# 1 Solutions of M3-4A16 Assessed Problems # 1

**Exercise 1.1 (The Lie group $SO_{\mathsf{K}}(3)$)**

*(A) Consider the zero locus map in the space of* $3 \times 3$ *real matrices,*

$$S_{\mathsf{K}} = \{ U \in GL(3, \mathbb{R}) | \, U^T \mathsf{K} U - \mathsf{K} = 0 \} \,, \qquad \textit{(locus map).}$$

*Explain why $S_{\mathsf{K}}$ is a manifold for $\mathsf{K} = \mathsf{K}^T \in GL(3, \mathbb{R})$. Hint: Is it a submersion?*

***Answer*** Following the hint in the problem statement, we begin by looking up the definition of *submersion*:

**Definition.** A ***submersion*** is a smooth map between smooth manifolds whose derivative is everywhere surjective.

Next, we observe that the map $S_{\mathsf{K}}$ is the zero locus, or zero level set, of the mapping

$$U \to (U^T K U - K) \,, \qquad \text{(locus map)}$$

In class, we learned that manifolds can arise as level sets

$$M = \{ x \big| f_i(x) = 0, \ i = 1, \dots, k \} \,, \qquad \text{(locus map)}$$

for a given set of smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, k$.

As we discussed in class, if the gradients $\nabla f_i$ are linearly independent, or more generally if the rank of $\{\nabla f(x)\}$ is a constant $r$ for all $x$, then $M$ is a smooth manifold of dimension $n - r$. (The proof uses the implicit function theorem and was not discussed in class.)
In particular, if $r = k = \dim M$, the map $\{f_i\} : \mathbb{R}^n \to M$ will be surjective.
Thus, the strategy suggested by the hint is to show that $r = k = \dim M$, so that the map $S_{\mathsf{K}}$ will a ***submersion*** and hence a submanifold. □

This strategy of connecting submanifolds with zero loci of surjective maps leads us to linearise the locus map. This is also part $(C)$ of the problem. Thus, beginning the proof of surjectivity of the locus map by linearising it in part $(A)$ sets up the answer to part $(C)$, too, where one evaluates the linearisation *at the identity*.

Let $U \in S_{\mathsf{K}}$, and let $\delta U$ be an arbitrary element of $R^{n \times n}$. Then linearise to find

$$(U + \delta U)^T K (U + \delta U) - K = (U^T K U - K) + \delta U^T K U + U^T K \delta U + O(\delta U)^2 \,.$$

We may conclude that $S_{\mathsf{K}}$ is a submanifold of $R^{n \times n}$ if we can show that the linearization of the locus map, namely the linear mapping defined by

$$L \equiv \delta U \to \delta U^T K U + U^T K \delta U \,, \qquad R^{n \times n} \to R^{n \times n}$$

has constant rank for all $U \in S_{\mathsf{K}}$.
**Lemma.** The linearization map $L$ is onto the space of $n \times n$ of symmetric matrices. Hence, it has constant rank and the original map is a submersion.
**Proof.**

- Both the original locus map and the image of $L$ lie in the subspace of $n \times n$ symmetric matrices.
- Indeed, given $U$ and any symmetric matrix $S$ we can find $\delta U$ such that

$$\delta U^T K U + U^T K \delta U = S \,.$$

Namely

$$\delta U = K^{-1} U^{-T} S / 2 \,.$$

- Thus, the linearization map $L$ is onto the space of $n \times n$ of symmetric matrices. That is, it has constant rank. This means the original locus map $U \to UKU^T - K$ to the space of symmetric matrices is a submersion.

■ ▲

(B) *Assuming that $S_{\mathsf{K}}$ is a manifold, prove that it is a matrix Lie group.*

**Answer** The smooth manifold $S_{\mathsf{K}}$ in the space $GL(3,\mathbb{R})$ of real $3 \times 3$ matrices is also a group, if it satisfies the group axioms. Then, being both a manifold and a matrix group, it will be a matrix Lie group.
Knowing that matrix multiplication is associative, we check that $S_{\mathsf{K}}$ satisfies the other three defining properties of a group:

- Identity: $I \in S_{\mathsf{K}}$ because $I^TKI = K$.
- Inverse: $U \in S_{\mathsf{K}} \Longrightarrow U^{-1} \in S_{\mathsf{K}}$ because

$$K = U^{-T}(U^TKU)U^{-1} = U^{-T}(K)U^{-1}.$$

- Closed under multiplication: $U, V \in S_{\mathsf{K}} \Longrightarrow UV \in S_{\mathsf{K}}$ because

$$(UV)^TKUV = V^T(U^TKU)V = V^T(K)V = K.$$

Hence, $S_{\mathsf{K}}$ is a subgroup of the matrix Lie group $GL(3,\mathbb{R})$.

▲

(C) *Write the defining relation for the tangent space to $S_{\mathsf{K}}$ at the identity, $T_I S_{\mathsf{K}}$.*

**Answer** The tangent space $T_I S_{\mathsf{K}}$ at the identity of the matrix Lie group $S_{\mathsf{K}}$ defined by $S_{\mathsf{K}} = \{U \in GL(n,\mathbb{R}) |\, UKU^T - K = 0\}$ is the linear space of matrices $A$ satisfying

$$A^TK + KA = 0$$

**Proof.** Near the identity the defining condition for $S_{\mathsf{K}}$ in part 1.1($a$) expands to

$$(I + \epsilon A^T + O(\epsilon^2))K(I + \epsilon A + O(\epsilon^2)) = K\,, \quad \text{for} \quad \epsilon \ll 1\,.$$

At linear order $O(\epsilon)$ one finds,

$$A^TK + KA = 0\,.$$

This relation defines the linear space of matrices $A \in T_I S_{\mathsf{K}}$. ▲

(D) *Show that for any pair of matrices $A, B \in T_I S_{\mathsf{K}}$, the matrix commutator satisfies*

$$[A, B] \equiv AB - BA \in T_I S_{\mathsf{K}}.$$

**Answer** Using $[A, B]^T = [B^T, A^T]$, we check *closure* by a direct computation,

$$\begin{aligned} [B^T, A^T]K + K[A, B] &= B^TA^TK - A^TB^TK + KAB - KBA \\ &= B^TA^TK - A^TB^TK - A^TKB + B^TKA \\ &= B^T(A^TK + KA) - A^T(B^TK + KB) = 0\,. \end{aligned}$$

Hence, the tangent space of $S_{\mathsf{K}}$ at the identity $T_I S_{\mathsf{K}}$ is closed under the matrix commutator $[\,\cdot\,,\,\cdot\,]$.

▲

(E) *Suppose the* $3 \times 3$ *matrices* $A$ *and* $K$ *satisfy*

$$AK + KA^T = 0\,.$$

*Show that* $\exp(At)K\exp(A^Tt) = K$ *for all* $t$.

**Answer** Compute

$$\frac{d}{dt}\Big(\exp(At)K\exp(A^Tt)\Big) = \exp(At)(AK + KA^T)\exp(A^Tt) = 0$$

Hence, the relation $\exp(At)K\exp(A^Tt) = K$ is preserved under this flow.

This result shows that the flow defined by exponentiation of matrices in $T_I S_{\mathsf{K}}$ preserves $K$, so it maps the manifold $S_{\mathsf{K}}$ smoothly into itself.

Is it *surprising*?

Imagine how surprised you would have been at finding this result, if you did not already know that Lie group transformations arise by exponentiating the infinitesimal transformations of their Lie algebras! ▲

(F) *Define the following* ***hat map*** *from basis vectors* $(\widehat{e}_1, \widehat{e}_2, \widehat{e}_3) \in \mathfrak{s}_{\mathsf{K}}$ *to basis vectors* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \in \mathbb{R}^3$,

$$\widehat{\ }: \mathfrak{s}_{\mathsf{K}} \to \mathbb{R}^3 \quad \textit{is defined by} \quad (\mathsf{K}\widehat{\mathbf{e}}\,)_{ij} = -\,\epsilon_{ijk}\mathsf{K}^{kl}\mathbf{e}_l = -\,c_{ij}{}^l\mathbf{e}_l = -\,[\mathbf{e}_i,\,\mathbf{e}_j]_{\mathsf{K}}\,.$$

*Under what conditions on* $\mathsf{K}$ *is the hat map* $(\,\widehat{\cdot}\,)$ *a linear isomorphism? (This is easy.)*

**Answer** For this variant of the hat map to be a linear isomorphism, the matrix $\mathsf{K}$ must be invertible. (The usual hat map is with $\mathsf{K} = \mathrm{Id}$.) ▲

(G) *For any vectors* $\mathbf{x} = x^i\mathbf{e}_i$, $\mathbf{y} = y^j\mathbf{e}_j \in \mathbb{R}^3$ *with components* $x^j$, $y^k$, *where* $j, k = 1, 2, 3$, *show that the Lie algebra structure* $[\mathbf{e}_i,\,\mathbf{e}_j]_{\mathsf{K}}$ *is represented on* $\mathbb{R}^3$ *by the vector product*

$$[\mathbf{x}, \mathbf{y}]_{\mathsf{K}} = \mathsf{K}(\mathbf{x} \times \mathbf{y})\,. \tag{1}$$

**Answer** The result follows from the following direct substitution in components:

$$[\mathbf{x}, \mathbf{y}]^l_{\mathsf{K}}\mathbf{e}_l = x^iy^jc_{ij}{}^l\mathbf{e}_l = x^iy^j\epsilon_{ijk}\mathsf{K}^{kl}\mathbf{e}_l = (\mathbf{x} \times \mathbf{y})_k\mathsf{K}^{kl}\mathbf{e}_l\,.$$

▲

(H) *Compute the Euler-Poincaré equation on* $\mathbb{R}^3$ *for a Lagrangian* $\ell : \mathfrak{s}_K \to \mathbb{R}$ *by using the hat map representation of* $\mathfrak{s}_K$ *on* $\mathbb{R}^3$ *in equation (1).*

