4 M4&5 A34 Enhanced Coursework $\boxed{\textbf{April 2012}}$

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

SOLVE FOUR OUT OF FIVE OF THE FOLLOWING PROBLEMS.

Exercise 4.1. Adjoint and coadjoint actions for $SE(2)$

(a) *Compute the adjoint and coadjoint actions AD, Ad, ad, Ad* and ad* for* $SE(2)$.

(b) *Show that*

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*_{(R_\theta(t),v(t))^{-1}}(\mu,\beta) = -\,\mathrm{ad}^*_{(\xi,\alpha)}(\mu,\beta)\,,$$

where, as before, we take $\dot{R}_\theta(t)|_{t=0} = \xi \in \mathbb{R}$, $\dot{v}(t)|_{t=0} = \alpha \in \mathbb{R}^2$ *and the pairing*

$$\langle\,\cdot\,,\,\cdot\,\rangle : se(2)^* \times se(2) \to \mathbb{R}$$

is given by the dot product of vectors in $\mathbb{R}^3$,

$$\Big\langle (\mu,\beta)\,,\,(\xi,\alpha)\Big\rangle = \mu\xi + \beta\cdot\alpha\,.$$

(c) *Compute the equations of motion for the dynamics on* $se(2)^*$ *resulting from Hamilton's principle* $\delta S = 0$ *with* $S = \int l(\xi,\alpha)\,dt$ *for the Lagrangian*

$$l(\xi,\alpha) = \frac{1}{2}A\xi^2 + \frac{1}{2}\alpha^T C\alpha$$

(d) *Derive the corresponding Lie-Poisson bracket for the Hamiltonian description of dynamics on* $se(2)^*$.

(e) *Sketch the coadjoint orbits in coordinates* $(\mu,\beta)\in\mathbb{R}^3$.

(f) *Work out the cotangent-lift momentum maps for the action of* $SE(2)$ *on* $\mathbb{R}^2$.

Exercise 4.2. *Determine the adjoint and coadjoint actions of the* $1+1$ *Poincaré group (the semidirect product* $SO(1,1)\,Ⓢ\,\mathbb{R}^{1,1}$*) and characterise its coadjoint orbits geometrically.*

(a) *Derive the* AD, Ad *and* ad *actions for the Poincaré group* $G = SO(1,1)\,Ⓢ\,\mathbb{R}^{1,1}$ *in* $1+1$ *dimensions.*

(b) *Introduce a natural pairing in which to define the dual Lie algebra and derive its* Ad^* *and* ad^* *actions.*

(c) *Lagrangians for all relativistic physical theories must invariant under the Poincaré group.*

Compute the coadjoint motion equations for any such theory as Euler-Poincaré equations for a Poincaré group reduced Lagrangian $\ell(\tilde{\lambda},\tilde{\nu})$.

(d) *Legendre transform and identify the corresponding Lie-Poisson brackets for coadjoint motion.*

(e) *Find a geometric expression for the coadjoint orbits of the Poincaré group.*

Hint: recall that the coadjoint orbits lie on level sets of the Casimirs for a Lie Poisson bracket, where a Casimir c *is defined as* $c : \{c,\,h\} = 0$ *for all* h.

Exercise 4.3. *Hamilton-Pontryagin metamorphosis*
Consider the left-invariant action S for Hamilton's principle $\delta S = 0$ given by

$$S = \int L(\Omega, \omega, g)\, dt = \int l(\Omega) + \frac{1}{2\sigma^2}\big|\omega - Ad_g\Omega\big|^2\, dt\,,$$

where $g \in G$ and $\omega = \dot{g}g^{-1} \in \mathfrak{g}$, for a matrix Lie group G and matrix Lie algebra $\mathfrak{g}$. Here $\sigma^2 \in \mathbb{R}$ is a positive constant and $|\cdot|$ is a Riemannian metric which defines a symmetric non-degenerate pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ between Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. (You may assume that $\mathfrak{g}^{**} \simeq \mathfrak{g}$.)

(a) Show that

$$(Ad_g\Omega)' = Ad_g\Omega' - ad_{Ad_g\Omega}\eta \quad \textit{with} \quad \eta = g'g^{-1}$$

(b) Write ω' in terms of η, $\dot{\eta}$ and ad_ω using cross-derivatives of $\dot{g} = \omega g$ and $g' = \eta g$.

(c) Derive the Euler-Poincaré equation for $\partial l/\partial\Omega$ from $\delta S = 0$.
(You may ignore endpoint terms when integrating by parts.)

(d) Interpret this Euler-Poincaré equation as a conservation law.

Exercise 4.4. *$GL(n,\mathbb{R})$-invariant motions Consider the Lagrangian*

$$L = \frac{1}{2}\,\mathrm{tr}\Big(\dot{S}S^{-1}\dot{S}S^{-1}\Big) + \frac{1}{2}\,\dot{\mathbf{q}}\cdot S^{-1}\dot{\mathbf{q}}\,,$$

where S is an $n \times n$ symmetric matrix and $\mathbf{q} \in \mathbb{R}^n$ is an $n-$component column vector.

