Stochastic advection by Lie transport (SALT)

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Outline

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Thinking lately about . . . Poincaré and Lichnerowicz

For Poincaré, the definition of a mathematical entity is the construction of the entity itself and not an expression of an underlying essence or existence.
This is to say that no mathematical object exists without human construction of it, both in mind and language.

https://en.wikipedia.org/wiki/Pre-intuitionism

You cannot apply mathematics you do not know.

– André Lichnerowicz (via Michael Ghil, 18 Sept 2019)

Reminder: What is advection, mathematically?

As in [Arnold1966], we consider Lagrangian trajectories as curves on M generated by the action $x_t = \phi_t(x)$ of diffeomorphisms ϕ_t parameterised by time t such that $x = \phi_0(x)$ at time $t = 0$; that is, $\phi_0 = Id$.

The *velocity* along the curve is defined as $\frac{d}{dt}\phi_t(x) =: u(t, \phi_t(x))$.

Smooth k-form $K(t,x)$ is *advected*, if *pull-back* $\phi_t^* K(t,x) := K(t, \phi_t(x))$ satisfies the following formula.

Definition (Deterministic Advection by Lie Transport (DALT))

$$\frac{d}{dt}(\phi_t^* K)(t,x) := \frac{d}{dt} K\big(t, \phi_t(x)\big) = \phi_t^*\Big(\partial_t K(t,x) + \mathcal{L}_u K(t,x)\Big) = 0\,.$$

Definition (Lie derivative)

$$\frac{d}{dt}\Big|_{t=0}(\phi_t^* K)(x) := \frac{d}{dt}\Big|_{t=0} K(\phi_t(x)) =: \mathcal{L}_u K(x) \text{ with } \frac{d\phi_t(x)}{dt}\Big|_{t=0} = u(x)$$

Thus, "advection" means "viewing conservation on particles from a fixed frame"; and doing so is governed by the Lie derivative.

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Examples of Deterministic Advection by Lie Transport

Definition (Lie derivative)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\phi_t^* K)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} K(\phi_t(x)) =: \mathcal{L}_u K(x)$$

$$\text{with } \frac{\mathrm{d}\phi_t(x)}{\mathrm{d}t}\Big|_{t=0} = u(x).$$

Example (Familiar examples from fluid dynamics:)

(Functions) $(\partial_t + \mathcal{L}_u) b(\mathbf{x}) = \partial_t b + \mathbf{u}\cdot\nabla b\,,$

(1-forms) $(\partial_t + \mathcal{L}_u)(\mathbf{v}(\mathbf{x})\cdot d\mathbf{x}) = \big(\partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\,\mathbf{v} + v_j \nabla u^j\big)\cdot d\mathbf{x}$
$= \big(\partial_t \mathbf{v} - \mathbf{u}\times \mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,,$

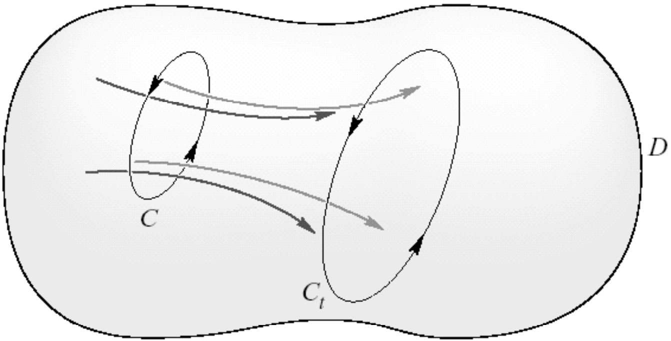
(2-forms) $(\partial_t + \mathcal{L}_u)(\boldsymbol{\omega}(\mathbf{x})\cdot d\mathbf{S}) = \big(\partial_t \boldsymbol{\omega} - \mathrm{curl}\,(\mathbf{u}\times\boldsymbol{\omega}) + \mathbf{u}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}\,,$

(3-forms) $(\partial_t + \mathcal{L}_u)(\rho(\mathbf{x})\,d^{\,3}x) = (\partial_t \rho + \mathrm{div}\,\rho\,\mathbf{u})\,d^{\,3}x\,.$

Advection in Kelvin's Circulation Theorem

The **deterministic** Kelvin circulation theorem follows from Newton's law for the evolution of momentum/mass $\mathbf{v}$ concentrated on an **advecting material loop**, $c_t = \phi_t c_0$ at velocity $\mathbf{u}$,

$$\frac{d}{dt}\oint_{c_t} \mathbf{v}\cdot d\mathbf{x} = \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$



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Reminder: Proof of the deterministic Kelvin's theorem

Proof.

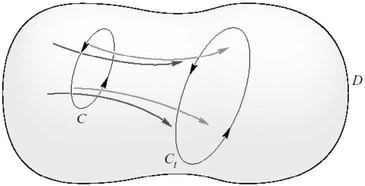
Consider a closed loop moving with the material flow $c_t = \phi_t c_0$ with Eulerian velocity $\frac{d}{dt}\phi_t(x) = \phi_t^* u(t,x) = u(t, \phi_t(x))$.

Compute the time derivative of the loop momentum/mass

$$
\begin{aligned}
\frac{d}{dt}\oint_{c_t} \mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x} &= \oint_{c_0} \frac{d}{dt}\Big(\phi_t^*\big(\mathbf{v}(t,\mathbf{x})\cdot d\mathbf{x}\big)\Big) \\
&= \oint_{c_0} \underbrace{\phi_t^*\Big(\big(\partial_t + \mathcal{L}_{u(t,\mathbf{x})}\big)(\mathbf{v}\cdot d\mathbf{x})\Big)}_{\text{Defines Lie derivative via product rule}} \\
&= \oint_{\phi_t c_0 = c_t} \big(\partial_t + \mathcal{L}_{u(t,\mathbf{x})}\big)(\mathbf{v}\cdot d\mathbf{x}) \\
&= \oint_{c_t} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}} = \oint_{c_0} \phi_t^*\Big(\mathbf{f}\cdot d\mathbf{x}\Big)
\end{aligned}
$$

□

What would a stochastic Kelvin's theorem look like?