**Answer** For $\mathbf{u}, \mathbf{v} \in \mathfrak{s}_K \cong \mathbb{R}^3$ and $\mathbf{X} \in \mathfrak{s}^*_K \cong \mathbb{R}^{3*} \cong \mathbb{R}^3$, one uses the Lie bracket on the matrix Lie algebra $\mathfrak{s}_K$ in vector form with the $\mathbb{R}^3$ dot-product pairing to define its ad and ad$^*$ operations by the formulas

$$\begin{aligned}\langle \mathbf{X}\,,\,[\,\mathbf{u}, \mathbf{v}\,]_{\mathsf{K}}\rangle &= \langle \mathbf{X}\,,\,\mathrm{ad}_{\mathbf{u}}\mathbf{v}\rangle = \langle \mathrm{ad}^*_{\mathbf{u}}\mathbf{X}\,,\,\mathbf{v}\rangle \\ &= \mathsf{K}\mathbf{X} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathsf{K}\mathbf{X} \times \mathbf{u}) \cdot \mathbf{v}\,.\end{aligned} \tag{2}$$

In particular, the vector forms of $\mathrm{ad}_{\mathbf{u}}\mathbf{v}$ and $\mathrm{ad}^*_{\mathbf{u}}\mathbf{X}$ are given explicitly by

$$\mathrm{ad}_{\mathbf{u}}\mathbf{v} = \mathsf{K}(\mathbf{u} \times \mathbf{v}) \quad \text{and} \quad \mathrm{ad}^*_{\mathbf{u}}\mathbf{X} = -\,\mathbf{u} \times \mathsf{K}\mathbf{X}\,. \tag{3}$$

These vector forms of the ad and ad$^*$ operations will yield a vector form of the Euler-Poincaré equation

$$\frac{d}{dt}\frac{\partial l}{\partial \mathbf{u}} = \mathrm{ad}^*_{\mathbf{u}}\frac{\partial l}{\partial \mathbf{u}}\,,$$

for the dynamics obtained from Hamilton's principle $\delta S = 0$ with $S = \int_a^b l(\mathbf{u})\,dt$ and a Lagrangian $l(\mathbf{u}) : \mathfrak{s}_K \cong \mathbb{R}^3 \to \mathbb{R}$ by identifying $\mathbf{X} = \partial l/\partial \mathbf{u} \in \mathfrak{s}_K^*$ in vector form. Namely,

$$\frac{d}{dt}\frac{\partial l}{\partial \mathbf{u}} = \mathrm{ad}^*_{\mathbf{u}}\frac{\partial l}{\partial \mathbf{u}} = -\,\mathbf{u}\times\left(\mathsf{K}\frac{\partial l}{\partial \mathbf{u}}\right).$$

▲

*(I) Legendre transform to the Hamiltonian side and compute the corresponding Lie-Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}_{\mathsf{K}}$ for smooth real functions $F, H : \mathbf{X} \in \mathbb{R}^3 \to \mathbb{R}$.*

***Answer*** The Legendre transform $\mathbf{u} \to \mathbf{X}$ yields the Hamiltonian

$$H(\mathbf{X}) = \mathbf{X}\cdot\mathbf{u} - l(\mathbf{u}) \quad \text{with dual relations} \quad \mathbf{X} = \frac{\partial l}{\partial \mathbf{u}} \quad \text{and} \quad \mathbf{u} = \frac{\partial H}{\partial \mathbf{X}}\,.$$

The Lie–Poisson bracket is defined on smooth functions $F, H : \mathfrak{s}_K^* \to \mathbf{R}$ as

$$\{F,\,H\}_{\mathsf{K}} := \left\langle \mathbf{X}\,,\,\mathrm{ad}_{\partial H/\partial \mathbf{X}}\frac{\partial F}{\partial \mathbf{X}}\right\rangle.$$

The corresponding Lie–Poisson dynamics is expressed in terms of the ad and ad$^*$ operations by

$$\begin{aligned}
\frac{dF}{dt} = \{F,\,H\}_{\mathsf{K}} &= \left\langle \mathbf{X}\,,\,\mathrm{ad}_{\partial H/\partial \mathbf{X}}\frac{\partial F}{\partial \mathbf{X}}\right\rangle \\
&= \left\langle \mathrm{ad}^*_{\partial H/\partial \mathbf{X}}\mathbf{X}\,,\,\frac{\partial F}{\partial \mathbf{X}}\right\rangle \\
&= -\,\frac{\partial F}{\partial \mathbf{X}}\cdot\frac{\partial H}{\partial \mathbf{X}}\times \mathsf{K}\mathbf{X}\,.
\end{aligned}$$

Consequently, the Lie–Poisson dynamics expresses itself as coadjoint motion on $\mathfrak{s}_K^*$, written in vector form as,

$$\frac{d\mathbf{X}}{dt} = \{\mathbf{X},\,H\}_{\mathsf{K}} = \mathrm{ad}^*_{\partial H/\partial \mathbf{X}}\mathbf{X} = -\,\frac{\partial H}{\partial \mathbf{X}}\times\mathsf{K}\mathbf{X}\,.$$

▲

*(J) Rewrite this Lie-Poisson bracket as a triple scalar product of gradients of smooth real functions on $\mathbb{R}^3$ and find its Casimir(s) $C : \{C(\mathbf{x}), H(\mathbf{x})\}_{\mathsf{K}} = 0$, for all $H$.*

***Answer*** By construction, the Lie-Poisson bracket, written as

$$\{F,\,H\}_{\mathsf{K}} = -\,\mathsf{K}\mathbf{X}\cdot\frac{\partial F}{\partial \mathbf{X}}\times\frac{\partial H}{\partial \mathbf{X}}$$

conserves the quadratic form, $C_{\mathsf{K}} = \frac{1}{2}\mathbf{X}\cdot\mathsf{K}\mathbf{X}$, which is the (quadratic) Casimir for the Lie-Poisson bracket $\{F,\,H\}_{\mathsf{K}}$. That is, $\{C_{\mathsf{K}},\,H\}_{\mathsf{K}} = 0$ for any Hamiltonian $H(\mathbf{X})$. Any differentiable function $\Phi$ of $C_{\mathsf{K}}$ is also a Casimir, since by the Leibniz rule for the Lie-Poisson bracket we have

$$\{\Phi(C_{\mathsf{K}}),\,H\}_{\mathsf{K}} = \Phi'(C_{\mathsf{K}})\{C_{\mathsf{K}},\,H\}_{\mathsf{K}} = 0\,.$$

▲

(K) *Compute the corresponding equations of motion for the Hamiltonian* $H = \frac{1}{2}\|\mathbf{X}\|^2$. *How are the resulting equations related to Euler's equations for rigid body motion?*

**Answer** For $H = \frac{1}{2}\|\mathbf{X}\|^2$ the Lie=Poisson equation becomes

$$\frac{d\mathbf{X}}{dt} = \{\mathbf{X},\, H\}_{\mathsf{K}} = \mathrm{ad}^*_{\partial H/\partial \mathbf{X}}\mathbf{X} = -\,\frac{\partial H}{\partial \mathbf{X}} \times \mathsf{K}\mathbf{X} \quad \Longrightarrow \quad \frac{d\mathbf{X}}{dt} = -\,\mathbf{X} \times \mathsf{K}\mathbf{X}\,.$$

Upon identifying $\mathbf{X} \to \mathbf{\Pi}$ and $\mathsf{K} \to \mathsf{I}^{-1}$, then $\mathsf{K}\mathbf{X} \to \mathsf{I}^{-1}\mathbf{\Pi} \to \mathbf{\Omega}$ in rigid body notation, for which Euler's equations are

$$\frac{d\mathbf{\Pi}}{dt} = +\,\mathbf{\Pi} \times \mathsf{I}^{-1}\mathbf{\Pi}$$

we see that coadjoint motion on $\mathfrak{s}^*_K$ with Hamiltonian $H = \frac{1}{2}\|\mathbf{X}\|^2$ is equivalent to Euler rigid body on $\mathfrak{so}(3)^*$ with the roles of the Casimir and Hamiltonian exchanged, so that the direction of time is reversed.

▲

**Exercise 1.2 (Noether's theorem)**

*(A)* ***Gauge invariance*** *Show that the Euler-Lagrange equations are unchanged under*

$$L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) \to L' = L + \frac{d}{dt}\gamma(\mathbf{q}(t),\dot{\mathbf{q}}(t))\,,$$

*for any function* $\gamma : \mathbb{R}^{6N} = \{(\mathbf{q},\dot{\mathbf{q}}) \mid \mathbf{q},\dot{\mathbf{q}} \in \mathbb{R}^{3N}\} \to \mathbb{R}$.

*Answer* Hamilton's principle for the difference is

$$0 = \delta\int_{t_1}^{t_2}\Big(L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) - L'(\mathbf{q}(t),\dot{\mathbf{q}}(t))\Big)\,dt = \delta\Big[\gamma(\mathbf{q}(t),\dot{\mathbf{q}}(t))\Big]_{t_1}^{t_2}$$

However, this vanishes for variations $\delta\mathbf{q}(t)$ that vanish at the endpoints in time. ▲

*(B)* ***Generalized coordinate theorem*** *Show that the Euler-Lagrange equations are* ***unchanged in form*** *under any smooth invertible mapping* $f : \{\mathbf{q} \mapsto \mathbf{s}\}$. *That is, with*

$$L(\mathbf{q}(t),\dot{\mathbf{q}}(t)) = \tilde{L}(\mathbf{s}(t),\dot{\mathbf{s}}(t))\,,$$

*show that*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial\mathbf{q}} = 0 \quad\Longleftrightarrow\quad \frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{\mathbf{s}}}\right) - \frac{\partial\tilde{L}}{\partial\mathbf{s}} = 0\,.$$

*Answer* This just amounts to a change of notation, so it clearly holds. ▲

*(C)* *How do the Euler-Lagrange equations transform under* $\mathbf{q}(t) = \mathbf{r}(t) + \mathbf{s}(t)$, *when* $\mathbf{r}(t)$ *and* $\mathbf{s}(t)$ *are independent of each other?*

*Answer* A sum of two separate Euler-Lagrange equations is obtained. ▲

*(D)* *State and prove Noether's theorem that each smooth symmetry of Hamilton's principle implies a conservation law for the corresponding Euler-Lagrange equations on the tangent space* $TM$ *of a smooth manifold* $M$.

