi$ represents the total time derivative. We could allow q-dependence in τ, but the result of the calculation would be morally the same, after keeping track of total time derivatives.
The variations in Hamilton's principle proceed as follows,

$$\begin{aligned}
0 = \delta S &= \frac{d}{ds}\bigg|_{s=0} \int_{t_1}^{t_2} L\left(Q(q,t,s), \frac{dQ(q,t,s)}{d\tau(t,s)}, \tau(t,s)\right)\, d\tau(t,s) \\
&= \int_{t_1}^{t_2} \left\{ \Big\langle \frac{\partial L}{\partial q},\, \xi(q,t) \Big\rangle + \Big\langle \frac{\partial L}{\partial \dot q},\, \dot\xi(q,t) - \dot q\dot\theta \Big\rangle + \frac{\partial L}{\partial t}\theta + L(q,\dot q,t)\dot\theta \right\} dt \\
&= \int_{t_1}^{t_2} \Big\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q},\, \Big(\xi(q,t) - \dot q\theta\Big) \Big\rangle dt + \left[\Big\langle \frac{\partial L}{\partial \dot q},\, \xi \Big\rangle - \Big(\Big\langle \frac{\partial L}{\partial \dot q},\, \dot q \Big\rangle - L \Big)\theta \right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \Big\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q},\, \Big(\delta q - \dot q\delta t\Big) \Big\rangle dt + \left[\Big\langle \frac{\partial L}{\partial \dot q},\, \delta q \Big\rangle - \Big(\Big\langle \frac{\partial L}{\partial \dot q},\, \dot q \Big\rangle - L \Big)\delta t \right]_{t_1}^{t_2}
\end{aligned}$$

with a few algebraic manipulations and integrations by parts in between the lines. (Of course, these should be checked!)
Thus, stationarity $\delta S = 0$ by symmetry and the Euler-Lagrange equations

$$[\,L\,]_{q^a} := \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$$

imply that the quantity

$$\begin{aligned}
C(t,q,\dot q) &= \Big\langle \frac{\partial L}{\partial \dot q},\, \delta q \Big\rangle - \Big(\Big\langle \frac{\partial L}{\partial \dot q},\, \dot q \Big\rangle - L \Big)\delta t & (1) \\
&=: \langle\, p\,,\, \delta q\,\rangle - E\,\delta t\,, & (2)
\end{aligned}$$

has the same value at every time along the solution path. That is, $C(t,q,\dot q)$ is a constant of the motion. This is Noether's theorem.

▲

5. *Show that* ***conservation of energy results from Noether's theorem*** *if, in Hamilton's principle, the variations of $L(q(t),\dot{q}(t))$ are chosen as*

$$\delta q(t) = \frac{d}{ds}\Big|_{s=0} q(t,s)\,,$$

corresponding to symmetry of the Lagrangian under reparametrisations of time $t \to \tau(t,s)$ so that $q(t) \to q(\tau(t,s))$ along a given curve $q(t)$.

Answer For reparametrisations of time, δq vanishes and δt is a function of time in the previous part; so stationarity of the action $\delta S = 0$ in the presence of time-reparametrisation symmetry implies that the quantity

$$C(t,q,\dot{q}) \quad = \quad \Big(\Big\langle \frac{\partial L}{\partial \dot{q}}, \dot{q}\Big\rangle - L\Big)\delta t =: E(t,q,\dot{q})\,\delta t(t)\,, \tag{3}$$

is a constant of motion along solutions of the Euler-Lagrange equations. This does not yet imply conservation of the energy E. For that, δt must be a constant.

For simple translations in time, δq again vanishes and δt is a *constant*; so stationarity of the action $\delta S = 0$ in the presence of *time-translation symmetry* implies that the energy

$$E(t,q,\dot{q}) := \Big\langle \frac{\partial L}{\partial \dot{q}}, \dot{q}\Big\rangle - L\,, \tag{4}$$

is a constant of motion along solutions of the Euler-Lagrange equations.

This energy is also the expression for the Legendre transform of the Lagrangian $L(t,q,\dot{q})$. ▲

Exercise 1.5 (Example Lagrangians)

(i) For the following Lagrangians, determine which of them are hyperregular. (A Lagrangian is hyperregular if its fibre derivative is invertible, so that the velocity may be expressed in terms of the position and canonical momentum.)

(ii) Write the Euler-Lagrange for these equations.

(iii) For the hyperregular Lagrangians apply the Legendre transformation to determine the Hamiltonian and Hamilton's canonical equations.