Q1: Would noise cause circulation in a fluid loop?

Q2: What do you mean by circulation?

A2: As usual, circulation means, "integral of the momentum per unit mass (a 1-form) around a closed loop moving with the fluid velocity".

A1: Ah! Circulation would still defined by the same formula, but now the loop would be moving with the fluid along a stochastic Lagrangian path?

Q3: Why would the loop stay together?

A3: Because the flow map in both cases preserves neighbours!

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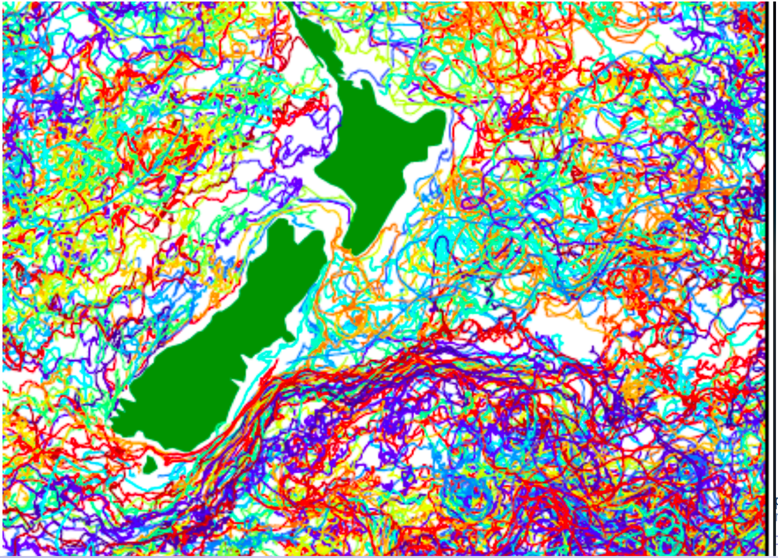
Intuition solves problems by envisioning the solution. What would a stochastic Lagrangian trajectory look like?



Path to the SALT algorithm for stochastic parameterisation

1. Hamilton's principle, constrained by a proposed stochastic decomposition of transport velocity implied an Euler fluid SPDE, in the Kelvin theorem form of Newton's law $\boxed{\surd}$ (Holm PRSA 2015)
2. After a slow-fast decomposition of the full flow map, multi-time homogenization was used to *derive* the stochastic decomposition

$$\tilde{u}(x_t, t) := \mathbf{d}x_t = u(x_t, t)dt + \sum_I \xi_i(x_t) \circ dW_t^i \,,$$

   (Holm 2015) proposed cf. (Cotter Gottwald Holm PRSA 2017) $\boxed{\surd}$
3. The SALT algorithm for determining the $\sum_I \xi_i(x_t)$ in this stochastic decomposition was developed for data assimilation in collaboration with Wei Pan, Igor Shevchenko, Colin Cotter and Dan Crisan. $\boxed{\surd}$ See arXiv:1802.05711, arXiv:1801.09729.

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What theoretical steps took us to the SALT algorithm?

Variational principle for stochastic fluid equations Holm PSRA 2015 $\boxed{\surd}$

Multi-time homogenization of slow-fast decomposition Cotter Gottwald Holm PRSA 2017 $\boxed{}$

$\longrightarrow$

Stochastic decomposition of transport velocity

(For data assimilation, $\sum_I \xi_i(x_t)$ was the key issue) $\boxed{}$

$+$

Stochastic Kelvin circulation theorem $\boxed{}$

Fluid motion equations from Hamilton's principle. $\boxed{}$

$\longrightarrow$

SALT Eulerian SPDEs with stochastic transport

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Multi-time fast-slow homogenization (CGH2017) Stochastic decomposition and the derivation of SALT flow

Write the fluid flow map for Lagrangian parcels $x_t := g_t x_0 \in \mathcal{D}$ with $g_0 x_0 = x_0$ as the composition of two time-dependent diffeomorphisms. Namely, with slow-fast $(t, t/\epsilon)$ composition of two maps denoted by (.)

$$g_{t,t/\epsilon} = \widetilde{g}_{t/\epsilon}.\bar{g}_t = (Id + \gamma_{t/\epsilon}).\bar{g}_t\,.$$

Upon writing $\bar{x}_t(x_0) = \bar{g}_t x_0$ we have at $O(1)$ from $u_t = \dot{g}_t g_t^{-1}$ that

$$\begin{aligned}\frac{d}{dt}(g_{t,t/\epsilon}x_0) &= u_t(\bar{g}_t x_0 + \gamma_{t/\epsilon}\bar{g}_t x_0) \\ &= \dot{\bar{x}}_t(x_0) + (\dot{\bar{x}}_t \cdot \nabla_{\bar{x}_t})\,\gamma_{t/\epsilon}(\bar{x}_t(x_0)) + \epsilon^{-1}\,\partial_{t/\epsilon}\gamma_{t/\epsilon}(\bar{x}_t(x_0)\,.\end{aligned}$$

Multi-time homogenisation in the limit $\epsilon \to 0$ shows that

$$\lim_{\epsilon\to 0} g_{t,t/\epsilon} = \phi_t := (Id + \gamma_{\circ W_t}).\bar{g}_t\,,$$

where $\circ W_t$ denotes Stratonovich stochastic time dependence. This is SALT flow.