*Answer* In the family of smoothly deformed curves $q_s(t) = Q(q,t,s)$ with $q_0 = Q(q,t,0) = q(t)$ during the time interval $t \in [t_1,\, t_2]$, the action $S = \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$ transforms to

$$S = \int_{t_1}^{t_2} L\left(Q(q,t,s),\frac{dQ(q,t,s)}{d\tau(t,s)},\tau(t,s)\right)\,d\tau(t,s)\,.$$

We denote

$$\delta q(t) = \frac{d}{ds}\Big|_{s=0} Q(q,t,s) = \xi(q(t),t)\,,\qquad \delta t = \frac{d}{ds}\Big|_{s=0}\tau(t,s) = \theta(t)\,,$$

so that at *linear order* in $s$ we have

$$Q(q,t,s) = q(t)+s\xi(q,t)\,,\qquad \tau(t,s) = t+s\theta(t)\,,\qquad \frac{dQ(q,t,s)}{d\tau(t,s)} = \frac{dq}{dt}+s\Big(\dot{\xi}(q,t)-\dot{q}\dot{\theta}\Big)\,.$$

Here the dot-notation as in $\dot{\xi}(q,t) = \partial_t\xi + \dot{q}\partial_q\xi$ represents the total time derivative. We could allow $q$-dependence in $\tau$, but the result of the calculation would be morally the same, after keeping track of total time derivatives.

The variations in Hamilton's principle proceed as follows,

$$
\begin{aligned}
0 = \delta S &= \frac{d}{ds}\Big|_{s=0} \int_{t_1}^{t_2} L\left( Q(q,t,s), \frac{dQ(q,t,s)}{d\tau(t,s)}, \tau(t,s) \right) d\tau(t,s) \\
&= \int_{t_1}^{t_2} \left\{ \Big\langle \frac{\partial L}{\partial q}, \xi(q,t) \Big\rangle + \Big\langle \frac{\partial L}{\partial \dot q}, \dot\xi(q,t) - \dot q \dot\theta \Big\rangle + \frac{\partial L}{\partial t}\theta + L(q,\dot q,t)\dot\theta \right\} dt \\
&= \int_{t_1}^{t_2} \Big\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}, \Big( \xi(q,t) - \dot q \theta \Big) \Big\rangle dt + \left[ \Big\langle \frac{\partial L}{\partial \dot q}, \xi \Big\rangle - \left( \Big\langle \frac{\partial L}{\partial \dot q}, \dot q \Big\rangle - L \right) \theta \right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \Big\langle \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}, \Big( \delta q - \dot q \delta t \Big) \Big\rangle dt + \left[ \Big\langle \frac{\partial L}{\partial \dot q}, \delta q \Big\rangle - \left( \Big\langle \frac{\partial L}{\partial \dot q}, \dot q \Big\rangle - L \right) \delta t \right]_{t_1}^{t_2}
\end{aligned}
$$

with a few algebraic manipulations and integrations by parts in between the lines. (Of course, these should be checked!)
Thus, stationarity $\delta S = 0$ by symmetry and the Euler-Lagrange equations

$$
[\, L\, ]_{q^a} := \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0
$$

imply that the quantity

$$
\begin{aligned}
C(t,q,\dot q) &= \Big\langle \frac{\partial L}{\partial \dot q}, \delta q \Big\rangle - \left( \Big\langle \frac{\partial L}{\partial \dot q}, \dot q \Big\rangle - L \right) \delta t && (4) \\
&=: \langle\, p \,,\, \delta q \,\rangle - E\, \delta t \,, && (5)
\end{aligned}
$$

has the same value at every time along the solution path. That is, $C(t,q,\dot q)$ is a constant of the motion. This is Noether's theorem.

▲

*(E) Show that* ***conservation of energy results from Noether's theorem*** *if, in Hamilton's principle, the variations of $L(q(t),\dot q(t))$ are chosen as*

$$
\delta q(t) = \frac{d}{ds}\Big|_{s=0} q(t,s)\,,
$$

*corresponding to symmetry of the Lagrangian under reparametrisations of time $t \to \tau(t,s)$ so that $q(t) \to q(\tau(t,s))$ along a given curve $q(t)$.*

***Answer*** For reparametrisations of time, $\delta q$ vanishes and $\delta t$ is a function of time in the previous part; so stationarity of the action $\delta S = 0$ in the presence of time-reparametrisation symmetry implies that the quantity

$$
C(t,q,\dot q) \;=\; \left( \Big\langle \frac{\partial L}{\partial \dot q}, \dot q \Big\rangle - L \right) \delta t =: E(t,q,\dot q)\, \delta t(t)\,, \qquad (6)
$$

is a constant of motion along solutions of the Euler-Lagrange equations. This does not yet imply conservation of the energy $E$. For that, $\delta t$ must be a constant.
For simple translations in time, $\delta q$ again vanishes and $\delta t$ is a *constant*; so stationarity of the action $\delta S = 0$ in the presence of *time-translation symmetry* implies that the energy

$$
E(t,q,\dot q) := \Big\langle \frac{\partial L}{\partial \dot q}, \dot q \Big\rangle - L\,, \qquad (7)
$$

is a constant of motion along solutions of the Euler-Lagrange equations.
This energy is also the expression for the Legendre transform of the Lagrangian $L(t,q,\dot q)$.

▲

**Exercise 1.3 (Examples: Geodesic motion $3X$, Magnetic $3X$ and Spherical Pendulum $2X$)**

*(i) For the following Lagrangians, determine which of them are hyperregular. (A Lagrangian is hyperregular if its fibre derivative is invertible, so that the velocity may be expressed in terms of the position and canonical momentum.)*

*(ii) Write the Euler-Lagrange for these equations.*

*(iii) For the hyperregular Lagrangians apply the Legendre transformation to determine the Hamiltonian and Hamilton's canonical equations.*

*(A) The kinetic energy Lagrangian $K(q,\dot{q}) = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j$ with $i,j = 1,2,\dots,N$, for a Riemannian manifold $Q$ with metric $g$, written as $(Q,g)$.*

*Answer*

(i) The fibre derivative in this case is $\mathbb{F}K(v_q) = g(q)(v_q,\cdot)$, for $v_q \in T_qQ$. In coordinates, this is

$$\mathbb{F}K(q,\dot{q}) = \left(q^i, \frac{\partial K}{\partial \dot{q}^i}\right) = (q^i, g_{ij}(q)\dot{q}^j) =: (q^i, p_i),$$

This Lagrangian is hyperregular for invertible $g(q)$; that is, when the metric is nondegenerate. In that case, one may solve for the velocity in terms of position and canonical momentum as

$$\dot{q}^i = (g^{-1}(q))^{ij}p_j$$

(ii) The Euler-Lagrange equations for this Lagrangian produce the ***geodesic equations*** for the metric $g$, and are given (for finite dimensional $Q$ in a local chart) by

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0, \quad i = 1,\dots n,$$

where the three-index quantities

$$\Gamma^h_{jk} = \frac{1}{2}g^{hl}\left(\frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{kl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right), \quad \text{with} \quad g_{ih}g^{hl} = \delta^l_i\,,$$

are the ***Christoffel symbols*** of the Levi-Civita connection on $(Q,g)$ and $g^{hl}$ is called the co-metric.
The calculation of these Euler-Lagrange equations, done in class, involves a step of symmetrising by using vanishing trace $\mathrm{Tr}(SA) = 0$ for the product of a symmetric matrix with an antisymmetric one.

(iii) The Legendre transform of this Lagrangian yields the corresponding Hamiltonian

$$H = \frac{1}{2}p_i g^{ij}(q)p_j$$

whose canonical equations are

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = g^{ij}(q)p_j\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} = -\,p_k\frac{\partial g^{kj}(q)}{\partial q^i}p_j\,.$$

▲

*(B)* $L(q,\dot{q}) = \left(g_{ij}(q)\dot{q}^i\dot{q}^j\right)^{1/2}$ *(Is it possible to assume that $L(q,\dot{q}) = 1$? Why?)*

*Answer*

(i) **Fibre derivative**

This Lagrangian is not hyperregular. Its fibre derivative begins well enough

$$\frac{\partial L}{\partial \dot q^i} = \frac{1}{\sqrt{g_{kl}(q)\dot q^k \dot q^l}}\, g_{ij}\dot q^j$$

The difficulty is that this Lagrangian is homogeneous of degree one in the velocities. Such functions satisfy Euler's relation,

$$\frac{\partial L}{\partial \dot q^i}\dot q^i - L = 0\,.$$

This already spells trouble, because its Legendre transform produces a Hamiltonian that vanishes identically

$$H = p_i\dot q^i - L \equiv 0$$

Taking another derivative of the Euler's relation yields

$$\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^j}\dot q^j = 0$$

so the Hessian of this Lagrangian $L$ with respect to the tangent vectors is singular (has zero determinant). This means the Legendre transformation for this Lagrangian is not invertible.

A singular Lagrangian might become problematic in some situations. However, there is a simple way of obtaining a regular Lagrangian from it whose trajectories, as we shall see, are the same as those for the singular Lagrangian.

The Lagrangian function in this part of the problem is related to the Lagrangian for geodesics in the previous part by

$$K(q,\dot q) = \frac{1}{2} g_{ij}(q)\dot q^i \dot q^j = \tfrac{1}{2}L^2(q,\dot q)\,.$$

Computing the Hessian with respect to the tangent vector yields the Riemannian metric,

$$\frac{1}{2}\frac{\partial^2 L^2}{\partial \dot q^i \partial \dot q^j} = g_{ij}(q)\,.$$

The emergence of a Riemannian metric from the Hessian of the square of a homogenous function of degree 1 is the hallmark of ***Finsler geometry***, of which Riemannian geometry is a special case. Finsler geometry, however, is beyond our present scope.