1. *The kinetic energy Lagrangian $K(q,\dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$ with $i,j = 1,2,\dots,N$, for a Riemannian manifold Q with metric g, written as (Q,g).*

Answer

(i) The fibre derivative in this case is $\mathbb{F}K(v_q) = g(q)(v_q,\cdot)$, for $v_q \in T_qQ$. In coordinates, this is

$$\mathbb{F}K(q,\dot{q}) = \Big(q^i, \frac{\partial K}{\partial \dot{q}^i}\Big) = (q^i, g_{ij}(q)\dot{q}^j) =: (q^i, p_i),$$

This Lagrangian is hyperregular for invertible $g(q)$; that is, when the metric is nondegenerate. In that case, one may solve for the velocity in terms of position and canonical momentum as

$$\dot{q}^i = (g^{-1}(q))^{ij}p_j$$

(ii) The Euler-Lagrange equations for this Lagrangian produce the ***geodesic equations*** for the metric g, and are given (for finite dimensional Q in a local chart) by

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0, \quad i = 1, \dots n,$$

where the three-index quantities

$$\Gamma^h_{jk} = \frac{1}{2}g^{hl}\left(\frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right), \quad \text{with} \quad g_{ih}g^{hl} = \delta^l_i\,,$$

are the ***Christoffel symbols*** of the Levi-Civita connection on (Q, g) and g^{hl} is called the co-metric.

The calculation of these Euler-Lagrange equations, done in class, involves a step of symmetrising by using vanishing trace $\mathrm{Tr}(SA) = 0$ for the product of a symmetric matrix with an antisymmetric one.

(iii) The Legendre transform of this Lagrangian yields the corresponding Hamiltonian

$$H = \frac{1}{2}p_i g^{ij}(q)p_j$$

whose canonical equations are

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = g^{ij}(q)p_j\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} = -\,p_k\frac{\partial g^{kj}(q)}{\partial q^i}p_j\,.$$

▲

2. $L(q, \dot{q}) = \Big(g_{ij}(q)\dot{q}^i\dot{q}^j\Big)^{1/2}$ *(Is it possible to assume that* $L(q, \dot{q}) = 1$*? Why?)*

Answer

(i) **Fibre derivative**

This Lagrangian is not hyperregular. Its fibre derivative begins well enough

$$\frac{\partial L}{\partial \dot{q}^i} = \frac{1}{\sqrt{g_{kl}(q)\dot{q}^k\dot{q}^l}}\,g_{ij}\dot{q}^j$$

The difficulty is that this Lagrangian is homogeneous of degree one in the velocities. Such functions satisfy Euler's relation,

$$\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = 0\,.$$

This already spells trouble, because its Legendre transform produces a Hamiltonian that vanishes identically

$$H = p_i\dot{q}^i - L \equiv 0$$

Taking another derivative of the Euler's relation yields

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\dot{q}^j = 0$$

so the Hessian of this Lagrangian L with respect to the tangent vectors is singular (has zero determinant). This means the Legendre transformation for this Lagrangian is not invertible.

A singular Lagrangian might become problematic in some situations. However, there is a simple way of obtaining a regular Lagrangian from it whose trajectories, as we shall see, are the same as those for the singular Lagrangian.

The Lagrangian function in this part of the problem is related to the Lagrangian for geodesics in the previous part by

$$K(q,\dot{q}) = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j = \tfrac{1}{2}L^2(q,\dot{q})\,.$$

Computing the Hessian with respect to the tangent vector yields the Riemannian metric,

$$\frac{1}{2}\frac{\partial^2 L^2}{\partial \dot{q}^i \partial \dot{q}^j} = g_{ij}(q)\,.$$

The emergence of a Riemannian metric from the Hessian of the square of a homogenous function of degree 1 is the hallmark of ***Finsler geometry***, of which Riemannian geometry is a special case. Finsler geometry, however, is beyond our present scope.

(ii) **Euler-Lagrange equations**

On setting $\sqrt{g_{kl}(q)\dot{q}^k\dot{q}^l} =: \|\dot{q}\|$, the Euler-Lagrange equations become

$$\frac{d}{dt}\left(\frac{1}{\|\dot{q}\|}\,g_{ij}\frac{dq^j}{dt}\right) = \frac{1}{2\|\dot{q}\|}\left(\frac{dq^k}{dt}\frac{\partial g_{kl}}{\partial q^i}\frac{dq^l}{dt}\right)$$

On dividing by $\|\dot{q}\|$ and setting $d\tau := \|\dot{q}\|dt$, this becomes

$$\frac{d}{d\tau}\left(g_{ij}\frac{dq^j}{d\tau}\right) = \frac{1}{2}\left(\frac{dq^k}{d\tau}\frac{\partial g_{kl}}{\partial q^i}\frac{dq^l}{d\tau}\right)\,,$$

which is again the geodesic equation, but now with a reparameterised time.