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Deriving the SALT vector field

The *stochastic vector-field* associated with stochastic flow $\{\phi_{s,t}\}_{0\le s\le t}$ from $\mathbf{d}\phi_t = \mathbf{d}(\bar{g}_t) + \mathbf{d}(\gamma_{\circ W_t}).\bar{g}_t)$

$$\mathbf{d}\phi_{s,t}(x) = u_t(\phi_{s,t}(x))\mathrm{d}t + \sum_k \xi^{(k)}(\phi_{s,t}(x)) \circ \mathrm{d}W_t, \quad \phi_{s,s}(x) = x \in M.$$

The spatial stochastic vector field $\mathbf{d}x_t$ on a given smooth manifold M which generates the SALT flow map is given by

$$\mathbf{d}x_t(x) = u_t(x)\mathrm{d}t + \sum_{k=1}^{\infty} \xi^{(k)}(x) \circ \mathrm{d}W_t^{(k)}, \ \ x \in M\,.$$

That is to say, $\mathbf{d}x_t = \mathbf{d}\phi_{0,t} \circ \phi_{0,t}^{-1}$ is the stochastic analogue of the usual Eulerian vector field.

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What was our next step?

Variational principle for stochastic fluid equations Holm PSRA 2015 $\boxed{\surd}$

Multi-time homogenization of slow-fast decomposition Cotter Gottwald Holm PRSA 2017 $\boxed{\surd}$

$\rightarrow$

Stochastic decomposition of transport velocity

(For data assimilation, $\sum_I \xi_i(x_t)$ was the key step) $\boxed{\surd}$

$+$

Interpret the stochastic decomposition of transport velocity in Kelvin theorem $\boxed{}$

$\rightarrow$

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SALT introduces a Stochastic Kelvin Circulation Theorem

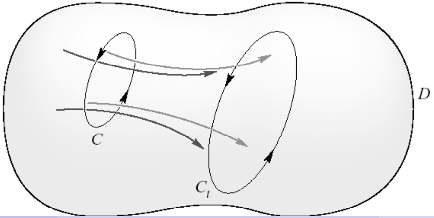
We derived the divergence-free advection velocity as the sum

$$\widetilde{u} := \underbrace{u(x,t)\,dt}_{\text{DRIFT}} + \sum_k \underbrace{\xi_k(x) \circ dW_k(t)}_{\text{NOISE}}, \quad \text{div}\,\widetilde{\mathbf{u}} = 0$$

Let $\mathbf{v}$ = momentum/mass. (In Hamilton's principle, $\mathbf{v} = D^{-1}\delta\ell/\delta\mathbf{u}$.)

The **stochastic Kelvin circulation theorem** represents **Newton's law** for the evolution of momentum concentrated on an advecting loop

$$\mathbf{d}\oint_{c(\widetilde{u})} \mathbf{v}\cdot d\mathbf{x} = \oint_{c(\widetilde{u})} \underbrace{(\mathbf{d} + \mathcal{L}_{\widetilde{u}})(\mathbf{v}\cdot d\mathbf{x})}_{\text{By KIW formula}} = \oint_{c(\widetilde{u})} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$



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The key was the Itô-Wentzell formula for transport of k-forms, Kunita (1984)

Let $\phi_t^* K(t,x) = K(t,\phi_t(x))$ denote change of variables by the pull-back $x \to \phi_t(x)$, $x \in \mathbb{R}^3$ of semimartingale flow ϕ_t

$$\phi_t(x) - \phi_0(x) = \int_0^t u(\phi_s(x), s)ds + \sum \int_0^t \xi_i(\phi_s(x)) \circ dW_s^i$$

acting on semimartingale k-form, $\mathbf{d}K(t,x) = G(t,x)dt + H(t,x) \circ dW_t$.

This is the Kunita-Itô-Wenzell (KIW) formula for tensor fields:

$$\phi_t^* K(t,x) - K(0,x) = \int_0^t \phi_s^* \big(\mathbf{d}K(s,x) + \mathcal{L}_{\mathbf{d}\phi_s(x)} K(s,x) \big) \,,$$

where $\mathcal{L}_{\mathbf{d}\phi_s(x)}$ is the Lie derivative by the vector field $\mathbf{d}\phi_s(x)$ whose time integral $\int_0^t \mathbf{d}\phi_s(x) = \phi_t(x) - \phi_0(x)$ generates the semimartingale flow ϕ_t.

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‘Transfer principle’ for Lagrangian fluid SPDE

Introducing differential notation for Kunita's Itô-Wenzell formula

We write the stochastic ‘fundamental theorem of calculus’ as

$$\phi_t^* K(t,x) - \phi_0^* K(0,x) := K(t,\phi_t(x)) - K(0,x) = \int_0^t \mathbf{d}\,(\phi_s^* K_s)$$

In this notation, the Kunita-Itô-Wenzell (KIW) formula is written

$$\int_0^t \mathbf{d}\,(\phi_s^* K_s) = \int_0^t \phi_s^* \big(\mathbf{d}K(s,x) + \mathcal{L}_{\mathbf{d}\phi_s(x)} K(s,x)\big)\,.$$

So the KIW formula ‘transfers’ to the equivalent *differential form*

$$\mathbf{d}\big(\phi_t^* K(t,x)\big) = \phi_t^* \Big(\mathbf{d}K(t,x) + \mathcal{L}_{\mathbf{d}\phi_t(x)} K(t,x)\Big)\,, \quad \text{a.s.}$$

where ϕ_t is the stochastic process obtained by homogenisation CGH2017

$$\mathbf{d}\phi_t(x) := u(\phi_t(x),t)dt + \sum_i \xi_i(\phi_t(x)) \circ dW_t^i\,.$$

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Kunita 1984 provided the key to stochastic advection

The Kunita Itô-Wentzell change of variables formula in *differential form* leads to the *stochastic advection law* (BdLHLT2019)

$$\mathbf{d}\big(\phi_t^* K(t,x)\big) = \phi_t^*\Big(\mathbf{d}K(t,x) + \mathcal{L}_{\mathbf{d}\phi_t(x)}K(t,x)\Big) = 0\,,\quad \text{a.s.}$$

where $\mathcal{L}_{\mathbf{d}\phi_t(x)}$ is the *Lie derivative* by the vector field $\mathbf{d}\phi_t(x)$ whose time integral $\int_0^t \mathbf{d}\phi_s(x) = \phi_t(x) - \phi_0(x)$ generates the semimartingale flow ϕ_t acting on semimartingale k-form, $\mathbf{d}K(t,x) = G(t,x)dt + H(t,x)\circ dW_t$.