(ii) **Euler-Lagrange equations**

On setting $\sqrt{g_{kl}(q)\dot q^k \dot q^l} =: \|\dot q\|$, the Euler-Lagrange equations become

$$\frac{d}{dt}\left(\frac{1}{\|\dot q\|}\, g_{ij}\frac{dq^j}{dt}\right) = \frac{1}{2\|\dot q\|}\left(\frac{dq^k}{dt}\frac{\partial g_{kl}}{\partial q^i}\frac{dq^l}{dt}\right)$$

On dividing by $\|\dot q\|$ and setting $d\tau := \|\dot q\| dt$, this becomes

$$\frac{d}{d\tau}\left(g_{ij}\frac{dq^j}{d\tau}\right) = \frac{1}{2}\left(\frac{dq^k}{d\tau}\frac{\partial g_{kl}}{\partial q^i}\frac{dq^l}{d\tau}\right)\,,$$

which is again the geodesic equation, but now with a reparameterised time.

Assuming that $L = \|\dot q\| = 1$ is not possible, because the value of $\|\dot q\|$ is not preserved by the flow.

(iii) **Hamiltonian and canonical equations**
Hamilton's canonical equations are problematic for a Hamiltonian that vanishes identically.

▲

*(C)* $L(\dot{\mathbf{q}}) = -\Big(1 - \dot{\mathbf{q}}\cdot\dot{\mathbf{q}}\Big)^{1/2}$ *for* $\dot{\mathbf{q}} \in \mathbb{R}^3$.

*Answer*

(i) **Fibre derivative**

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\dot{\mathbf{q}}}{\sqrt{1 - \dot{\mathbf{q}}\cdot\dot{\mathbf{q}}}} =: \gamma\,\dot{\mathbf{q}} \quad\Longrightarrow\quad \dot{\mathbf{q}} = \pm\frac{\mathbf{p}}{\sqrt{1 + \mathbf{p}\cdot\mathbf{p}}}$$

so this Lagrangian is hyperregular, after making a choice of sign convention, that $\mathbf{p}\cdot\dot{\mathbf{q}} > 0$, for example; so that $\gamma = \sqrt{1+\mathbf{p}\cdot\mathbf{p}} = 1/\sqrt{1-\dot{\mathbf{q}}\cdot\dot{\mathbf{q}}}$.

(ii) **Euler-Lagrange equations**

$$\frac{d(\gamma\,\dot{\mathbf{q}})}{dt} = 0$$

(iii) **Hamiltonian and canonical equations**
The Hamiltonian for this system is

$$H = \mathbf{p}\cdot\dot{\mathbf{q}} - L = \sqrt{1+|\mathbf{p}|^2} = \gamma$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{\sqrt{1+|\mathbf{p}|^2}}\,, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} = 0$$

**This is geodesic motion in $\mathbb{R}^3$ for a relativistic particle of unit rest mass.**

▲

*(D) The Lagrangian for a free particle of unit mass relative to a moving frame is obtained by setting*

$$L(\dot{\mathbf{q}}, \mathbf{q}, t) = \frac{1}{2}\|\dot{\mathbf{q}} + \mathbf{R}(\mathbf{q})\|^2$$

*for a function* $\mathbf{R}(\mathbf{q},t)$ *which governs the space and time dependence of the moving frame velocity. For example, a frame rotating with time-dependent frequency* $\Omega(t)$ *about the vertical axis* $\hat{\mathbf{z}}$ *is obtained by choosing* $\mathbf{R}(\mathbf{q},t) = \mathbf{q}\times\Omega(t)\hat{\mathbf{z}}$.

*Answer*

(i) **Fibre derivative**
The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} + \mathbf{R}(\mathbf{q})$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

$$\frac{d}{dt}(\dot{q}_i + R_i(\mathbf{q})) \quad = \quad (\dot{q}_j + R_j(\mathbf{q}))\frac{\partial R^j}{\partial q^i}$$

or

$$\ddot{q}_i = (R_{j,i} - R_{i,j})\dot{q}^j + \frac{\partial}{\partial q^i}(\tfrac{1}{2}|\mathbf{R}|^2)$$

In vector form, this is

$$\ddot{\mathbf{q}} = \dot{\mathbf{q}} \times 2\boldsymbol{\Omega} + \frac{\partial}{\partial \mathbf{q}}(\tfrac{1}{2}|\mathbf{R}|^2) \quad \text{with} \quad 2\boldsymbol{\Omega} := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{R}(\mathbf{q})$$

and the terms on the right comprise the sum of the Coriolis and centrifugal forces.

(iii) **Hamiltonian and canonical equations**

The Hamiltonian for this system is

$$H = \mathbf{p}\cdot\dot{\mathbf{q}} - L = \tfrac{1}{2}|\mathbf{p}|^2 - \mathbf{p}\cdot\mathbf{R}(\mathbf{q})$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p} - \mathbf{R}(\mathbf{q})\,, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} = p_j \frac{\partial}{\partial \mathbf{q}} R^j(\mathbf{q})$$

▲

*(E) The Lagrangian for a charged particle of mass $m$ in a magnetic field $\mathbf{B} = \mathrm{curl}\mathbf{A}$ is*

$$L(q,\dot{q}) = \frac{m}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} + \frac{e}{c}\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}),$$

*for constants $m$, $e$, $c$ and prescribed function $\mathbf{A}(\mathbf{q})$.*

*How do the Euler-Lagrange equations for this Lagrangian differ from those of the previous part for free motion in a moving frame with velocity $\frac{e}{mc}\mathbf{A}(\mathbf{q})$?*

*Answer*

(i) **Fibre derivative**

The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A}(\mathbf{q})$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

In vector form, this is

$$\ddot{\mathbf{q}} = \frac{e}{mc}\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{with} \quad \mathbf{B}(\mathbf{q}) := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{A}(\mathbf{q})$$

and the terms on the right comprise the Lorentz force.

(iii) **Hamiltonian and canonical equations**

The Hamiltonian for this system is

$$H = \mathbf{p}\cdot\dot{\mathbf{q}} - L = \frac{1}{2m}\Big|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q})\Big|^2$$

and its canonical equations are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}\Big(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q})\Big)\,, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} = \frac{e}{mc}\Big(p_j - \frac{e}{c}A_j(\mathbf{q})\Big)\frac{\partial}{\partial \mathbf{q}}A^j(\mathbf{q})$$

**These are the same equations as in the previous part, modulo the relation $\mathbf{R} = e\mathbf{A}/mc$ and neglect of centrifugal force.**

▲

*(F) Let $Q$ be the manifold $\mathbb{R}^3 \times S^1$ with variables $(\mathbf{q}, \theta)$. Introduce the Lagrangian $L : TQ \simeq T\mathbb{R}^3 \times TS^1 \mapsto \mathbb{R}$ as*

$$L(\mathbf{q}, \theta, \dot{\mathbf{q}}, \dot{\theta}) = \frac{m}{2}\|\dot{\mathbf{q}}\|^2 + \frac{e}{2c}\Big(\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}) + \dot{\theta}\Big)^2 .$$

*The Lagrangian $L$ is positive definite in $(\dot{\mathbf{q}}, \dot{\theta})$; so it may be regarded as the kinetic energy of a metric.*

*(a) Interpret the motion as geodesic.*

*(b) Identify how the Euler-Lagrange equations for this Lagrangian differ from those of the previous part for a charged particle with mass moving in a magnetic field?*

**Answer**

(i) **Fibre derivative**

in this example, we have two fibre derivatives that each give a linear relation

$$\begin{aligned} \mathbf{p} &= \frac{\partial L}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}} + \frac{e}{c}\Big(\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}) + \dot{\theta}\Big)\mathbf{A}(\mathbf{q}) = m\dot{\mathbf{q}} + p_\theta \mathbf{A}(\mathbf{q}) \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \frac{e}{c}\Big(\dot{\mathbf{q}} \cdot \mathbf{A}(\mathbf{q}) + \dot{\theta}\Big) \end{aligned}$$

so this Lagrangian is hyperregular.

(ii) **Euler-Lagrange equations**

$$\begin{aligned} \ddot{\mathbf{q}} &= \frac{p_\theta}{m}\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{with} \quad \mathbf{B}(\mathbf{q}) := \frac{\partial}{\partial \mathbf{q}} \times \mathbf{A}(\mathbf{q}) \\ \frac{dp_\theta}{dt} &= 0 \end{aligned}$$

**This is the same as the previous part, on setting $p_\theta = e/c$.**

(iii) **Hamiltonian and canonical equations**

The Hamiltonian $H$ associated to $L$ by the Legendre transformation for this Lagrangian is

$$\begin{aligned} H(\mathbf{q}, \theta, \mathbf{p}, p_\theta) &= \mathbf{p} \cdot \dot{\mathbf{q}} + p_\theta \dot{\theta} - L(\mathbf{q}, \dot{\mathbf{q}}, \theta, \dot{\theta}) \\ &= \mathbf{p} \cdot \frac{1}{m}(\mathbf{p} - p_\theta \mathbf{A}) + p_\theta(p_\theta - \mathbf{A} \cdot \dot{\mathbf{q}}) \\ &\qquad - \frac{1}{2}m|\dot{\mathbf{q}}|^2 - \frac{1}{2}p_\theta^2 \\ &= \mathbf{p} \cdot \frac{1}{m}(\mathbf{p} - p_\theta \mathbf{A}) + \frac{1}{2}p_\theta^2 \\ &\qquad - p_\theta \mathbf{A} \cdot \frac{1}{m}(\mathbf{p} - p_\theta \mathbf{A}) - \frac{1}{2m}|\mathbf{p} - p_\theta \mathbf{A}|^2 \\ &= \frac{1}{2m}|\mathbf{p} - p_\theta \mathbf{A}|^2 + \frac{1}{2}p_\theta^2 . \end{aligned} \tag{8}$$

**Remarks**

(a) This example provides an easy but fundamental illustration of the geometry of (Lagrangian) reduction by symmetry. The canonical equations for the Hamiltonian $H$ now reproduce Newton's equations for the Lorentz force law, reinterpreted as geodesic motion with respect to the metric defined by the Lagrangian on the tangent bundle $TQ \simeq T\mathbb{R}^3 \times TS^1$.