Assuming that $L = \|\dot{q}\| = 1$ is not possible, because the value of $\|\dot{q}\|$ is not preserved by the flow.

(iii) **Hamiltonian and canonical equations**

Hamilton's canonical equations are problematic for a Hamiltonian that vanishes identically.

▲

3. $L(\dot{\mathbf{q}}) = -\Big(1 - \dot{\mathbf{q}}\cdot\dot{\mathbf{q}}\Big)^{1/2}$ *for* $\dot{\mathbf{q}} \in \mathbb{R}^3$.

Answer

(i) **Fibre derivative**

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\dot{\mathbf{q}}}{\sqrt{1-\dot{\mathbf{q}}\cdot\dot{\mathbf{q}}}} =: \gamma\,\dot{\mathbf{q}} \quad\Longrightarrow\quad \dot{\mathbf{q}} = \pm\frac{\mathbf{p}}{\sqrt{1+\mathbf{p}\cdot\mathbf{p}}}$$

so this Lagrangian is hyperregular, after making a choice of sign convention, that $\mathbf{p}\cdot\dot{\mathbf{q}} > 0$, for example; so that $\gamma = \sqrt{1+\mathbf{p}\cdot\mathbf{p}} = 1/\sqrt{1-\dot{\mathbf{q}}\cdot\dot{\mathbf{q}}}$.

(ii) **Euler-Lagrange equations**

$$\frac{d(\gamma\,\dot{\mathbf{q}})}{dt} = 0$$

(iii) **Hamiltonian and canonical equations**

The Hamiltonian for this system is

$$H = \mathbf{p}\cdot\dot{\mathbf{q}} - L = \sqrt{1+|\mathbf{p}|^2} = \gamma$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{\sqrt{1+|\mathbf{p}|^2}}\,, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} = 0$$

This is geodesic motion in $\mathbb{R}^3$ for a relativistic particle of unit mass.

▲

4. *The Lagrangian for a free particle of unit mass relative to a moving frame is obtained by setting*

$$L(\dot{\mathbf{q}}, \mathbf{q}, t) = \frac{1}{2}\|\dot{\mathbf{q}} + \mathbf{R}(\mathbf{q})\|^2$$

for a function $\mathbf{R}(\mathbf{q}, t)$ *which governs the space and time dependence of the moving frame velocity. For example, a frame rotating with time-dependent frequency* $\Omega(t)$ *about the vertical axis* $\hat{\mathbf{z}}$ *is obtained by choosing* $\mathbf{R}(\mathbf{q}, t) = \mathbf{q} \times \Omega(t)\hat{\mathbf{z}}$.

Answer

(i) **Fibre derivative**
The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} + \mathbf{R}(\mathbf{q})$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

$$\frac{d}{dt}(\dot{q}_i + R_i(\mathbf{q})) \quad = \quad (\dot{q}_j + R_j(\mathbf{q}))\frac{\partial R^j}{\partial q^i}$$

or

$$\ddot{q}_i = (R_{j,i} - R_{i,j})\dot{q}^j + \frac{\partial}{\partial q^i}(\tfrac{1}{2}|\mathbf{R}|^2)$$

In vector form, this is

$$\ddot{\mathbf{q}} = \dot{\mathbf{q}} \times 2\mathbf{\Omega} + \frac{\partial}{\partial \mathbf{q}}(\tfrac{1}{2}|\mathbf{R}|^2) \quad \text{with} \quad 2\mathbf{\Omega} := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{R}(\mathbf{q})$$

and the terms on the right comprise the sum of the Coriolis and centrifugal forces.

(iii) **Hamiltonian and canonical equations**
The Hamiltonian for this system is

$$H = \mathbf{p} \cdot \dot{\mathbf{q}} - L = \tfrac{1}{2}|\mathbf{p}|^2 - \mathbf{p} \cdot \mathbf{R}(\mathbf{q})$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p} - \mathbf{R}(\mathbf{q})\,, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} = p_j \frac{\partial}{\partial \mathbf{q}} R^j(\mathbf{q})$$

▲

5. *The Lagrangian for a charged particle of mass* m *in a magnetic field* $\mathbf{B} = \mathrm{curl}\mathbf{A}$ *is*

$$L(q, \dot{q}) = \frac{m}{2}\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{e}{c}\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}),$$

for constants m, e, c *and prescribed function* $\mathbf{A}(\mathbf{q})$.