Choose ϕ_t as the stochastic process obtained by homogenisation CGH2017

$$\mathbf{d}\phi_t(x) := u(\phi_t(x),t)dt + \sum_i \xi_i(\phi_t(x))\circ dW_t^i\,.$$

The Lie derivative $\mathcal{L}_{\mathbf{d}\phi_t}K$ has both a dynamic and a geometric definition

$$\mathcal{L}_{\mathbf{d}\phi_t}K = \lim_{\Delta s\to 0}\frac{1}{\Delta s}(\phi_{\Delta s}^*K - K) = \mathbf{d}\phi_t \,\lrcorner\, dK + d(\mathbf{d}\phi_t \,\lrcorner\, K)\ (Cartan)$$

Now we assemble the stochastic fluid equations

$$\mathbf{d}\oint_{c(\mathbf{d}\phi_t)} \mathbf{v}\cdot d\mathbf{x} = \oint_{c(\mathbf{d}\phi_t)} \underbrace{(\mathbf{d} + \mathcal{L}_{\mathbf{d}\phi_t})(\mathbf{v}\cdot d\mathbf{x})}_{\text{KIW formula}} = \oint_{c(\mathbf{d}\phi_t)} \underbrace{\mathbf{f}\cdot d\mathbf{x}}_{\text{Newton's Law}}$$

This corresponds to the *motion equation* derived from Hamilton's principle

$$\Big(\mathbf{d} + \mathcal{L}_{\mathbf{d}\phi_t}\Big)\Big(\frac{1}{D}\frac{\delta\ell}{\delta\mathbf{u}}\cdot d\mathbf{x}\Big) = \mathbf{f}\cdot d\mathbf{x}\,,$$

with the *advection of mass* expressed in KIW form

$$\Big(\mathbf{d} + \mathcal{L}_{\mathbf{d}\phi_t}\Big)\big(Dd^3x\big) = 0\,,$$

where the *flow velocity* is given by the stochastic vector field

$$\mathbf{d}\phi_t(x) := u(\phi_t(x),t)dt + \sum_i \xi_i(\phi_t(x))\circ dW_t^i\,.$$

Now we understand the stochastic Kelvin's circulation theorem.
It's the rate of change of momentum of a ***stochastically*** *moving loop.*

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'Transfer principle' for **Eulerian** fluid SPDE (**SALT**)

Consider the stochastic *Eulerian* divergence-free velocity vector field as,

$$(\phi_t^{-1})^* \mathbf{d}\phi_t(x) =: \mathbf{d}x_t(x,t) = u(x,t)dt + \sum_i \xi_i(x) \circ dW_t^i, \quad \mathrm{div}(\mathbf{d}x_t) = 0\,,$$

assumed in Holm [2015] then derived from homogenization in Cotter et al. [2017]. The KIW formula expresses the Euler-Poincaré SPDE as

$$\begin{aligned} &\mathbf{d}u + \mathbf{d}x_t \cdot \nabla u + u_j \nabla \mathbf{d}x_t{}^j = -\nabla p\, dt \quad (u \text{ is 3D Euler velocity}) \\ \Longrightarrow\ &\mathbf{d}\omega + \mathbf{d}x_t \cdot \nabla\omega - \omega \cdot \nabla \mathbf{d}x_t = 0 \quad (\omega = \mathrm{curl}\, u \text{ is vorticity}) \end{aligned}$$

as the 3D Euler fluid motion and vorticity equations with SALT.

2D versions of these 3D Euler vorticity equations have appeared previously, e.g., in Brzézniak, Capínski and Flandoli (1991) and Mémin (2014).

There is other history here, which would require its own lecture. Maybe Franco will discuss it!

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More about the Stochastic 3D Euler equations

For $\quad \widetilde{u} := \underbrace{u(x,t)\, dt}_{\text{DRIFT}} + \underbrace{\sum_k \xi_k(x) \circ dW_k(t)}_{\text{NOISE}} \quad$ with $\quad \operatorname{div} \widetilde{\mathbf{u}} = 0.$

The stochastic Euler equation of motion with $\mathbf{v}$ = momentum/mass is

$$\mathbf{d}\mathbf{v} - \widetilde{\mathbf{u}} \times \operatorname{curl} \mathbf{v} = -\nabla(p dt + \widetilde{\mathbf{u}} \cdot \mathbf{v})\,.$$

We take the curl to find the equation for vorticity $\boldsymbol{\omega} := \operatorname{curl}\mathbf{v}$,

$$\mathbf{d}\boldsymbol{\omega} = \operatorname{curl}(\widetilde{\mathbf{u}} \times \boldsymbol{\omega}) = -\widetilde{\mathbf{u}} \cdot \nabla\boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \widetilde{\mathbf{u}} \quad \text{or} \quad \mathbf{d}\omega = -\big[\widetilde{u}\,, \omega\big]\,.$$

Thus, the stochastic Euler equation keeps its deterministic form!
Only the transport velocity changes, to become a stochastic process!

Crisan, Flandoli, Holm (J Nonlin Sci 2018) have proven these equations have local-in-time existence, uniqueness and Beale-Kato-Majda regularity. (Same properties as the deterministic Euler equations!)

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Interim summary: What were our (theoretical) steps?