(b) On the constant level set $p_\theta = e/c$, this Hamiltonian $H$ is a function of only the variables $(\mathbf{q}, \mathbf{p})$ and is equal (up to an additive constant) to the Hamiltonian for charged particle motion under the Lorentz force.

▲

(G) *Consider a Lagrangian containing a penalty that consistent with a constraint imposed by a Lagrange multiplier,* $\pi$.

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z \cdot \mathbf{q} - \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2 + \frac{1}{\epsilon}\pi(\mathbf{q}\cdot\dot{\mathbf{q}})$$

*for a particle with coordinates* $\mathbf{q} \in \mathbb{R}^3$, *constants* $g$, $\epsilon$ *and vertical unit vector* $\mathbf{e}_z$. *Let* $\gamma_\epsilon(t)$ *be the curve in* $\mathbb{R}^3$ *obtained by solving the Euler-Lagrange equations for* $L_\epsilon$ *with the initial conditions* $\mathbf{q}_0 = \gamma_\epsilon(0)$, $\dot{\mathbf{q}}_0 = \dot{\gamma}_\epsilon(0)$.

*Show that*

(a) *In the limit*

$$\lim_{g\to 0, \epsilon\to 0} \gamma_\epsilon(t)$$

*the motion is along is a great circle on the two-sphere* $S^2$, *provided that the initial conditions satisfy* $\|\mathbf{q}_0\|^2 = 1$ *and* $\mathbf{q}_0 \cdot \dot{\mathbf{q}}_0 = 0$.

(b) *For constant* $g > 0$ *the limit*

$$\lim_{\epsilon\to 0} \gamma_\epsilon(t)$$

*recovers the dynamics of a spherical pendulum.*

***Answer***

(i) **Fibre derivative**

The fibre derivative gives a linear relation

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} + \frac{\pi}{2}\,\mathbf{q}$$

which is solvable for the velocity $\dot{\mathbf{q}}(\mathbf{p}, \mathbf{q})$ as $\dot{\mathbf{q}} = \mathbf{p} - \pi\,\mathbf{q}/2$, **but we do not know the Lagrange multiplier,** $\pi$**.**

This means the Lagrangian is not hyperregular.

(ii) **Euler-Lagrange equations**

$$\ddot{\mathbf{q}} = -g\hat{\mathbf{e}}_3 + \frac{1}{\epsilon}(\dot{\pi} - 1 + \|\mathbf{q}\|^2)\mathbf{q}\,. \tag{9}$$

Imposing $\frac{d}{dt}(\mathbf{q}\cdot\dot{\mathbf{q}}) = 0$ yields

$$\frac{1}{\epsilon}(\dot{\pi} + 1 - \|\mathbf{q}\|^2) = \frac{1}{\|\mathbf{q}\|^2}\,(g\hat{\mathbf{e}}_3 \cdot \mathbf{q} - \|\dot{\mathbf{q}}\|^2)$$

which only determines $\pi(t)$ after the motion for $\mathbf{q}(t)$ is already obtained, from,

$$\ddot{\mathbf{q}} = -g\hat{\mathbf{e}}_3 + \frac{1}{\|\mathbf{q}\|^2}\,(g\hat{\mathbf{e}}_3 \cdot \mathbf{q} - \|\dot{\mathbf{q}}\|^2)\mathbf{q}\,. \tag{10}$$

This equation **reduces to spherical pendulum motion when** $\|\mathbf{q}\|^2 = 1$. However, we know that the Lagrange multiplier $\pi$ imposes $d\|\mathbf{q}\|^2/dt = 0$. So if we set $\|\mathbf{q}\|^2 = 1$ initially, it's value will be preserved. Therefore, the rest of the solution of this problem follows the spherical pendulum solution.

(iii) **Hamiltonian and canonical equations**

The corresponding Hamiltonian is obtained by the Legendre transformation as,

$$H(\mathbf{q}, \mathbf{p}) = \tfrac{1}{2}\|\mathbf{p} - \frac{\pi}{\epsilon}\mathbf{q}\|^2 + g\hat{\mathbf{e}}_3 \cdot \mathbf{q} + \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2\,, \tag{11}$$

in which the variable $\mathbf{p}$ is the momentum canonically conjugate to the radial position $\mathbf{q}$. The canonical equations on $(1 - \|\mathbf{q}\|^2) = 0$, are

$$\begin{aligned}\dot{\mathbf{q}} &= \{\mathbf{q}, H\} = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{p} - \frac{\pi}{\epsilon}\mathbf{q} \\ \dot{\mathbf{p}} &= \{\mathbf{p}, H\} = -\frac{\partial H}{\partial \mathbf{q}} = -g\hat{\mathbf{e}}_3 + (\mathbf{p} - \frac{\pi}{\epsilon}\mathbf{q})\frac{\pi}{\epsilon} + \frac{1}{\epsilon}(1 - \|\mathbf{q}\|^2)\mathbf{q}\,.\end{aligned}$$

These equations are equivalent to the spherical pendulum equations for any value of $\epsilon$. Hence, items (a) and (b) above are answered by the spherical pendulum solution.

(iv) **Two penalties, instead of a penalty and a constraint.**
The story would have been different, if we had chosen the Lagrangian as

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z\cdot\mathbf{q} - \frac{1}{4\epsilon}(1 - \|\mathbf{q}\|^2)^2 - \frac{1}{2\epsilon}(\mathbf{q}\cdot\dot{\mathbf{q}})^2\,.$$

In this case, the fibre derivative is

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}} - \frac{1}{\epsilon}\,(\mathbf{q}\cdot\dot{\mathbf{q}})\mathbf{q} = \Big(\mathrm{Id} - \frac{1}{\epsilon}\,\mathbf{q}\otimes\mathbf{q}\Big)\cdot\dot{\mathbf{q}}$$

This relation can NOT be solved for the velocity as $\dot{\mathbf{q}}(\mathbf{p}, \mathbf{q})$ unless $\mathbf{q}\cdot\dot{\mathbf{q}} = 0$, exactly. Consequently, this Lagrangian is not hyperregular. ▲

*(H) How does the motion in the previous part differ from that obtained via Hamilton's principle for the following Lagrangian?*

$$L_\epsilon(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 - g\mathbf{e}_z\cdot\mathbf{q} - \mu(1 - \|\mathbf{q}\|^2)$$

*where $\mu$ is called a **Lagrange multiplier** and must be determined as part of the solution.*

*Answer* See **Exercise 1.6** about the spherical pendulum. ▲

**Exercise 1.4 (Poisson brackets)**

*(A) Show that the canonical Poisson bracket is bilinear, skew symmetric, satisfies the Jacobi identity and acts as a derivation on products of functions in phase space.*

**Answer** This is easy, but the last property requires tedious algebraic manipulation, if one tries to do it directly. It can go much faster, if one first identifies Poisson brackets with Hamiltonian vector fields by

$$X_H = \{\,\cdot\,, H\}$$

and uses the map

$$X_F X_G X_H = -\{F,\, \{G,\, \{H,\, \cdot\,\}\}\},$$

to get some clarity in the notation.

▲

*(B) Given two constants of motion, what does the Jacobi identity imply about additional constants of motion associated with their Poisson bracket?*

**Answer** The Poisson bracket of two constants of motion is another one. ▲

*(C) Compute the Poisson brackets among the $\mathbb{R}^3$-valued functions*

$$J_i = \epsilon_{ijk} q_j p_k$$

*for* $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3$.

**Answer**

$$\{J_i,\, J_j\} = \epsilon_{ijk} J_k$$

▲

*(D) Verify that Hamilton's equations for the function*

$$J^\xi(\mathbf{q}, \mathbf{p}) = \langle J(z), \xi \rangle = \boldsymbol{\xi} \cdot (\mathbf{q} \times \mathbf{p})$$

*with* $z := (\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3$ *and* $\boldsymbol{\xi} \in \mathbb{R}^3$ *give infinitesimal rotations of* $\mathbf{q}$ *and* $\mathbf{p}$ *about the* $\boldsymbol{\xi}$*-axis.*

**Answer** The Hamiltonian vector field for $J^\xi$ is

$$\begin{aligned} X_{J^\xi} := \{\,\cdot\,, J^\xi\} &= \frac{\partial J^\xi}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{q}} - \frac{\partial J^\xi}{\partial \mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{p}} \\ &= \boldsymbol{\xi} \times \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{q}} + \boldsymbol{\xi} \times \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}}, \end{aligned}$$

whose coefficients are the infinitesimal rotations of $\mathbf{q}$ and $\mathbf{p}$ about the $\boldsymbol{\xi}$-axis. ▲

### 1.4a Poisson brackets – revised question

(a) Show that the canonical Poisson bracket is bilinear, skew symmetric, satisfies the Jacobi identity and acts as a derivation on products of functions in phase space.

**Answer** As before, or as a direct calculation. ▲

(b) Given two constants of motion, what does the Jacobi identity imply about additional constants of motion associated with their Poisson bracket?

**Answer**

This is the Bruns theorem: Poisson brackets of constants of motion are conserved. ▲

(c) Compute the Poisson brackets among the $\mathbb{R}^3$-valued functions of $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^3$

$$J_i = \epsilon_{ijk} p_j q_k \quad \text{or} \quad \mathbf{J} = \mathbf{p} \times \mathbf{q} \quad \text{in vector notation.}$$

**Answer**

$$\{J_i,\, J_j\} = -\,\epsilon_{ijk} J_k\,.$$

▲

(d) Answer the following questions about these Poisson brackets.