How do the Euler-Lagrange equations for this Lagrangian differ from those of the previous part for free motion in a moving frame with velocity $\frac{e}{mc}\mathbf{A}(\mathbf{q})$*?*

$\boxed{\textit{Answer}}$

(i) **Fibre derivative**
The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A}(\mathbf{q})$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**
In vector form, this is

$$\ddot{\mathbf{q}} = \frac{e}{mc}\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{with} \quad \mathbf{B}(\mathbf{q}) := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{A}(\mathbf{q})$$

and the terms on the right comprise the Lorentz force.

(iii) **Hamiltonian and canonical equations**
The Hamiltonian for this system is

$$H = \mathbf{p}\cdot\dot{\mathbf{q}} - L = \frac{1}{2m}\Big|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q})\Big|^2$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}\Big(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q})\Big)\,, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} = \frac{e}{mc}\Big(p_j - \frac{e}{c}A_j(\mathbf{q})\Big)\frac{\partial}{\partial \mathbf{q}}A^j(\mathbf{q})$$

These are the same equations as in the previous part, modulo the relation $\mathbf{R} = e\mathbf{A}/mc$ and neglect of centrifugal force.

▲

6. *Let Q be the manifold $\mathbb{R}^3 \times S^1$ with variables $(\mathbf{q}, \theta)$. Introduce the Lagrangian $L : TQ \simeq T\mathbb{R}^3 \times TS^1 \mapsto \mathbb{R}$ as*

$$L(\mathbf{q}, \theta, \dot{\mathbf{q}}, \dot{\theta}) = \frac{m}{2}\|\dot{\mathbf{q}}\|^2 + \frac{e}{2c}\Big(\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}) + \dot{\theta}\Big)^2 .$$

The Lagrangian L is positive definite in $(\dot{\mathbf{q}}, \dot{\theta})$; so it may be regarded as the kinetic energy of a metric.

(i) Interpret the motion as geodesic.

(ii) Identify how the Euler-Lagrange equations for this Lagrangian differ from those of the previous part for a charged particle with mass moving in a magnetic field?

$\boxed{\textit{Answer}}$

(i) **Fibre derivative**
in this example, we have two fibre derivatives that each give a linear relation

$$\begin{aligned} \mathbf{p} &= \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\Big(\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}) + \dot{\theta}\Big)\mathbf{A}(\mathbf{q}) = m\dot{\mathbf{q}} + p_\theta\mathbf{A}(\mathbf{q}) \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \frac{e}{c}\Big(\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}) + \dot{\theta}\Big) \end{aligned}$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

$$\begin{aligned} \ddot{\mathbf{q}} &= \frac{p_\theta}{m}\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{with} \quad \mathbf{B}(\mathbf{q}) := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{A}(\mathbf{q}) \\ \frac{dp_\theta}{dt} &= 0 \end{aligned}$$

This is the same as the previous part, on setting $p_\theta = e/c$.

(iii) **Hamiltonian and canonical equations**
The Hamiltonian H associated to L by the Legendre transformation for this Lagrangian is

$$\begin{aligned}
H(\mathbf{q},\theta,\mathbf{p},p_\theta) &= \mathbf{p}\cdot\dot{\mathbf{q}} + p_\theta\dot\theta - L(\mathbf{q},\dot{\mathbf{q}},\theta,\dot\theta) \\
&= \mathbf{p}\cdot\frac{1}{m}(\mathbf{p}-p_\theta\mathbf{A}) + p_\theta(p_\theta - \mathbf{A}\cdot\dot{\mathbf{q}}) \\
&\quad - \frac{1}{2}m|\dot{\mathbf{q}}|^2 - \frac{1}{2}p_\theta^2 \\
&= \mathbf{p}\cdot\frac{1}{m}(\mathbf{p}-p_\theta\mathbf{A}) + \frac{1}{2}p_\theta^2 \\
&\quad - p_\theta\mathbf{A}\cdot\frac{1}{m}(\mathbf{p}-p_\theta\mathbf{A}) - \frac{1}{2m}|\mathbf{p}-p_\theta\mathbf{A}|^2 \\
&= \frac{1}{2m}|\mathbf{p}-p_\theta\mathbf{A}|^2 + \frac{1}{2}p_\theta^2 .
\end{aligned} \tag{5}$$

Remarks

(i) This example provides an easy but fundamental illustration of the geometry of (Lagrangian) reduction by symmetry. The canonical equations for the Hamiltonian H now reproduce Newton's equations for the Lorentz force law, reinterpreted as geodesic motion with respect to the metric defined by the Lagrangian on the tangent bundle $TQ \simeq T\mathbb{R}^3 \times TS^1$.

