3 M3-4-5 A34 Assessed Problems # 3 Mar 2012

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 3.1. Exterior calculus operations

Vector notation for differential basis elements: *One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk} dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$, in vector notation as*

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\
d\mathbf{S} &= (dS_1, dS_2, dS_3) \\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,, \\
dS_i &:= \frac{1}{2}\epsilon_{ijk} dx^j \wedge dx^k\,, \\
d^3x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}
$$

(a) ***Vector algebra operations***

(i) Show that contraction with the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ recovers the following familiar operations among vectors

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\
(or, \quad X \,\lrcorner\, dS_i &= \epsilon_{ijk} X^j dx^k) \\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\
X \,\lrcorner\, d^3x &= \mathbf{X} \cdot d\mathbf{S} = X^k dS_k\,, \\
Y \,\lrcorner\, X \,\lrcorner\, d^3x &= \mathbf{X} \times \mathbf{Y} \cdot d\mathbf{x} = \epsilon_{ijk} X^i Y^j dx^k\,, \\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^3x &= \mathbf{X} \times \mathbf{Y} \cdot \mathbf{Z}\,.
\end{aligned}
$$

(ii) Show that these are consistent with

$$
X \,\lrcorner\, (\alpha \wedge \beta) = (X \,\lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \,\lrcorner\, \beta)\,,
$$

for a k-form α.

(iii) Use (ii) to compute $Y \,\lrcorner\, X \,\lrcorner\, (\alpha \wedge \beta)$ and $Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, (\alpha \wedge \beta)$.

(b) ***Exterior derivative examples in vector notation***
Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$
\begin{aligned}
df &= f_{,j}\, dx^j =: \nabla f \cdot d\mathbf{x} \\
0 = d^2 f &= f_{,jk}\, dx^k \wedge dx^j \\
df \wedge dg &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k =: (\nabla f \times \nabla g) \cdot d\mathbf{S} \\
df \wedge dg \wedge dh &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \wedge h_{,l}\, dx^l =: (\nabla f \cdot \nabla g \times \nabla h)\, d^3x
\end{aligned}
$$

Likewise, show that

$$
\begin{aligned}
d(\mathbf{v} \cdot d\mathbf{x}) &= (\mathrm{curl}\, \mathbf{v}) \cdot d\mathbf{S} \\
d(\mathbf{A} \cdot d\mathbf{S}) &= (\mathrm{div}\, \mathbf{A})\, d^3x\,.
\end{aligned}
$$

(c) Verify the compatibility condition $d^2 = 0$ for these forms as

$$
\begin{aligned}
0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathrm{grad}\, f) \cdot d\mathbf{S}\,, \\
0 = d^2(\mathbf{v} \cdot d\mathbf{x}) = d\big((\mathrm{curl}\, \mathbf{v}) \cdot d\mathbf{S}\big) &= (\mathrm{div}\,\mathrm{curl}\, \mathbf{v})\, d^3x\,.
\end{aligned}
$$

(d) Verify the exterior derivatives of the following contraction formulas for $X = \mathbf{X}\cdot\nabla$

(i) $d(X \lrcorner\, \mathbf{v}\cdot d\mathbf{x}) = d(\mathbf{X}\cdot\mathbf{v}) = \nabla(\mathbf{X}\cdot\mathbf{v})\cdot d\mathbf{x}$

(ii) $d(X \lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}) = d(\boldsymbol{\omega}\times\mathbf{X}\cdot d\mathbf{x}) = \mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X})\cdot d\mathbf{S}$

(iii) $d(X \lrcorner\, f\, d^3x) = d(f\mathbf{X}\cdot d\mathbf{S}) = \mathrm{div}\,(f\mathbf{X})\, d^3x$

(e) Use Cartan's formula,

$$\pounds_X\alpha = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha)$$

for a $k-$*form* α, $k = 0,1,2,3$ *in* $\mathbb{R}^3$ *to verify the Lie derivative formulas:*

(i) $\pounds_X f = X \lrcorner\, df = \mathbf{X}\cdot\nabla f$

(ii) $\pounds_X\,(\mathbf{v}\cdot d\mathbf{x}) = \big(-\ \mathbf{X}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot d\mathbf{x}$

(iii) $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S}) = \big(\mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X}) + \mathbf{X}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}$
$= \big(-\boldsymbol{\omega}\cdot\nabla\mathbf{X} + \mathbf{X}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{X}\big)\cdot d\mathbf{S}$

(iv) $\pounds_X(f\, d^3x) = (\mathrm{div}\, f\mathbf{X})\, d^3x$

(v) Derive these formulas from the dynamical definition of Lie derivative.