(i) Do the Poisson brackets $\{J_l, J_m\}$ close among themselves?

**Answer** Yes, namely $\{J_i,\, J_j\} = -\,\epsilon_{ijk} J_k$. ▲

(ii) Write the Poisson bracket $\{F(\mathbf{J}), H(\mathbf{J})\}$ for the restriction to functions of $\mathbf{J} = (J_1, J_2, J_3)$.

**Answer**

$$\{F(\mathbf{J}), H(\mathbf{J})\} = -\,\mathbf{J} \cdot \frac{\partial F}{\partial \mathbf{J}} \times \frac{\partial H}{\partial \mathbf{J}}\,.$$

▲

(iii) Write in vector notation the equation $\dot{\mathbf{J}} = \{\mathbf{J}, H(\mathbf{J})\}$ for any Hamiltonian function $H(\mathbf{J})$.

**Answer**

$$\dot{\mathbf{J}} = \{\mathbf{J}, H(\mathbf{J})\} = \mathbf{J} \times \frac{\partial H}{\partial \mathbf{J}}$$

▲

(iv) Compute the dynamical equation for the Hamiltonian function

$$H(\mathbf{J}) = J^{\xi} = \boldsymbol{\xi} \cdot \mathbf{J}$$

for any vector $\boldsymbol{\xi} \in \mathbb{R}^3$. Interpret the solutions for this flow geometrically.

**Answer**

$$\dot{\mathbf{J}} = \{\mathbf{J}, H(\mathbf{J})\} = \mathbf{J} \times \frac{\partial H}{\partial \mathbf{J}} = -\,\boldsymbol{\xi} \times \mathbf{J}$$

This is *clockwise* rotation of vector $\mathbf{J}$ at angular frequency $\boldsymbol{\xi}$. Since $\mathbf{J} = \mathbf{p} \times \mathbf{q}$, the vectors $\mathbf{p}$ and $\mathbf{q}$ rotate the same way. That is, $\dot{\mathbf{p}} = -\,\boldsymbol{\xi} \times \mathbf{p}$ and $\dot{\mathbf{q}} = -\,\boldsymbol{\xi} \times \mathbf{q}$. ▲

**Exercise 1.5 (Nambu Poisson brackets on $\mathbb{R}^3$)**

(a) *Show that for smooth functions $c, f, h : \mathbb{R}^3 \to \mathbb{R}$, the $\mathbb{R}^3$-bracket defined by*

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h$$

*satisfies the defining properties of a Poisson bracket. Is it also a derivation satisfying the Leibnitz relation for a product of functions on $\mathbb{R}^3$? If so, why?*

**Answer** The $\mathbb{R}^3$-bracket is plainly a skew-symmetric bilinear Leibniz operator. Its Hamiltonian vector fields are divergence free vector fields in $\mathbb{R}^3$. These vector fields in $\mathbb{R}^3$ satisfy the Jacobi identity under commutation. The identification of the $\mathbb{R}^3$-bracket with its Hamiltonian vector fields shows that it satisfies Jacobi. This will be made clearer below. ▲

(b) *How is the $\mathbb{R}^3$-bracket related to the canonical Poisson bracket in the plane?*

**Answer** The canonical Poisson bracket in the $(x, y)$-plane is given by the particular choice of the $\mathbb{R}^3$-bracket

$$\{f, h\} = -\nabla z \cdot \nabla f \times \nabla h$$

▲

(c) *The Casimirs (or distinguished functions, as Lie called them) of a Poisson bracket satisfy*

$$\{c, h\}(\mathbf{x}) = 0\,, \quad \textit{for all} \quad h(\mathbf{x})$$

*Part 5 verifies that the $\mathbb{R}^3$-bracket satisfies the defining properties of a Poisson bracket. What are the Casimirs for the $\mathbb{R}^3$ bracket?*

**Answer** Smooth functions of $c$ are Casimirs for the $\mathbb{R}^3$-bracket given by

$$\{f, h\} = -\nabla c \cdot \nabla f \times \nabla h.$$

▲

(d) *Write the motion equation for the $\mathbb{R}^3$-bracket*

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\}$$

*in vector form using gradients and cross products. Show that the corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence.*

**Answer**

$$\dot{\mathbf{x}} = \{\mathbf{x}, h\} = \nabla c \times \nabla h$$

The corresponding Hamiltonian vector field $X_h = \{\,\cdot\,, h\}$ has zero divergence because the vector $\nabla c \times \nabla h$ has zero divergence. (It's a curl.) ▲

(e) *Show that under the $\mathbb{R}^3$-bracket, the Hamiltonian vector fields $X_f = \{\,\cdot\,, f\}$, $X_h = \{\,\cdot\,, h\}$ satisfy the following anti-homomorphism that relates the commutation of vector fields to the $\mathbb{R}^3$-bracket operation between smooth functions on $\mathbb{R}^3$,*

$$[X_f, X_h] = -X_{\{f,h\}}.$$

*Hint: commutation of divergenceless vector fields does satisfy the Jacobi identity.*

***Answer*** **Lemma.** The $\mathbb{R}^3$-bracket defined on smooth functions $(C, F, H)$ by

$$\{F, H\} = -\nabla C \cdot \nabla F \times \nabla H$$

may be identified with the divergenceless vector fields by

$$[X_G, X_H] = -X_{\{G,H\}}\,, \tag{12}$$

where $[X_G, X_H]$ is the Jacobi-Lie bracket of vector fields $X_G$ and $X_H$.
**Proof.** Equation (12) may be verified by a direct calculation,

$$\begin{aligned} [X_G, X_H] &= X_G X_H - X_H X_G \\ &= \{G,\,\cdot\,\}\{H,\,\cdot\,\} - \{H,\,\cdot\,\}\{G,\,\cdot\,\} \\ &= \{G,\,\{H,\,\cdot\,\}\} - \{H,\,\{G,\,\cdot\,\}\} \\ &= \{\{G,\,H\},\,\cdot\,\} = -X_{\{G,H\}}\,. \end{aligned}$$

■

**Remark.** The last step in the proof of the Lemma uses the Jacobi identity for the $\mathbb{R}^3$-bracket, which follows from the Jacobi identity for divergenceless vector fields, since

$$X_F X_G X_H = -\{F,\,\{G,\,\{H,\,\cdot\,\}\}\}$$

▲

*(f) Show that the motion equation for the $\mathbb{R}^3$-bracket is invariant under a certain linear combination of the functions $c$ and $h$. Interpret this invariance geometrically.*

***Answer***

$$\nabla(\alpha c + \beta h) \times \nabla(\gamma c + \epsilon h) = \nabla c \times \nabla h \quad \text{for constants satisfying} \quad \alpha\epsilon - \beta\gamma = 1.$$

Under such a (volume-preserving) transformation, the level sets change, but their intersections remain invariant.

▲

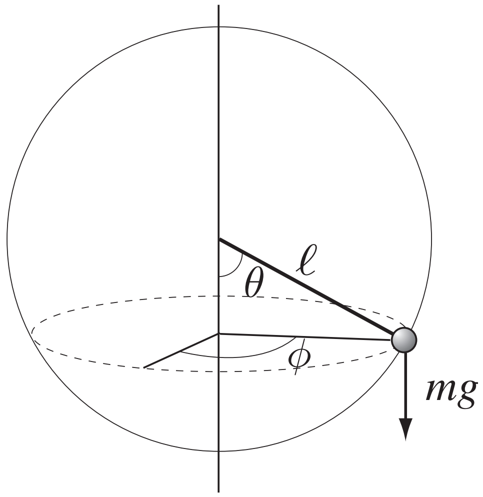


Figure 1: Spherical pendulum: $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\theta$, $z = -R\cos\theta$.

**Exercise 1.6 (Spherical pendulum)**

*A spherical pendulum of length L swings from a fixed point of support under the constant downward force of gravity mg.*

*Use spherical coordinates with azimuthal angle $0 \le \phi < 2\pi$ and polar angle $0 \le \theta < \pi$ measured from the* ***downward*** *vertical defined in terms of Cartesian coordinates by (note minus sign in z)*

*(A) Find its equations of motion according to the approaches of*

- *(a) Newton,*
- *(b) Lagrange and*
- *(c) Hamilton.*

**Answer** See Appendix A, Section A.1.2 of the text.

**The Lagrangian approach**

We use spherical coordinates with azimuthal angle $0 \le \phi < 2\pi$ and polar angle $0 \le \theta < \pi$ measured from the downward vertical defined in terms of Cartesian coordinates by (note minus sign in $z$)

$$
\begin{aligned}
x &= R\sin\theta\cos\phi\,,\\
y &= R\sin\theta\sin\theta\,,\\
z &= -R\cos\theta\,.
\end{aligned}
$$

**Kinetic energy** In Cartesian coordinates, the kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,.$$

Upon translating into spherical polar coordinates, the velocity components become

$$
\begin{aligned}
\dot{x} &= R\dot{\theta}\cos\theta\cos\phi - R\dot{\phi}\sin\theta\sin\phi\,,\\
\dot{y} &= R\dot{\theta}\cos\theta\sin\phi - R\dot{\phi}\sin\theta\cos\phi\,,\\
\dot{z} &= R\dot{\theta}\sin\theta\,,
\end{aligned}
$$

and the kinetic energy becomes

$$T = \frac{mR^2}{2}(\dot\theta^2 + \dot\phi^2 \sin^2\theta)\,.$$

This is the standard form for the particle kinetic energy in spherical coordinates.