Variational principle for stochastic fluid equations Holm PSRA 2015 $\boxed{\surd}$

Multi-time homogenization of slow-fast decomposition Cotter Gottwald Holm PRSA 2017 $\boxed{\surd}$

$\longrightarrow$

Stochastic decomposition of transport velocity (Using data to find $\xi_i(x_t)$ was the key issue) $\boxed{\surd}$

$+$

Stochastic Kelvin theorem. $\boxed{\surd}$

Fluid equations derived from Hamilton's principle. $\boxed{}$

$\longrightarrow$

SALT Eulerian SPDEs with stochastic transport

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For Hamilton's principle, we need the diamond operation

Definition

The operation $\diamond : V \times V^* \to \mathfrak{X}^*$ between tensor space elements $a \in V^*$ and $b \in V$ produces an element of $\mathfrak{X}(\mathcal{D})^*$, a one-form density, defined by

$$\Big\langle b \diamond a, u \Big\rangle_{\mathfrak{X}} = -\int_{\mathcal{D}} b \cdot \mathcal{L}_u \, a =: \Big\langle b \, , \, -\mathcal{L}_u \, a \Big\rangle_V \, ,$$

where $\langle \cdot \, , \, \cdot \rangle_{\mathfrak{X}}$ denotes the symmetric, non-degenerate L^2 pairing between vector fields and one-form densities, which are dual with respect to this pairing. Likewise, $\langle \cdot \, , \, \cdot \rangle_V$ represents the corresponding L^2 pairing between dual elements of V and V^*.

Also, $\mathcal{L}_u a$ stands for the Lie derivative of an element $a \in V^*$ with respect to a vector field $u \in \mathfrak{X}(\mathcal{D})$, and $b \cdot \mathcal{L}_u \, a$ denotes the contraction between elements $b \in V$ and elements $a \in V^*$.

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Reminder of Lie derivatives examples in fluid dynamics

Example (Lie derivatives examples in fluid dynamics)

(Functions) $(\partial_t + \mathcal{L}_u)\theta(\mathbf{x}) = \partial_t\theta + \mathbf{u}\cdot\nabla\theta\,,$

(1-forms) $$(\partial_t + \mathcal{L}_u)(\mathbf{v}(\mathbf{x})\cdot d\mathbf{x}) = \big(\partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\,\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x} = \big(\partial_t\mathbf{v} - \,\mathbf{u}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,,$$

(2-forms) $$(\partial_t + \mathcal{L}_u)(\boldsymbol{\omega}(\mathbf{x})\cdot d\mathbf{S}) = \big(\partial_t\boldsymbol{\omega} - \,\mathrm{curl}\,(\mathbf{u}\times\boldsymbol{\omega}) + \mathbf{u}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}\,,$$

(3-forms) $$(\partial_t + \mathcal{L}_u)(\rho(\mathbf{x})\,d^{\,3}x) = (\partial_t\rho + \mathrm{div}\,\rho\,\mathbf{u})\,d^{\,3}x\,.$$

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Reminder of examples of the diamond operation

Example (Calculating diamond ($\diamond$) with vector calculus)

In $\mathbb{R}^3$, for $a = \theta \in \Lambda^0, \mathrm{D} = Dd^3x \in \Lambda^3, A = \mathbf{A} \cdot d\mathbf{x} \in \Lambda^1, B = \mathbf{B} \cdot d\mathbf{S} \in \Lambda^2$.

$$\begin{aligned} \left(\frac{\delta \ell}{\delta a} \diamond a \right) = \Bigg[- & \frac{\delta \ell}{\delta \theta} \nabla \theta \cdot d\mathbf{x} + D \nabla \left(\frac{\delta \ell}{\delta D} \right) \cdot d\mathbf{x} \\ & + \left(\frac{\delta \ell}{\delta \mathbf{A}} \times \mathrm{curl}\ \mathbf{A} - \mathbf{A}\ \mathrm{div} \frac{\delta \ell}{\delta \mathbf{A}} \right) \cdot d\mathbf{x} \\ & + \left(\mathrm{curl} \frac{\delta \ell}{\delta \mathbf{B}} \times \mathbf{B} - \frac{\delta \ell}{\delta \mathbf{B}} \mathrm{div} \mathbf{B} \right) \cdot d\mathbf{x} \,. \end{aligned}$$

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Variational principle for stochastic advection

Theorem (Variational principle for stochastic continuum dynamics)

Consider a cylindrically stochastic Stratonovich path $x_t = \phi_t X$ with $\phi_t \in \mathrm{Diff}(\mathcal{D})$. The following two statements are equivalent:

(i) *The Clebsch-constrained Hamilton's variational principle holds on $\mathfrak{X}(\mathcal{D}) \times V^*$,*

$$\delta S := \delta \int_{t_1}^{t_2} l(\phi_t^* u, \phi_t^* a) + \Big\langle \phi_t^* b \,,\, \mathbf{d}\,(\phi_t^* a) \Big\rangle_V \, dt = 0\,.$$

(ii) *The Euler–Poincaré equations for continua hold, in the form*

$$\mathbf{d}\left(\phi_t^* \frac{\delta l}{\delta u}\right) = \phi_t^* \left(\mathbf{d} \frac{\delta l}{\delta u} + \mathcal{L}_{\mathbf{d}x_t} \frac{\delta l}{\delta u} \right) = \phi_t^* \left(\frac{\delta l}{\delta a} \diamond a \right) \mathrm{d}t\,,$$

$$\mathbf{d}\big(\phi_t^* a_t\big) = \phi_t^* \Big(\mathbf{d} a_t + \mathcal{L}_{\mathbf{d}x_t} a_t \Big) = 0\,.$$

Proof of the variational principle for stochastic fluids

Proof.