(f) Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:

(i) $\pounds_{fX}\alpha = f\pounds_X\alpha + df\wedge(X \lrcorner\, \alpha)$

(ii) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

(iii) $\pounds_X(X \lrcorner\, \alpha) = X \lrcorner\, \pounds_X\alpha$

(iv) $\pounds_X(Y \lrcorner\, \alpha) = (\pounds_X Y) \lrcorner\, \alpha + Y \lrcorner\, (\pounds_X\alpha)$

(v) $\pounds_X(\alpha\wedge\beta) = (\pounds_X\alpha)\wedge\beta + \alpha\wedge\pounds_X\beta$

Exercises in exterior calculus operations

Answer

Problems *(a)-(c)* are easily verified by direct computation, as are parts *(i-iii)* in problem *(d)*.

However, the linked parts *(iv & v)* in problem *(d)* require a bit more thought, although both of them are easy from the dynamical viewpoint, by differentiating the properties of the pull-back ϕ_t^*, which commutes with exterior derivative, wedge product and contraction. That is, for $m\in M$,

$$\begin{aligned} d(\phi_t^*\alpha) &= \phi_t^* d\alpha\,,\\ \phi_t^*(\alpha\wedge\beta) &= \phi_t^*\alpha\wedge\phi_t^*\beta\,,\\ \phi_t^*(X(m) \lrcorner\, \alpha) &= X(\phi_t(m)) \lrcorner\, \phi_t^*\alpha\,. \end{aligned}$$

Setting the dynamical definition of Lie derivative equal to its geometrical definition by Cartan's formula yields

$$\begin{aligned} \pounds_X\alpha &= \frac{d}{dt}\Big|_{t=0}(\phi_t^*\alpha)\\ &= X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha)\,, \end{aligned}$$

where α is a k-form on a manifold M and X is a smooth vector field with flow ϕ_t on M. Informed by these identities and this equality, one may now derive *(d)* The general form of the relation required in part *(iv)* follows immediately from the product rule for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y(m) \,\lrcorner\, \alpha) = Y(\phi_t(m)) \,\lrcorner\, \phi_t^*\alpha \,.$$

A direct computation using the dynamical definition of the Lie derivative above

$$\pounds_Y \alpha = \frac{d}{dt}\Big|_{t=0} (\phi_t^*\alpha) \,,$$

then yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \phi_t^*(Y \,\lrcorner\, \alpha) \quad &= \quad \Big(\frac{d}{dt}\Big|_{t=0} Y(\phi_t(m))\Big) \,\lrcorner\, \alpha \\ &\qquad + Y \,\lrcorner\, \Big(\frac{d}{dt}\Big|_{t=0} \phi_t^*\alpha\Big) \,. \end{aligned}$$

Hence, we recognise that the desired formula in part *(iv)* is the ***product rule***:

$$\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha) \,.$$

Part *(v)* in problem *(d)* is again simply a product rule, proved the same way.

▲

Exercise 3.2. ***Operations among vector fields***

The Lie derivative of one vector field by another is called the ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y \cdot X - \nabla X \cdot Y = -\,\pounds_Y X$$

In components, the Jacobi-Lie bracket is

$$[X\,,\,Y] = \Big[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\Big] = \Big(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\Big)\frac{\partial}{\partial x^l}$$

The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

Verify the following formulas

(a) $X \,\lrcorner\, (Y \,\lrcorner\, \alpha) = -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha)$

(b) $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$*, for zero-forms (functions) and one-forms.*

(c) $\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$*, as a result of (b). Use (c) to verify the Jacobi identity.*

(d) Verify formula (b) for arbitrary $k-$*forms.*

(e) For a top form α *and divergenceless vector fields* X *and* Y*, show that*

$$[X\,,\,Y] \,\lrcorner\, \alpha = d(X \,\lrcorner\, (Y \,\lrcorner\, \alpha))$$

and write its equivalent as a formula in vector calculus.