**Potential energy** The potential energy of the spherical pendulum is

$$V = mgz = -\,mgR\cos\theta\,.$$

**Lagrangian** Its Lagrangian is similar to Equation (**??**) for the rotating hoop:

$$L = T - V = \frac{mR^2}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) + mgR\cos\theta\,.$$

$\theta$ **equation:** The Euler–Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0\,,$$

in which for the spherical pendulum Lagrangian,

$$\begin{aligned}
\frac{\partial L}{\partial\dot\theta} &= mR^2\dot\theta\,,\\
\frac{\partial L}{\partial\theta} &= \dot\phi^2 mR^2\sin\theta\cos\theta - mgR\sin\theta\,.
\end{aligned}$$

Consequently, one finds the motion equation

$$mR^2\ddot\theta - \dot\phi^2 mR^2\sin\theta\cos\theta + mgR\sin\theta = 0\,.$$

$\phi$ **equation:** The Euler–Lagrange equation in $\phi$ is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} - \frac{\partial L}{\partial\phi} = 0\,.$$

Consequently, one computes that

$$\begin{aligned}
\frac{\partial L}{\partial\dot\phi} &= mR^2\dot\phi\sin^2\theta\,,\\
\frac{\partial L}{\partial\phi} &= 0\,,\\
\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} &= \frac{d}{dt}(mR^2\dot\phi\sin^2\theta) = 0\,.
\end{aligned}$$

Thus, as guaranteed by Noether's theorem, azimuthal symmetry of the Lagrangian (that is, $L$ being independent of $\phi$) implies conservation of the azimuthal angular momentum.

### The Hamiltonian approach

One computes the canonical momenta and solves for velocities in terms of momenta and coordinates as

$$\begin{aligned}
P_\theta &= \frac{\partial L}{\partial\dot\theta} = mR^2\dot\theta\,, \qquad \text{so} \qquad \dot\theta = \frac{P_\theta}{mR^2}\,,\\
P_\phi &= \frac{\partial L}{\partial\dot\phi} = mR^2\dot\phi\sin^2\theta\,, \qquad \text{so} \qquad \dot\phi = \frac{P_\phi}{mR^2\sin^2\theta}\,.
\end{aligned}$$

The Hamiltonian is obtained by Legendre-transforming the Lagrangian as

$$
\begin{aligned}
H &= P_\theta \dot\theta + P_\phi \dot\phi - L \\
&= \frac{P_\theta^2}{2mR^2} + \frac{P_\phi^2}{2mR^2 \sin^2\theta} - mgl\cos\theta\,.
\end{aligned}
$$

This Hamiltonian has canonical motion equations,

$$
\begin{aligned}
\dot P_\theta &= -\frac{\partial H}{\partial \theta} = -\,mgl\sin\theta + \frac{P_\phi^2}{mR^2}\frac{\cos\theta}{\sin^3\theta}\,, \\
\dot P_\phi &= -\frac{\partial H}{\partial \phi} = 0\,.
\end{aligned}
$$

The angular frequencies are recovered in their canonical form from the Hamiltonian as

$$
\begin{aligned}
\dot\theta &= \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mR^2}\,, \\
\dot\phi &= \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mR^2\sin^2\theta}\,.
\end{aligned}
$$

By Noether's theorem, the azimuthal angular momentum $P_\phi$ is conserved because the Lagrangian (and hence the Hamiltonian) of the spherical pendulum are independent of $\phi$. This symmetry also allows further progress toward characterising the spherical pendular motion. In particular, the equilibria are azimuthally symmetric.

Substituting for $\dot\phi^2$ in Equation (A) from (13) yields

$$
mR^2\ddot\theta = -\,mgR\sin\theta + \Big(\frac{P_\phi}{mR^2\sin^2\theta}\Big)^2 mR^2\sin\theta\cos\theta\,.
$$

This may be rewritten in terms of an effective potential $V_{eff}(\theta)$ as

$$
\ddot\theta = -\,\frac{\partial V_{eff}(\theta)}{\partial\theta}\,,
$$

with

$$
V_{eff}(\theta) = -\,(g/R)\cos\theta + \frac{P_\phi^2}{2mR^2\sin^2\theta}\,.
$$

This approach enables a phase-plane analysis in $(\theta,\, P_\theta)$. Combining this with conservation of energy defined as

$$
E/(mR^2) = \dot\theta^2/2 + V_{eff}(\theta)
$$

▲

*(B) Write the constrained Lagrangian for the* $L(\mathbf{x},\dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ *as*

$$
L(\mathbf{x},\dot{\mathbf{x}}) = \tfrac12|\dot{\mathbf{x}}|^2 - g\hat{\mathbf{e}}_3\cdot\mathbf{x} - \tfrac12\mu(1-|\mathbf{x}|^2), \tag{13}
$$

*in which the Lagrange multiplier* $\mu$ *constrains the motion to remain on the sphere* $S^2$ *by enforcing* $(1-|\mathbf{x}|^2)=0$ *when it is varied in Hamilton's principle.*

*(a) Compute the variations in Hamilton's principle and write the Euler-Lagrange equations for the spherical pendulum on* $T\mathbb{R}^3$*.*

*(b) Solve for the Lagrange multiplier by requiring that* $TS^2$ *is preserved by this motion on* $T\mathbb{R}^3$*.*

**Answer**

(a) The corresponding Euler-Lagrange equations are

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \mu\mathbf{x}\,. \tag{14}$$

(b) This equation preserves *both* of the $TS^2$ defining relations $1-|\mathbf{x}|^2 = 0$ and $\mathbf{x}\cdot\dot{\mathbf{x}} = 0$, provided the Lagrange multiplier is given by

$$\mu = g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\dot{\mathbf{x}}|^2\,. \tag{15}$$

**Proof.** Compute time derivatives of $1 - |\mathbf{x}|^2 = 0$ and $\mathbf{x}\cdot\dot{\mathbf{x}} = 0$ and substitute the motion equation (14) into the latter and solve for $\mu$ given by (15).

**Remark** Regroup (14) using (15) for $\mu$ to find an interpretation of the forces

$$\ddot{\mathbf{x}} = -g\,\underbrace{\Big(\hat{\mathbf{e}}_3 - (\hat{\mathbf{e}}_3\cdot\mathbf{x})\mathbf{x}\Big)}_{\text{Projects }\hat{\mathbf{e}}_3\text{ on }TS^1} \quad + \quad \underbrace{(-|\dot{\mathbf{x}}|^2\mathbf{x})}_{\text{Centripetal force}}\quad, \tag{16}$$

where $(\hat{\mathbf{e}}_3 - (\hat{\mathbf{e}}_3\cdot\mathbf{x})\mathbf{x}) = \hat{\mathbf{e}}_3\cdot(\mathrm{Id} - \mathbf{x}\otimes\mathbf{x})$ and $(\mathrm{Id} - \mathbf{x}\otimes\mathbf{x})\cdot\mathbf{x} = (1-|\mathbf{x}|^2)\mathbf{x} = 0$, because of the constraint.

■

▲

*(C) Find the Hamiltonian and its canonical equations.*

**Answer** The fibre derivative of the constrained Lagrangian $L$ in (13) is

$$\mathbf{y} = \frac{\partial L}{\partial\dot{\mathbf{x}}} = \dot{\mathbf{x}}\,. \tag{17}$$

The corresponding Hamiltonian is obtained by the Legendre transformation as,

$$H(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}|\mathbf{y}|^2 + g\hat{\mathbf{e}}_3\cdot\mathbf{x} + \tfrac{1}{2}(g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)(1-|\mathbf{x}|^2)\,, \tag{18}$$

in which the variable $\mathbf{y}$ is the momentum canonically conjugate to the radial position $\mathbf{x}$. The canonical equations on $(1-|\mathbf{x}|^2) = 0$, are

$$\dot{\mathbf{x}} = \{\mathbf{x}, H\} = \frac{\partial H}{\partial\mathbf{y}} = \mathbf{y} \quad\text{and}\quad \dot{\mathbf{y}} = \{\mathbf{y}, H\} = -\frac{\partial H}{\partial\mathbf{x}} = -g\hat{\mathbf{e}}_3 + (g\hat{\mathbf{e}}_3\cdot\mathbf{x} - |\mathbf{y}|^2)\mathbf{x}\,. \tag{19}$$

▲

*(D) A convenient choice of basis for the algebra of polynomials in $(\mathbf{x},\mathbf{y})$ that are $S^1$-invariant under rotations about the 3-axis is given by*

$$\begin{aligned} \sigma_1 &= x_3 & \sigma_3 &= y_1^2 + y_2^2 + y_3^2 & \sigma_5 &= x_1y_1 + x_2y_2 \\ \sigma_2 &= y_3 & \sigma_4 &= x_1^2 + x_2^2, & \sigma_6 &= x_1y_2 - x_2y_1 \end{aligned}$$

*(a) Find the cubic relation that these $S^1$-invariants satisfy and express the defining relations for $TS^2$ in terms of them.*

*(b) Use these relations to eliminate $\sigma_4$ and $\sigma_5$ in favour of $\{\sigma_1, \sigma_2, \sigma_3, \sigma_6\}$ and find the cubic relation satisfied among $\{\sigma_1, \sigma_2, \sigma_3, \sigma_6\}$.*

**Answer**

(a) These six $S^1$-invariants satisfy the cubic algebraic relation

$$\sigma_5^2 + \sigma_6^2 = \sigma_4(\sigma_3 - \sigma_2^2)\,. \tag{20}$$

Hence, they also satisfy the positivity conditions

$$\sigma_4 \geq 0, \quad \sigma_3 \geq \sigma_2^2. \tag{21}$$

In these variables, the defining relations for $TS^2$ become

$$\sigma_4 + \sigma_1^2 = 1 \quad \text{and} \quad \sigma_5 + \sigma_1\sigma_2 = 0\,. \tag{22}$$

(b) Using the relations in (22) to eliminate $\sigma_4$ and $\sigma_5$ from (20) yields the cubic relation

$$C(\sigma_1,\sigma_2,\sigma_3,\sigma_6) = \sigma_2^2 + \sigma_6^2 - \sigma_3(1-\sigma_1^2) = 0 \tag{23}$$

Thus, the motion takes place on the following family of surfaces depending on $(\sigma_1,\sigma_2,\sigma_3) \in \mathbb{R}^3$ and parameterised by the conserved value of $\sigma_6^2 = J_3^2$,

$$\sigma_3 = \frac{\sigma_2^2 + J_3^2}{1-\sigma_1^2}\,. \tag{24}$$

Vertical planar slices through the surface $C = 0$ are parabolic at $\sigma_1 = 0$. They are U-shaped at $\sigma_2 = 0$ and diverging at $\sigma_1^2 = 1$, unless $\sigma_6 = 0 = J_3$, in which case the spherical pendulum swings in a single plane.

