# 3 M3-4-5 A34 Assessed Problems # 3 $\boxed{\textbf{Mar 2012}}$

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

**Exercise 3.1. Exterior calculus operations**

**Vector notation for differential basis elements:** *One denotes differential basis elements $dx^i$ and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$, in vector notation as*

$$\begin{aligned} d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\ d\mathbf{S} &= (dS_1, dS_2, dS_3) \\ &:= (dx^2 \wedge dx^3,\, dx^3 \wedge dx^1,\, dx^1 \wedge dx^2)\,, \\ dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,, \\ d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,. \end{aligned}$$

*(a)* ***Vector algebra operations***

*(i) Show that contraction with the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ recovers the following familiar operations among vectors*

$$\begin{aligned} X \lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\ X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,, \\ (or,\quad X \lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k) \\ Y \lrcorner\, X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,, \\ X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,, \\ Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,, \\ Z \lrcorner\, Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot\mathbf{Z}\,. \end{aligned}$$

*(ii) Show that these are consistent with*

$$X \lrcorner\,(\alpha\wedge\beta) = (X \lrcorner\,\alpha)\wedge\beta + (-1)^k\alpha\wedge(X \lrcorner\,\beta)\,,$$

*for a $k$-form $\alpha$.*

*(iii) Use (ii) to compute $Y \lrcorner\, X \lrcorner\,(\alpha\wedge\beta)$ and $Z \lrcorner\, Y \lrcorner\, X \lrcorner\,(\alpha\wedge\beta)$.*

*(b)* ***Exterior derivative examples in vector notation***
*Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation*

$$\begin{aligned} df &= f_{,j}\,dx^j =: \nabla f\cdot d\mathbf{x} \\ 0 = d^2 f &= f_{,jk}\,dx^k\wedge dx^j \\ df\wedge dg &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k =: (\nabla f\times\nabla g)\cdot d\mathbf{S} \\ df\wedge dg\wedge dh &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k\wedge h_{,l}\,dx^l =: (\nabla f\cdot\nabla g\times\nabla h)\,d^{\,3}x \end{aligned}$$

*Likewise, show that*

$$\begin{aligned} d(\mathbf{v}\cdot d\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S} \\ d(\mathbf{A}\cdot d\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\,d^{\,3}x\,. \end{aligned}$$

*(c) Verify the compatibility condition $d^2 = 0$ for these forms as*

$$\begin{aligned} 0 = d^2 f = d(\nabla f\cdot d\mathbf{x}) &= (\mathrm{curl\,grad}\,f)\cdot d\mathbf{S}\,, \\ 0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S}\big) &= (\mathrm{div\,curl}\,\mathbf{v})\,d^{\,3}x\,. \end{aligned}$$

*(d) Verify the exterior derivatives of the following contraction formulas for* $X = \mathbf{X}\cdot\nabla$

*(i)* $d(X \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) = d(\mathbf{X}\cdot\mathbf{v}) = \nabla(\mathbf{X}\cdot\mathbf{v})\cdot d\mathbf{x}$

*(ii)* $d(X \,\lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}) = d(\boldsymbol{\omega}\times\mathbf{X}\cdot d\mathbf{x}) = \mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X})\cdot d\mathbf{S}$

*(iii)* $d(X \,\lrcorner\, f\,d^{\,3}x) = d(f\mathbf{X}\cdot d\mathbf{S}) = \mathrm{div}\,(f\mathbf{X})\,d^{\,3}x$

*(e) Use Cartan's formula,*

$$\pounds_X\alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$$

*for a* $k-$*form* $\alpha$, $k = 0,1,2,3$ *in* $\mathbb{R}^3$ *to verify the Lie derivative formulas:*

*(i)* $\pounds_X f = X \,\lrcorner\, df = \mathbf{X}\cdot\nabla f$

*(ii)* $\pounds_X\,(\mathbf{v}\cdot d\mathbf{x}) = \big(-\,\mathbf{X}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot d\mathbf{x}$

*(iii)* $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S}) = \big(\mathrm{curl}\,(\boldsymbol{\omega}\times\mathbf{X}) + \mathbf{X}\,\mathrm{div}\,\boldsymbol{\omega}\big)\cdot d\mathbf{S}$
$\qquad = \big(-\,\boldsymbol{\omega}\cdot\nabla\mathbf{X} + \mathbf{X}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{X}\big)\cdot d\mathbf{S}$

*(iv)* $\pounds_X(f\,d^{\,3}x) = (\mathrm{div}\,f\mathbf{X})\,d^{\,3}x$

*(v) Derive these formulas from the dynamical definition of Lie derivative.*

*(f) Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:*

*(i)* $\pounds_{fX}\alpha = f\pounds_X\alpha + df\wedge(X \,\lrcorner\, \alpha)$

*(ii)* $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$

*(iii)* $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$

*(iv)* $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$

*(v)* $\pounds_X(\alpha\wedge\beta) = (\pounds_X\alpha)\wedge\beta + \alpha\wedge\pounds_X\beta$

**Exercise 3.2.** ***Operations among vector fields***

*The Lie derivative of one vector field by another is called the* ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y\cdot X - \nabla X\cdot Y = -\,\pounds_Y X$$

*In components, the Jacobi-Lie bracket is*

$$[X\,,\,Y] = \Big[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\Big] = \left(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\right)\frac{\partial}{\partial x^l}$$

*The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,*

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

*Verify the following formulas*

*(a)* $X \,\lrcorner\, (Y \,\lrcorner\, \alpha) = -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha)$

*(b)* $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$*, for zero-forms (functions) and one-forms.*

*(c)* $\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$*, as a result of (b). Use (c) to verify the Jacobi identity.*

*(d) Verify formula (b) for arbitrary* $k-$*forms.*