(a) Legendre transform to construct the corresponding Hamiltonian and canonical equations.

(b) Show that the Lagrangian and Hamiltonian are invariant under the group action

$$\mathbf{q} \to G\mathbf{q} \quad \textit{and} \quad S \to GSG^T$$

for any constant invertible $n \times n$ matrix, G.

(c) Compute the infinitesimal generator for this group action and construct its corresponding momentum map. Is this momentum map equivariant? Prove it.

(d) Verify directly that this momentum map is a conserved $n\times n$ matrix quantity by using the equations of motion.

Exercise 4.5. ***Maxwell form of Euler's fluid equations***

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity $\mathbf{u}$ *satisfying* $\mathrm{div}\mathbf{u} = 0$ *in a rotating frame with tme-independent Coriolis parameter* $\mathrm{curl}\mathbf{R}(\mathbf{x}) = 2\boldsymbol{\Omega}$ *are given in the form of Newton's Law of Force by*

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{Acceleration} = \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{Coriolis} - \underbrace{\nabla p}_{Pressure} . \tag{1}$$

(a) *Show that this Newton's Law equation for Euler fluid motion in a rotating frame may be expressed as,*

$$\partial_t \mathbf{v} - \mathbf{u}\times\boldsymbol{\omega} + \nabla\Big(p + \tfrac{1}{2}|\mathbf{u}|^2\Big) = 0\,, \quad \textit{with} \quad \nabla\cdot\mathbf{u} = 0\,, \tag{2}$$

where we denote,

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}\,, \quad \boldsymbol{\omega} = \mathrm{curl}\mathbf{v} = \mathrm{curl}\mathbf{u} + 2\boldsymbol{\Omega}\,.$$

(b) *[Kelvin's circulation theorem]*

Show that the Euler equations (2) preserve the circulation integral $I(t)$ *defined by*

$$I(t) = \oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x}\,,$$

where $c(\mathbf{u})$ *is a closed circuit moving with the fluid at velocity* $\mathbf{u}$.

(c) *[Stokes theorem for vorticity of a rotating fluid]*

Show that the Euler equations (2) satisfy

$$\frac{d}{dt}\iint_{S(\mathbf{u})} \mathrm{curl}\mathbf{v}\cdot d\mathbf{S} = 0\,,$$

where the surface $S(\mathbf{u})$ *is bounded by an arbitrary circuit* $\partial S = c(\mathbf{u})$ *moving with the fluid.*

(d) *The* ***Lamb vector***,

$$\boldsymbol{\ell} := -\,\mathbf{u}\times\boldsymbol{\omega}\,,$$

represents the nonlinearity in Euler's fluid equation (2).

Show that by making the following identifications

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\omega} + \mathrm{curl}\,\mathbf{A}_0 \\
\mathbf{E} &= \boldsymbol{\ell} + \nabla\big(p + \tfrac{1}{2}|\mathbf{u}|^2\big) + \big(\nabla\phi - \partial_t\mathbf{A}_0\big) \\
\mathbf{D} &= \boldsymbol{\ell} \\
\mathbf{H} &= \nabla\psi\,,
\end{aligned} \tag{3}$$

the Euler fluid equations (2) imply the ***Maxwell form***

$$\begin{aligned}
\partial_t\mathbf{B} &= -\,\mathrm{curl}\,\mathbf{E} \\
\partial_t\mathbf{D} &= \mathrm{curl}\,\mathbf{H} + \mathbf{J} \\
\mathrm{div}\,\mathbf{B} &= 0 \\
\mathrm{div}\,\mathbf{E} &= 0 \\
\mathrm{div}\,\mathbf{D} &= \rho\ = -\,\Delta\big(p + \tfrac{1}{2}|\mathbf{u}|^2\big) \\
\mathbf{J} &= \mathbf{E}\times\mathbf{B} + (\mathrm{curl}^{-1}\mathbf{E})\times\mathrm{curl}\,\mathbf{B}\,,
\end{aligned} \tag{4}$$

provided the (smooth) gauge functions ϕ *and* $\mathbf{A}_0$ *satisfy* $\Delta\phi - \partial_t\mathrm{div}\mathbf{A}_0 = 0$ *with* $\partial_n\phi = \hat{\mathbf{n}}\cdot\partial_t\mathbf{A}_0$ *at the boundary and* ψ *may be arbitrary. What role is played by* $\mathbf{H}$ *as far as waves are concerned?*

(e) *Show that Euler's fluid equations (2) imply the following two elegant relations,*

$$dF = 0 \quad \textit{and} \quad dG = J\,,$$

where the 2-forms F, G and the 3-form J are given as

$$\begin{aligned}
F &= \boldsymbol{\ell}\cdot d\mathbf{x}\wedge dt + \boldsymbol{\omega}\cdot d\mathbf{S},\\
G &= \boldsymbol{\ell}\cdot d\mathbf{S},\\
J &= \mathbf{J}\cdot d\mathbf{S}\wedge dt + \rho\, d^3x,
\end{aligned}$$

and ρ and $\mathbf{J}$ are defined as in equations (4).