▲

(E) *(a) Find the Poisson bracket relations among the remaining quadratic invariant variables* $\{\sigma_1,\sigma_2,\sigma_3,\sigma_6\}$

*(b) Explain how this Poisson bracket is related to the* $\mathbb{R}^3$*-bracket.*

**Answer**

(a) We begin by computing the Poisson bracket relations among the $\sigma$'s from their definitions in terms of the canonically conjugate variables $(\mathbf{x},\mathbf{y})$, *before* we insert the relations (22).

| $\{\cdot,\cdot\}$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ |
|---|---|---|---|---|---|---|
| $\sigma_1$ | 0 | 1 | $2\sigma_2$ | 0 | 0 | 0 |
| $\sigma_2$ | $-1$ | 0 | 0 | 0 | 0 | 0 |
| $\sigma_3$ | $-2\sigma_2$ | 0 | 0 | $-2\sigma_5$ | $-2(\sigma_3-\sigma_2^2)$ | 0 |
| $\sigma_4$ | 0 | 0 | $2\sigma_5$ | 0 | $2\sigma_4$ | 0 |
| $\sigma_5$ | 0 | 0 | $2(\sigma_3-\sigma_2^2)$ | $-2\sigma_4$ | 0 | 0 |
| $\sigma_6$ | 0 | 0 | 0 | 0 | 0 | 0 |

However, *after* the relations in (22) have been used to write $\sigma_4$ and $\sigma_5$ as functions of $\sigma_1$ and $\sigma_2$, the remaining variables are no longer independent, and the Poisson bracket relations among the quadratic invariant variables $\{\sigma_1,\sigma_2,\sigma_3,\sigma_6\}$ must preserve the $C = 0$ family of surfaces $\sigma_3 = F(\sigma_1,\sigma_2)$ parameterised by constant $J_3$ in equation (24). Consequently, up to a constant factor (that may be absorbed into the unit of time, $t$) the Poisson bracket must be given by

$$\{\sigma_i,\,\sigma_j\} = -\,\epsilon_{ijk}\frac{\partial C}{\partial \sigma_k}\,. \tag{25}$$

In components, this Poisson bracket is expressed as

| $\{\cdot,\cdot\}$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_6$ |
|---|---|---|---|---|
| $\sigma_1$ | 0 | $1-\sigma_1^2$ | $2\sigma_2$ | 0 |
| $\sigma_2$ | $-1+\sigma_1^2$ | 0 | $-2\sigma_1\sigma_3$ | 0 |
| $\sigma_3$ | $-2\sigma_2$ | $2\sigma_1\sigma_3$ | 0 | 0 |
| $\sigma_6$ | 0 | 0 | 0 | 0 |

(b) The Poisson bracket amongst $\{\sigma_1, \sigma_2, \sigma_3\}$ defines an $\mathbb{R}^3$-bracket. (This part of the question was just a clue to align people's thinking for the rest.)

▲

*(F) Write their dynamics on $TS^2$ in Hamiltonian form.*

**Answer** In Hamiltonian form the dynamics on $TS^2$ (which is preserved by the motion) simplifies because the spherical pendulum Hamiltonian in (18) becomes linear in the $S^1$-invariants

$$H|_{TS^2} = \tfrac{1}{2}\sigma_3 + g\sigma_1\,. \tag{26}$$

Hence the dynamics becomes

$$\dot\sigma_i = \{\sigma_i, H\} = \epsilon_{ijk}\frac{\partial C}{\partial \sigma_j}\frac{\partial H}{\partial \sigma_k}$$

or explicitly,

$$\dot\sigma_1 = \{\sigma_1, H\} = -\sigma_2\,, \qquad \dot\sigma_2 = \{\sigma_2, H\} = \sigma_1\sigma_3 + g(1-\sigma_1^2)\,, \qquad \dot\sigma_3 = \{\sigma_3, H\} = 2g\sigma_2\,,$$

and $\dot\sigma_6 = \{\sigma_6, H\} = 0$ because the Poisson bracket with $\sigma_6$ vanishes with all the other $S^1$-invariants. If in the previous equations of motion we restrict to a constant level surface of $H|_{TS^2}$, then $\sigma_3$ may be eliminated from the equations in favour of $H|_{TS^2}$ and we find

$$\dot\sigma_1 = \{\sigma_1, H\} = -\sigma_2\,, \qquad \dot\sigma_2 = \sigma_1(2H|_{TS^2} - g\sigma_1) + g(1-\sigma_1^2)$$

Hence,

$$\ddot\sigma_1 = -\dot\sigma_2 = -(2H|_{TS^2})\sigma_1 + 2g\sigma_1^2 - g = -\frac{d}{d\sigma_1}\left(H|_{TS^2}\,\sigma_1^2 - \frac{3}{2}g\sigma_1^3 + g\sigma_1\right)$$

This is the particle in the cubic potential whose phase plane orbits are shaped like a fish swimming to the left. ▲

**Exercise 1.7 (The Hopf map)**

*In coordinates* $(a_1, a_2) \in \mathbb{C}^2$, *the Hopf map* $\mathbb{C}^2/S^1 \to S^3 \to S^2$ *is obtained by transforming to the four quadratic* $S^1$*-invariant quantities*

$$(a_1, a_2) \to Q_{jk} = a_j a_k^* \,, \quad \textit{with} \quad j,k = 1,2\,.$$

*Let the* $\mathbb{C}^2$ *coordinates be expressed as*

$$a_j = q_j + ip_j$$

*in terms of canonically conjugate variables satisfying the fundamental Poisson brackets*

$$\{q_k, p_m\} = \delta_{km} \quad \textit{with} \quad k, m = 1, 2.$$

*(A) Compute the Poisson brackets* $\{a_j, a_k^*\}$ *for* $j,k = 1,2$.

**Answer** The $\mathbb{C}^2$ coordinates $a_j = q_j + ip_j$ satisfy the Poisson bracket

$$\{a_j \,,\, a_k^*\} = -2i\,\delta_{jk}\,, \quad \text{for } j,k = 1,2\,.$$

▲

*(B) Is the transformation* $(q,p) \to (a, a^*)$ *canonical? Explain why or why not.*

**Answer** The transformation $(q,p) \mapsto (a, a^*)$ is indeed canonical. The constant $(-2i)$ is inessential for Hamiltonian dynamics, because it can be absorbed into the definition of time.

▲

*(C) Compute the Poisson brackets among* $Q_{jk}$, *with* $j,k = 1,2$.

**Answer** The quadratic $S^1$ invariants on $\mathbb{C}^2$ given by $Q_{jk} = a_j a_k^*$ satisfy the Poisson bracket relations,

$$\{Q_{jk} \,,\, Q_{lm}\} \;=\; 2i\,(\delta_{kl}Q_{jm} - \delta_{jm}Q_{kl})\,, \quad j,k,l,m = 1,2\,.$$

Thus, they do close among themselves, but they do not satisfy canonical Poisson bracket relations.

▲

*(D) Make the linear change of variables,*

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = 2Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

*and compute the Poisson brackets among* $(X_0, X_1, X_2, X_3)$.

**Answer** The quadratic $S^1$ invariants $(X_0, X_1, X_2, X_3)$ given by

$$X_0 = Q_{11} + Q_{22}\,, \quad X_1 + iX_2 = 2Q_{12}\,, \quad X_3 = Q_{11} - Q_{22}\,,$$

may be expressed in terms of the $a_j$, $j = 1,2$ as

$$X_0^2 = |a_1|^2 + |a_2|^2\,, \quad X_1 + iX_2 = 2a_1 a_2^*\,, \quad X_3 = |a_1|^2 - |a_2|^2\,.$$

These satisfy the Poisson bracket relations,

$$\{X_0 \,,\, X_k\} = 0\,, \quad \{X_j \,,\, X_k\} = -\,\epsilon_{jkl} X_l$$

▲

(E) *Express the Poisson bracket* $\{F(\mathbf{X}), H(\mathbf{X})\}$ *in vector form among functions* $F$ *and* $H$ *of* $\mathbf{X} = (X_1, X_2, X_3)$.

**Answer** The Poisson bracket $\{F(\mathbf{X}), H(\mathbf{X})\}$ is given in vector form as

$$\{F(\mathbf{X}), H(\mathbf{X})\} = -\,\mathbf{X}\cdot\frac{\partial F}{\partial \mathbf{X}}\times\frac{\partial H}{\partial \mathbf{X}}\,.$$

It's the same as the Poisson bracket for the rigid body.

▲

(F) *Show that the quadratic invariants* $(X_0, X_1, X_2, X_3)$ *themselves satisfy a quadratic relation. How is this relevant to the Hopf map?*

**Answer** The quadratic invariants $(X_0, X_1, X_2, X_3)$ satisfy the quadratic relation

$$X_0^2(\mathbf{X}) = X_1^2 + X_2^2 + X_3^2 = |\mathbf{X}|^2\,.$$

This relation is relevant. It completes the Hopf map, because level sets of $X_0$ are spheres $S^2 \in S^3$. Moreover, it is relevant to the Poisson bracket in vector form above, which may be written using this relation as

$$\{F(\mathbf{X}), H(\mathbf{X})\} = -\,\frac{1}{2}\frac{\partial X_0^2}{\partial \mathbf{X}}\cdot\frac{\partial F}{\partial \mathbf{X}}\times\frac{\partial H}{\partial \mathbf{X}}\,.$$

▲