(ii) On the constant level set $p_\theta = e/c$, this Hamiltonian H is a function of only the variables $(\mathbf{q},\mathbf{p})$ and is equal to the Hamiltonian for charged particle motion under the Lorentz force up to an additive constant.

▲

7. *Consider the Lagrangian*

$$L_\epsilon(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z\cdot\mathbf{q} - \frac{1}{4\epsilon}(1-\|\mathbf{q}\|^2)^2 + \frac{1}{\epsilon}\pi(\mathbf{q}\cdot\dot{\mathbf{q}})$$

for a particle with coordinates $\mathbf{q}\in\mathbb{R}^3$, constants g, ϵ and vertical unit vector $\mathbf{e}_z$. Let $\gamma_\epsilon(t)$ be the curve in $\mathbb{R}^3$ obtained by solving the Euler-Lagrange equations for L_ϵ with the initial conditions $\mathbf{q}_0=\gamma_\epsilon(0)$, $\dot{\mathbf{q}}_0 = \dot\gamma_\epsilon(0)$.

Show that

(a) *In the limit*

$$\lim_{g\to0,\epsilon\to0}\gamma_\epsilon(t)$$

the motion is along is a great circle on the two-sphere S^2, provided that the initial conditions satisfy $\|\mathbf{q}_0\|^2 = 1$ and $\mathbf{q}_0\cdot\dot{\mathbf{q}}_0 = 0$.

(b) *For constant $g>0$ the limit*

$$\lim_{\epsilon\to0}\gamma_\epsilon(t)$$

recovers the dynamics of a spherical pendulum.

Answer

(i) **Fibre derivative**
The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{q}}} = \dot{\mathbf{q}}$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

$$\ddot{\mathbf{q}} = -g\hat{\mathbf{e}}_3 + \frac{1}{\epsilon}(\dot{\pi} + 1 - \|\mathbf{q}\|^2)\mathbf{q}\,. \tag{6}$$

Imposing $\frac{d}{dt}(\mathbf{q}\cdot\dot{\mathbf{q}}) = 0$ yields

$$\frac{1}{\epsilon}(\dot{\pi} + 1 - \|\mathbf{q}\|^2) = \|\dot{\mathbf{q}}\|^2 - g\hat{\mathbf{e}}_3\cdot\mathbf{q}$$

which determines $\pi(t)$ once the motion for $\mathbf{q}(t)$ is known.

(iii) **Hamiltonian and canonical equations**

The corresponding Hamiltonian is obtained by the Legendre transformation as,

$$H(\mathbf{q},\mathbf{p}) = \tfrac{1}{2}\|\mathbf{p} - \frac{\pi}{\epsilon}\mathbf{q}\|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{q} + \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2\,, \tag{7}$$

in which the variable $\mathbf{p}$ is the momentum canonically conjugate to the radial position $\mathbf{q}$. The canonical equations on $(1 - \|\mathbf{q}\|^2) = 0$, are

$$\begin{aligned}
\dot{\mathbf{q}} &= \{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p} - \frac{\pi}{\epsilon}\mathbf{q} \\
\dot{\mathbf{p}} &= \{\mathbf{p}, H\} = -\frac{\partial H}{\partial \mathbf{q}} = -g\hat{\mathbf{e}}_3 + (\mathbf{p} - \frac{\pi}{\epsilon}\mathbf{q})\frac{\pi}{\epsilon} + \frac{1}{\epsilon}(1 - \|\mathbf{q}\|^2)\mathbf{q}\,.
\end{aligned}$$

These equations seem to be equivalent to the spherical pendulum equations for any value of ϵ. Hence, items (a) and (b) above seem to be answered by the spherical pendulum solution. ▲

8. *How does the motion in the previous part differ from that obtained via Hamilton's principle for the following Lagrangian?*

$$L_\epsilon(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z\cdot\mathbf{q} - \mu(1 - \|\mathbf{q}\|^2)$$

where μ is called a ***Lagrange multiplier*** *and must be determined as part of the solution.*

Answer See **Exercise 1.9** about the spherical pendulum. ▲

Exercise 1.6 (Poisson brackets)

1. *Show that the canonical Poisson bracket is bilinear, skew symmetric, satisfies the Jacobi identity and acts as a derivation on products of functions in phase space.*

Answer This is easy for all but the last property, which is a bit tedious. ▲

2. *Given two constants of motion, what does the Jacobi identity imply about additional constants of motion associated with their Poisson bracket?*

Answer The Poisson bracket of two constants of motion is another one. ▲

3. *Compute the Poisson brackets among the $\mathbb{R}^3$-valued functions*

$$J_i = \epsilon_{ijk}p_j q_k$$

for $(\mathbf{q},\mathbf{p}) \in T^*\mathbb{R}^3$.