Evaluating the variational derivatives at fixed time t and coordinate X yields the following relations:

$$
\begin{aligned}
\delta(\phi_t^* b) &: 0 = \mathbf{d}\,(\phi_t^* a) = \phi_t^*\Big(\mathbf{d} a_t + \mathcal{L}_{\mathbf{d}x_t} a_t\Big),\\
\delta(\phi_t^* a) &: 0 = -\,\mathbf{d}\,(\phi_t^* b) + \phi_t^*\left(\frac{\delta l}{\delta a}\right)\,\mathrm{d}t\,,\\
\delta(\phi_t^* u) &: 0 = \frac{\delta l}{\delta(\phi_t^* u)} - (\phi_t^* b)\diamond(\phi_t^* a)\,.
\end{aligned}
$$

One then computes the motion equation via

$$
\mathbf{d}\,\frac{\delta l}{\delta(\phi_t^* u)} = \mathbf{d}\,(\phi_t^* b)\diamond(\phi_t^* a) + (\phi_t^* b)\diamond\mathbf{d}\,(\phi_t^* a)\,,
$$

$$
\text{leading to}\quad \phi_t^*\left(\mathbf{d}\frac{\delta l}{\delta u} + \mathcal{L}_{\mathbf{d}x_t}\frac{\delta l}{\delta u}\right) = \phi_t^*\left(\frac{\delta l}{\delta a}\diamond a\right)\,\mathrm{d}t\,,
$$

after using the KIW pull-back formula. □

Example: stochastic magnetohydrodynamics

Example (Adiabatic compressible stochastic MHD, BdLHLT2019)

In the case of adiabatic compressible stochastic magnetohydrodynamics (MHD), the action in Hamilton's principle is given by

$$S = \int l(\mathbf{u}, D, s, \mathbf{B})\, \mathrm{d}t = \int \left(\frac{D}{2}\, |\mathbf{u}|^2 - De(D,s) - \frac{1}{2}|\mathbf{B}|^2 \right) d^3x\, \mathrm{d}t\,.$$

Thermodynamic First Law, for mass density D and entropy/ mass s,

$$de = -p\, d(1/D) + T ds\,,$$

with pressure $p(D,s)$ and temperature $T(D,s)$. In 3D vector form, the motion equation is

$$\mathbf{du} + (\mathbf{dx}_t \cdot \nabla)\mathbf{u} + (\nabla \mathbf{u})^T \cdot \mathbf{dx}_t = -\Big(\frac{1}{D}\nabla p\Big) dt - \Big(\frac{1}{D}\mathbf{B} \times \mathrm{curl}\ \mathbf{B}\Big) dt\,.$$

where $\mathbf{dx}_t := \mathbf{u}(t, \mathbf{x}_t)\, \mathrm{d}t + \boldsymbol{\xi}(\mathbf{x}_t) \circ \mathrm{d}W_t$ is the stochastic Lagrangian trajectory.

Example (SMHD advected variables and conservation laws)

By definition, the advected variables $\{s, \mathbf{B}, D\}$ satisfy the following Lie-derivative relations which close the ideal MHD system, by applying the KIW formula for the advective dynamics,

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}x_t})\, s = 0, \quad \text{or} \quad \mathbf{d}s = -\,\mathbf{d}\mathbf{x}_t \cdot \nabla\, s\,,$$

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}x_t})\,(\mathbf{B}\cdot d\mathbf{S}) = 0, \quad \text{or} \quad \mathbf{d}\mathbf{B} = \mathrm{curl}\,(\mathbf{d}\mathbf{x}_t \times \mathbf{B}),$$

$$(\mathbf{d} + \mathcal{L}_{\mathbf{d}x_t})\,(D\, d^3x) = 0\,, \quad \text{or} \quad \mathbf{d}D = -\,\nabla\cdot(D\,\mathbf{d}\mathbf{x}_t)\,,$$

and the pressure is a function $p(D,s) = D^2\partial e/\partial D$ specified by giving the equation of state of the fluid, $e = e(D,s)$.

These stochastic MHD equations preserve magnetic helicity and entropy

$$\Lambda_{mag} = \int \mathbf{B}\cdot\mathrm{curl}^{-1}\mathbf{B}\, d^3x, \qquad \mathcal{S} = \int D\Phi(s)\, d^3x\,,$$

provided $\mathrm{d}\mathbf{x}_t$ and $\mathbf{B}$ are tangent to the boundary.

What is the next step?

Hamilton's principle $\boxed{\surd}$

$+$

Multi-time fast-slow homogenization $\boxed{\surd}$

$\rightarrow$

Stochastic decomposition $\boxed{\surd}$

$+$

Kunita's Itô-Wenzell (KIW) formula $\boxed{\surd}$

$\rightarrow$

Eulerian fluid SPDEs with stochastic transport $\boxed{\surd}$

What is the next step? $\boxed{}$

Hamiltonian SALT

The SALT equations read

$$\mathbf{d}\frac{\delta \ell}{\delta u} + \mathcal{L}_{\mathbf{d}x_t}\frac{\delta \ell}{\delta u} \overset{\mathfrak{X}^*}{=} \frac{\delta \ell}{\delta a}\diamond a\, dt \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}x_t} a \overset{V^*}{=} 0,$$

where $\mathbf{d}x_t := u(t, x_t)\,\mathrm{d}t + \xi(x_t)\circ \mathrm{d}W_t$ is the stochastic transport vector field along the Lagrangian trajectory.