Operations among vector fields

Answer

(a) By direct substitution

$$\begin{aligned} X \lrcorner (Y \lrcorner \alpha) &= X^l Y^m \alpha_{mli_3\dots i_k} dx^{i_3} \wedge \dots \wedge dx^{i_k} \\ &= -X^l Y^m \alpha_{lmi_3\dots i_k} dx^{i_3} \wedge \dots \wedge dx^{i_k} \\ &= -Y \lrcorner (X \lrcorner \alpha)\,, \end{aligned}$$

by antisymmetry of $\alpha_{mli_3\dots i_k}$ in its first two indices.

(b) For zero-forms (functions) all terms in the formula vanish identically. So that's easy enough.
For a 1-form $\alpha = \mathbf{v} \cdot d\mathbf{x}$ the formula

$$[X\,,\,Y] \lrcorner \alpha = \pounds_X (Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)\,,$$

is seen to hold by comparing

$$[X\,,\,Y] \lrcorner \alpha \;=\; (X^k Y^l_{,k} - Y^k X^l_{,k}) v_l\,,$$

with

$$\begin{aligned} &\pounds_X (Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha) \\ &\qquad = X^k \partial_k (Y^l v_l) - Y^l (X^k v_{l,k} + v_j X^j_{,l})\,, \end{aligned}$$

(c) Given $[X\,,\,Y] \lrcorner \alpha = \pounds_X (Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)$, as verified in part *(b)* for zero-forms (functions) and one-forms, we use Cartan's formula to compute

$$\begin{aligned} \pounds_{[X\,,\,Y]} \alpha &= d([X\,,\,Y] \lrcorner \alpha) + [X\,,\,Y] \lrcorner d\alpha \\ &= d\big(\pounds_X (Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)\big) \\ &\quad + \pounds_X (Y \lrcorner d\alpha) - Y \lrcorner (\pounds_X d\alpha) \\ &= \pounds_X d(Y \lrcorner \alpha) - d(Y \lrcorner (\pounds_X \alpha) \\ &\quad + \pounds_X (Y \lrcorner d\alpha) - Y \lrcorner d(\pounds_X \alpha) \\ &= \pounds_X (\pounds_Y \alpha) - \pounds_Y (\pounds_X \alpha)\,, \end{aligned}$$

as required. Thus, the product rule for Lie derivative of a contraction obtained in answering problem *(b)* provides the key to solving *(c)*. ■

Consequently,

$$\begin{aligned} \pounds_{[\,Z\,,[X\,,\,Y]\,]} \alpha &= \pounds_Z \pounds_X \pounds_Y \alpha - \pounds_Z \pounds_Y \pounds_X \alpha \\ &\quad - \pounds_X \pounds_Y \pounds_Z \alpha + \pounds_Y \pounds_X \pounds_Z \alpha\,, \end{aligned}$$

and summing over cyclic permutations immediately verifies that

$$\pounds_{[\,Z\,,[X\,,\,Y]\,]}\, \alpha + \pounds_{[\,X\,,[Y\,,\,Z]\,]}\, \alpha + \pounds_{[\,Y\,,[Z\,,\,X]\,]}\, \alpha = 0\,.$$

This is the ***Jacobi identity for the Lie derivative***.

(d) The product rule

$$\pounds_X (Y \lrcorner \alpha) = (\pounds_X Y) \lrcorner \alpha + Y \lrcorner (\pounds_X \alpha)\,,$$

found in part *(d)*, subpart *(iv)* of the previous problem has already solved this part, since $\pounds_X Y = [X,\,Y]$ allows us to rearrange this product rule as

$$[X,\,Y] \lrcorner \alpha = \pounds_X (Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)\,,$$

as required, for any arbitrary k-form α.

(e) From formula *(b)* we have

$$
\begin{aligned}
[X\,,\,Y] \,\lrcorner\, \alpha &= \pounds_X (Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \\
&= d(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) + X \,\lrcorner\, d(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \\
&= d(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) + X \,\lrcorner\, (\pounds_Y \alpha - Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \\
&= d(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) \\
&\text{since} \quad \pounds_X \alpha = 0 = \pounds_Y \alpha \text{ for divergenceless vector fields } X \text{ and } Y \\
&\text{and} \quad d\alpha = 0 \text{ for a top form } \alpha
\end{aligned}
$$

This is equivalent to the vector calculus formula

$$
(\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X})\cdot d\mathbf{S} = -\,\mathrm{curl}\,(\mathbf{X}\times\mathbf{Y})\cdot d\mathbf{S} \quad \text{for} \quad \nabla\cdot\mathbf{X} = 0 = \nabla\cdot\mathbf{Y}
$$

▲