Answer

$$\{J_i,\, J_j\} = \epsilon_{ijk} J_k$$

▲

4. *Verify that Hamilton's equations for the function*

$$J^\xi(\mathbf{q},\mathbf{p}) = \langle J(z), \xi\rangle = \boldsymbol{\xi}\cdot(\mathbf{q}\times\mathbf{p})$$

with $z := (\mathbf{q},\mathbf{p}) \in T^*\mathbb{R}^3$ *and* $\boldsymbol{\xi}\in\mathbb{R}^3$ *give infinitesimal rotations of* $\mathbf{q}$ *and* $\mathbf{p}$ *about the* $\boldsymbol{\xi}$*-axis.*

Answer The Hamiltonian vector field for J^ξ is

$$\begin{aligned} X_{J^\xi} := \{\,\cdot\,, J^\xi\} &= \frac{\partial J^\xi}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{q}} - \frac{\partial J^\xi}{\partial \mathbf{q}}\cdot\frac{\partial}{\partial \mathbf{p}} \\ &= \boldsymbol{\xi}\times\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{q}} + \boldsymbol{\xi}\times\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{p}}\,, \end{aligned}$$

whose coefficients are the infinitesimal rotations of $\mathbf{q}$ and $\mathbf{p}$ about the $\boldsymbol{\xi}$-axis. ▲

5. *Show that for smooth functions* $c, f, h : \mathbb{R}^3 \to \mathbb{R}$*, the* $\mathbb{R}^3$*-bracket defined by*

$$\{f, h\} = -\nabla c\cdot\nabla f\times\nabla h$$

satisfies the defining properties of a Poisson bracket. Is it also a derivation satisfying the Leibnitz relation for a product of functions on $\mathbb{R}^3$*? If so, why?*

Answer The $\mathbb{R}^3$-bracket is plainly a skew-symmetric bilinear Leibniz operator. Its Hamiltonian vector fields are divergence free vector fields in $\mathbb{R}^3$. These vector fields in $\mathbb{R}^3$ satisfy the Jacobi identity under commutation. The identification of the $\mathbb{R}^3$-bracket with its Hamiltonian vector fields shows that it satisfies Jacobi. This will be made clearer below. ▲

6. *How is the* $\mathbb{R}^3$*-bracket related to the canonical Poisson bracket in the plane?*

Answer The canonical Poisson bracket in the (x, y)-plane is given by the particular choice of the $\mathbb{R}^3$-bracket

$$\{f, h\} = -\nabla z\cdot\nabla f\times\nabla h$$

▲

7. *The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy*

$$\{c, h\}(\mathbf{x}) = 0\,, \quad \textit{for all} \quad h(\mathbf{x})$$

Part 5 verifies that the $\mathbb{R}^3$*-bracket satisfies the defining properties of a Poisson bracket. What are the Casimirs for the* $\mathbb{R}^3$ *bracket?*

Answer Smooth functions of c are Casimirs for the $\mathbb{R}^3$-bracket given by

$$\{f, h\} = -\nabla c\cdot\nabla f\times\nabla h.$$

▲

8. *Write the motion equation for the* $\mathbb{R}^3$*-bracket*

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\}$$

in vector form using gradients and cross products. Show that the corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ *has zero divergence.*

Answer

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h$$

The corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence because the vector $\nabla c \times \nabla h$ has zero divergence. (It's a curl.) ▲

9. *Show that under the* $\mathbb{R}^3$*-bracket, the Hamiltonian vector fields* $X_f = \{\,\cdot\,, f\}$, $X_h = \{\,\cdot\,, h\}$ *satisfy the following anti-homomorphism that relates the commutation of vector fields to the* $\mathbb{R}^3$*-bracket operation between smooth functions on* $\mathbb{R}^3$*,*

$$[X_f, X_h] = -X_{\{f,h\}}.$$

Hint: commutation of divergenceless vector fields does satisfy the Jacobi identity.

Answer

Lemma 1.7 *The* $\mathbb{R}^3$*-bracket defined on smooth functions* (C, F, H) *by*

$$\{F, H\} = -\,\nabla C \cdot \nabla F \times \nabla H$$

may be identified with the divergenceless vector fields by

$$[X_G, X_H] = -X_{\{G,H\}}\,, \tag{8}$$

where $[X_G, X_H]$ *is the Jacobi-Lie bracket of vector fields* X_G *and* X_H*.*

Proof. Equation (8) may be verified by a direct calculation,

$$\begin{aligned}
[X_G, X_H] &= X_G X_H - X_H X_G \\
&= \{G,\,\cdot\,\}\{H,\,\cdot\,\} - \{H,\,\cdot\,\}\{G,\,\cdot\,\} \\
&= \{G,\,\{H,\,\cdot\,\}\} - \{H,\,\{G,\,\cdot\,\}\} \\
&= \{\{G,\,H\},\,\cdot\,\} = -X_{\{G,H\}}\,.
\end{aligned}$$