The Legendre transform from the Lagrangian side to the Hamiltonian side for SALT is given by $\mu = \delta\ell/\delta u$ and

$$\begin{aligned} \mathbf{d}h(\mu, a) &= \big\langle \mu\,,\, \mathbf{d}x_t \big\rangle - \ell(u,a)dt \\ &= \big\langle \mu\,,\, u \big\rangle dt - \ell(u,a)dt + \big\langle \mu\,,\, \xi(x_t)\big\rangle \circ \mathrm{d}W_t \\ &= h(\mu,a)dt + \big\langle \mu\,,\, \xi(x_t)\big\rangle \circ \mathrm{d}W_t\,, \end{aligned}$$

so that,

$$\delta(\mathbf{d}h) = \underbrace{\delta h(\mu,a)\,dt}_{\text{DALT!}} + \underbrace{\big\langle \delta\mu\,,\, \xi(x_t)\big\rangle \circ \mathrm{d}W_t}_{\text{New!}}$$

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Lie–Poisson bracket for SALT

With $\delta(\mathbf{d}h)/\delta\mu = \mathbf{d}x_t := u(t,x_t)\,\mathrm{d}t + \xi(x_t)\circ \mathrm{d}W_t$ and $\delta(\mathbf{d}h)/\delta a = -\,(\delta\ell/\delta a)\,\mathrm{d}t$, the SALT Euler–Poincaré equations

$$\mathbf{d}\frac{\delta\ell}{\delta u} + \mathcal{L}_{\mathbf{d}x_t}\frac{\delta\ell}{\delta u} \overset{\mathfrak{X}^*}{=} \frac{\delta\ell}{\delta a}\diamond a\,dt \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}x_t}a \overset{V^*}{=} 0,$$

translate into the Lie–Poisson Hamiltonian form, as

$$\mathbf{d}\begin{bmatrix}\mu\\ a\end{bmatrix} = -\begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a\\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\delta(\mathbf{d}h)/\delta\mu\\ \delta(\mathbf{d}h)/\delta a\end{bmatrix} = -\begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a\\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\mathbf{d}x_t\\ \delta h/\delta a\,\mathrm{d}t\end{bmatrix}$$

The definition of the diamond operator $(\diamond)$ will ensure that the Lie–Poisson matrix operator is skew-symmetric in L^2 pairing under integration by parts.

$$\mathbf{d}f(\mu,a) = -\left\langle \begin{bmatrix}\delta f/\delta\mu\\ \delta f/\delta a\end{bmatrix}^T, \begin{bmatrix}\mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a\\ \mathcal{L}_{(\,\cdot\,)}a & 0\end{bmatrix}\begin{bmatrix}\delta(\mathbf{d}h)/\delta\mu\\ \delta(\mathbf{d}h)/\delta a\end{bmatrix}\right\rangle =: \Big\{f,h\Big\},$$

where $\{f,h\}$ is the *same* Lie–Poisson bracket as for DALT.

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Casimirs for SALT

Theorem

SALT dynamics preserves the same Casimirs as for DALT dynamics.

Proof.

A functional $C[\mu, a]$ whose variational derivatives $[\delta C/\delta\mu, \delta C/\delta a]^T$ comprise a null eigenvector of the Lie–Poisson matrix operator is called a *Casimir functional* for that Lie–Poisson system.
SALT and DALT have the same Lie–Poisson matrix operator.

Therefore, Casimir functionals for DALT are preserved for SALT, since they satisfy the corresponding equation,

$$\mathbf{d}C(\mu, a) = \left\langle \begin{bmatrix} \delta(\mathbf{d}h)/\delta\mu \\ \delta(\mathbf{d}h)/\delta a \end{bmatrix}^T, \begin{bmatrix} \mathrm{ad}^*_{(\,\cdot\,)}\mu & (\,\cdot\,)\diamond a \\ \mathcal{L}_{(\,\cdot\,)}a & 0 \end{bmatrix} \begin{bmatrix} \delta C/\delta\mu \\ \delta C/\delta a \end{bmatrix} \right\rangle =: \Big\{ C, \mathbf{d}h \Big\} = 0,$$

so that $C[\mu_t, a_t] = C[\mu_0, a_0]$ is conserved *for any Hamiltonian* $\mathbf{d}h[\mu, a]$. □

SALT RSW motion

SALT RSW motion is governed by the following nondimensional equations for horizontal fluid velocity $\mathbf{v} = \epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})$ with $\mathrm{curl}\mathbf{R}(\mathbf{x}) = 2\Omega(\mathbf{x})\hat{\mathbf{z}}$ and depth D,

$$\mathsf{d}\mathbf{v} - \mathsf{d}\mathbf{x}_t \times \mathrm{curl}\mathbf{v} + \nabla\psi = 0\,, \qquad \mathsf{d}D + \nabla\cdot(D\mathsf{d}\mathbf{x}_t) = 0\,,$$

with notation

$$\psi = \left(\frac{D-B}{\epsilon\mathcal{F}} + \frac{\epsilon}{2}|\mathbf{u}|^2\right)dt + \mathbf{v}\cdot\xi(x_t)\circ \mathrm{d}W_t\,,$$

and variable Coriolis parameter $2\Omega(\mathbf{x})$, bottom topography $B = B(\mathbf{x})$, Rossby number ϵ and rotational Froude number $\mathcal{F}$,

$$\epsilon = \frac{\mathcal{U}_0}{f_0 L} \ll 1 \quad \text{and} \quad \mathcal{F} = \frac{f_0^2 L^2}{gB_0} = O(1)\,.$$

The dimensional scales $(B_0, L, \mathcal{U}_0, f_0, g)$ denote equilibrium fluid depth, horizontal length scale, horizontal fluid velocity, reference Coriolis parameter, and gravitational acceleration, respectively.

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Homework #2.1 SALT RSW

(a) Show that the SALT RSW equations arise as Euler–Poincaré equations from Hamilton's principle with action integral,

$$S_{\rm RSW} = \int \ell(\mathbf{u}, D)\, dt = \int \Big[D\mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{(D-B)^2}{2\epsilon\mathcal{F}} + \frac{\epsilon}{2} D|\mathbf{u}|^2 \Big] dx^1\wedge dx^2\, {\rm d}t\,,$$

where $(\mathbf{d}D + \nabla\cdot(D\mathbf{d}\mathbf{x}_t))dx^1\wedge dx^2 = 0$.
Hint: first identify the momentum and advected quantity, so $(\diamond)$ may be computed.

(b) Write the Kelvin circulation theorem for SALT RSW.

(c) Legendre transform to compute the Hamiltonian.

(d) Compute the Lie–Poisson form of the SALT RSW equations.

(e) Compute the Casimirs for the Lie–Poisson bracket.