■

Remark 1.8 The last step in the proof of Lemma 1.7 uses the Jacobi identity for the $\mathbb{R}^3$-bracket, which follows from the Jacobi identity for divergenceless vector fields, since

$$X_F X_G X_H = -\,\{F,\,\{G,\,\{H,\,\cdot\,\}\}\}$$

▲

10. *Show that the motion equation for the* $\mathbb{R}^3$*-bracket is invariant under a certain linear combination of the functions c and h. Interpret this invariance geometrically.*

Answer

$$\nabla(\alpha c + \beta h) \times \nabla(\gamma c + \epsilon h) = \nabla c \times \nabla h \quad \text{for constants satisfying} \quad \alpha\epsilon - \beta\gamma = 1.$$

Under such a (volume-preserving) transformation, the level sets change, but their intersections remain invariant.

▲

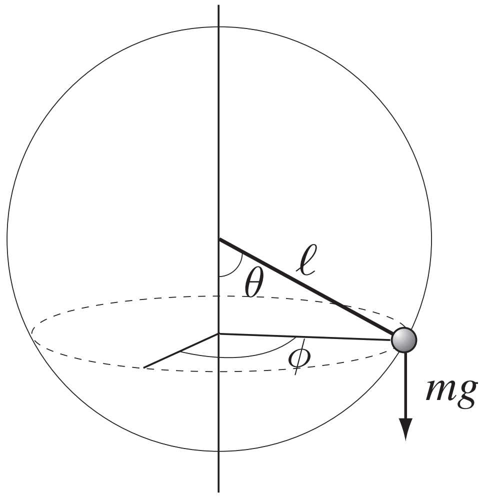

Figure 1: Spherical pendulum: $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\theta$, $z = -R\cos\theta$.

Exercise 1.9 (Spherical pendulum)

A spherical pendulum of length L swings from a fixed point of support under the constant downward force of gravity mg.

Use spherical coordinates with azimuthal angle $0 \le \phi < 2\pi$ and polar angle $0 \le \theta < \pi$ measured from the ***downward*** *vertical defined in terms of Cartesian coordinates by (note minus sign in z)*

1. *Find its equations of motion according to the approaches of*
   - *(a) Newton,*
   - *(b) Lagrange and*
   - *(c) Hamilton.*

   $\boxed{\textit{Answer}}$ This is the content of Appendix A, Section A.1.2 of the text. The remainder of the problem was solved in class, as follows. ▲

2. *Write the constrained Lagrangian for the $L(\mathbf{x}, \dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ as*

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \tfrac{1}{2}|\dot{\mathbf{x}}|^2 - g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \tfrac{1}{2}\mu(1 - |\mathbf{x}|^2), \tag{9}$$

   in which the Lagrange multiplier μ constrains the motion to remain on the sphere S^2 by enforcing $(1 - |\mathbf{x}|^2) = 0$ when it is varied in Hamilton's principle.

   - *(a) Compute the variations in Hamilton's principle and write the Euler-Lagrange equations for the spherical pendulum on $T\mathbb{R}^3$.*
   - *(b) Solve for the Lagrange multiplier by requiring that TS^2 is preserved by this motion on $T\mathbb{R}^3$.*

   $\boxed{\textit{Answer}}$

   (a) The corresponding Euler-Lagrange equations are

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \mu\mathbf{x}\,. \tag{10}$$

   (b) This equation preserves *both* of the TS^2 defining relations $1 - |\mathbf{x}|^2 = 0$ and $\mathbf{x}\cdot\dot{\mathbf{x}} = 0$, provided the Lagrange multiplier is given by

$$\mu = g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - |\dot{\mathbf{x}}|^2\,. \tag{11}$$

▲

3. *Find the Hamiltonian and its canonical equations.*

Answer The fibre derivative of the constrained Lagrangian L in (9) is

$$\mathbf{y} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}}\,. \tag{12}$$

The corresponding Hamiltonian is obtained by the Legendre transformation as,

$$H(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}|\mathbf{y}|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{x} + \tfrac{1}{2}(g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)(1-|\mathbf{x}|^2)\,, \tag{13}$$

in which the variable $\mathbf{y}$ is the momentum canonically conjugate to the radial position $\mathbf{x}$. The canonical equations on $(1-|\mathbf{x}|^2)=0$, are

$$\dot{\mathbf{x}} = \{\mathbf{x}, H\} = \frac{\partial H}{\partial \mathbf{y}} = \mathbf{y} \quad\text{and}\quad \dot{\mathbf{y}} = \{\mathbf{y}, H\} = -\frac{\partial H}{\partial \mathbf{x}} = -g\hat{\mathbf{e}}_3 + (g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)\mathbf{x}\,. \tag{14}$$