(f) Explain how the Casimirs are related to PV and depth.

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Homework #2.2 Euler, SALT & LA SALT Rigid Body

(1) The deterministic Euler Rigid Body equations for body angular momentum $\Pi \in so(3)^* \equiv \mathbb{R}^3$ and body angular velocity $\mathbb{I}^{-1}\Pi = \Omega \in \mathbb{R}^3$ may be expressed as

$$\frac{d\Pi}{dt} = \Pi \times \frac{\partial h}{\partial \Pi} \quad \text{with} \quad h(\Pi) = \tfrac{1}{2}\Pi \cdot \mathbb{I}^{-1}\Pi\,.$$

Discuss the solutions.This is classical.

(2) The SALT Rigid Body equations may be expressed as

$$\mathbf{d}\Pi = \Pi \times \frac{\partial(\mathbf{d}h)}{\partial \Pi} \quad \text{with} \quad \mathbf{d}h(\Pi) = h(\Pi)\,dt + \Pi \cdot \xi \circ dW_t\,,$$

for a constant $\xi \in so(3) \equiv \mathbb{R}^3$. Discuss the solutions. See arXiv:1601.02249 or https://doi.org/10.1007/s00332-017-9404-3.

(3) The LA SALT Rigid Body equations may be expressed as

$$\mathbf{d}\Pi = \Pi \times \mathbb{E}\left[\frac{\partial h}{\partial \Pi}\right] dt + \Pi \times \xi \circ dW_t\,,$$

for a constant $\xi \in so(3) \equiv \mathbb{R}^3$. Discuss the solutions. See arXiv:1908.11481

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What's next? Do these ideas apply to climate modelling?

"Climate is what you expect. Weather is what you get." [1]

> *There are many questions regarding climate whose answers remain elusive.*
>
> *For example, there is the question of determinism; was it somehow inevitable at some earlier time that the climate now would be as it actually is?*

An *almost intransitive system* is one that can undergo two or more distinct types of behaviour, and will exhibit one type for a long time, but not forever . . . . Such a system is still deterministic. We are stochastic!

[1]Lorenz, E. N., 1995: Climate is what you expect. Unpublished, available at `http://eaps4.mit.edu/research/Lorenz/Climate_expect.pdf`

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One candidate is Lagrangian Averaged (LA) SALT

The LA SALT equations substitute $u_t \to \mathbb{E}\,[u_t]$ in the Lagrangian path

$$\oint_{C\left(\mathbf{d}x_t = u_t dt + \xi(x)\circ dW_t\right)} u_t \cdot dx \quad \Longrightarrow \quad \oint_{C\left(\mathbf{d}X_t = \mathbb{E}[u_t]dt + \xi(x)\circ dW_t\right)} u_t \cdot dx \,.$$

For example, in the Euler fluid case the modified Kelvin theorem reads,

$$\mathbf{d}\oint_{C\left(\mathbf{d}X_t\right)} u_t \cdot dx = \oint_{C\left(\mathbf{d}X_t\right)} \left[\mathbf{d}u_t \cdot dx + \mathcal{L}_{\mathbf{d}X_t}(u_t \cdot dx)\right] = 0\,,$$

where $\mathcal{L}_{\mathbf{d}X_t}(u_t \cdot dx)$ denotes the Lie derivative of the 1-form $(u_t \cdot dx)$ with respect to the vector field $\mathbf{d}X_t$ given by

$$\mathbf{d}X_t = \mathbb{E}\,[u_t]\,dt + \sum_k \xi^{(k)}(x) \circ dW_t \,.$$

The corresponding Euler–Poincaré form of the equations is

$$\mathbf{d}\frac{\delta \ell}{\delta u} + \mathcal{L}_{\mathbf{d}X_t}\frac{\delta \ell}{\delta u} = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\,dt \quad \text{and} \quad \mathbf{d}a + \mathcal{L}_{\mathbf{d}X_t} a = 0\,.$$

What does LA SALT tell us about extreme events?

When the *expected* Euler–Poincaré equations are written out in $\boxed{\text{Itô form}}$, with $\mu := \frac{\delta \ell}{\delta u}$, we find generalised NS and advected-diffusive equations

$$\frac{\partial}{\partial t}\mathbb{E}[\mu] + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mathbb{E}[\mu] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}[\mu]) = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond \mathbb{E}[a] + \mathbb{E}[\mathbb{F}_\mu] \,,$$

$$\frac{\partial}{\partial t}\mathbb{E}[a] + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mathbb{E}[a] - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mathbb{E}[a]) = \mathbb{E}[\mathbb{F}_a] \quad \boxed{\text{Climate}}\,.$$

These $\boxed{\text{Climate}}$ equations predict the expectations $\mathbb{E}[\mu]$ and $\mathbb{E}[a]$ throughout the domain of flow. The Itô $\boxed{\text{Weather}}$ equations for the fluctuations are *linear* drift/stochastic transport relations:

$$\mathbf{d}\mu + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}\mu + \sum_k \mathcal{L}_{\xi^{(k)}}\mu\, dW_t - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}\mu)\, dt = \mathbb{E}\left[\frac{\delta \ell}{\delta a}\right] \diamond a\, dt + \mathbb{F}_\mu$$

$$\mathbf{d}a + \mathcal{L}_{\mathbb{E}[\mathbf{d}X_t]}a + \sum_k \mathcal{L}_{\xi^{(k)}}a\, dW_t - \frac{1}{2}\sum_k \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}a)\, dt = \mathbb{F}_a \quad \boxed{\text{Weather}}\,.$$

The risk of extreme events EVOLVES: $\frac{d}{dt}\mathbb{E}\left[\langle |\mu - \mathbb{E}[\mu]|^2\rangle\right] = RHS$

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What's next? Over to you! Any questions?

[CGH17],[CFH17],[AGH17],[HT16a],[HT16b],[ACH16],[Hol15]

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