▲

4. *A convenient choice of basis for the algebra of polynomials in $(\mathbf{x},\mathbf{y})$ that are S^1-invariant under rotations about the 3-axis is given by*

$$\begin{array}{llllll}
\sigma_1 & = x_3 & \sigma_3 & = y_1^2+y_2^2+y_3^2 & \sigma_5 & = x_1y_1+x_2y_2 \\
\sigma_2 & = y_3 & \sigma_4 & = x_1^2+x_2^2, & \sigma_6 & = x_1y_2-x_2y_1
\end{array}$$

(a) *Find the cubic relation that these S^1-invariants satisfy and express the defining relations for TS^2 in terms of them.*

(b) *Use these relations to eliminate σ_4 and σ_5 in favour of $\{\sigma_1,\sigma_2,\sigma_3,\sigma_6\}$ and find the cubic relation satisfied among $\{\sigma_1,\sigma_2,\sigma_3,\sigma_6\}$.*

Answer

(a) These six S^1-invariants satisfy the cubic algebraic relation

$$\sigma_5^2 + \sigma_6^2 = \sigma_4(\sigma_3 - \sigma_2^2)\,. \tag{15}$$

Hence, they also satisfy the positivity conditions

$$\sigma_4 \ge 0, \quad \sigma_3 \ge \sigma_2^2. \tag{16}$$

In these variables, the defining relations for TS^2 become

$$\sigma_4 + \sigma_1^2 = 1 \quad\text{and}\quad \sigma_5 + \sigma_1\sigma_2 = 0\,. \tag{17}$$

(b) Using the relations in (17) to eliminate σ_4 and σ_5 from (15) yields the cubic relation

$$C(\sigma_1,\sigma_2,\sigma_3,\sigma_6) = \sigma_2^2 + \sigma_6^2 - \sigma_3(1-\sigma_1^2) = 0 \tag{18}$$

▲

5. (a) *Find the Poisson bracket relations among the remaining quadratic invariant variables $\{\sigma_1,\sigma_2,\sigma_3,\sigma_6\}$*

(b) *Explain how this Poisson bracket is related to the $\mathbb{R}^3$-bracket.*

Answer

(a) The Poisson bracket relations among the remaining quadratic invariant variables $\{\sigma_1, \sigma_2, \sigma_3, \sigma_6\}$ may be computed from their definitions in terms of the canonically conjugate variables $(\mathbf{x}, \mathbf{y})$, as

$\{\cdot\,,\,\cdot\}$	σ_1	σ_2	σ_3	σ_6
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$	0
σ_2	$-1+\sigma_1^2$	0	$-\,2\sigma_1\sigma_3$	0
σ_3	$-\,2\sigma_2$	$2\sigma_1\sigma_3$	0	0
σ_6	0	0	0	0

(b) The Poisson bracket amongst $\{\sigma_1, \sigma_2, \sigma_3\}$ defines an $\mathbb{R}^3$-bracket, given by

$$\{\sigma_i,\,\sigma_j\} = -\,\epsilon_{ijk}\frac{\partial C}{\partial \sigma_k}\,. \tag{19}$$

▲

6. *Write their dynamics on TS^2 in Hamiltonian form.*

Answer In Hamiltonian form the dynamics on TS^2 (which is preserved by the motion) simplifies because the spherical pendulum Hamiltonian in (13) becomes linear in the S^1-invariants

$$H|_{TS^2} = \tfrac{1}{2}\sigma_3 + g\sigma_1\,. \tag{20}$$

Hence the dynamics becomes

$$\dot\sigma_i = \{\sigma_i, H\} = \epsilon_{ijk}\frac{\partial C}{\partial \sigma_j}\frac{\partial H}{\partial \sigma_k}$$

or explicitly,

$$\dot\sigma_1 = \{\sigma_1, H\} = -\sigma_2\,, \qquad \dot\sigma_2 = \{\sigma_2, H\} = \sigma_1\sigma_3 + g(1-\sigma_1^2)\,, \qquad \dot\sigma_3 = \{\sigma_3, H\} = 2g\sigma_2\,,$$

and $\dot\sigma_6 = \{\sigma_6, H\} = 0$ because the Poisson bracket with σ_6 vanishes with all the other S^1-invariants.

▲