2 M3-4-5 A34 Assessed Problems # 2 Mar 2012

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 2.1. ***Adjoint and coadjoint actions of semidirect product*** $(S,T)\circledS\mathbb{R}$

Compute the adjoint and coadjoint actions for the semidirect-product group $(S,T)\circledS\mathbb{R}$ *obtained from the action of scaling* S *and translations* $T\in\mathbb{R}$ *on the real line.*

The group composition rule is

$$(\tilde{S},\tilde{v})(S,v)=(\tilde{S}S,\tilde{S}v+\tilde{v})\,, \tag{1}$$

which can be represented by multiplication of 2×2 *matrices. That is, the action of* G *on* $\mathbb{R}^2$ *has a matrix representation, given by*

$$(S,v)\mapsto\begin{pmatrix} S & v \\ 0 & 1\end{pmatrix}, \tag{2}$$

where $S\in\mathbb{R}$*. The matrix multiplication*

$$\begin{pmatrix} \tilde{S} & \tilde{v} \\ 0 & 1\end{pmatrix}\begin{pmatrix} S & v \\ 0 & 1\end{pmatrix}=\begin{pmatrix} \tilde{S}S & \tilde{S}v+\tilde{v} \\ 0 & 1\end{pmatrix} \tag{3}$$

agrees with the notation for $SE(2)=SO(2)\circledS\mathbb{R}^2$ *and* $SL(2,\mathbb{R})\circledS\mathbb{R}^2$*, except that rotations or* $SL(2,\mathbb{R})$ *actions on* $\mathbb{R}^2$ *in those cases are replaced here by a simple scaling of the real line. The inverse group element is given by*

$$(\tilde{S},\tilde{v})^{-1}=(\tilde{S}^{-1},-\tilde{S}^{-1}\tilde{v}) \tag{4}$$

and the identity element is $(S,v)_{\rm Id}=(1,0)$*.*

The semidirect-product group action of (S,T) *on* $\mathbb{R}$ *may be represented by the action of the* 2×2 *matrix representation of this group in (2) on an* extended *vector* $(r,1)^T\in$ *as*

$$\begin{pmatrix} S & v \\ 0 & 1\end{pmatrix}\begin{pmatrix} r \\ 1\end{pmatrix}=\begin{pmatrix} S\,r+v \\ 1\end{pmatrix},$$

The Lie group $G=(S,T)\circledS\mathbb{R}$ *has two parameters, the scale factor* $S\in\mathbb{R}$ *and the translation* $v\in\mathbb{R}$*.*

Problem statement

(a) *Derive the* AD*,* Ad *and* ad *actions for* $(S,T)\circledS\mathbb{R}$*. Use the notation* $(S'(0),v'(0))=(\sigma,\nu)$ *for Lie algebra elements.*

(b) *Introduce a natural pairing in which to define the dual Lie algebra and derive its* Ad^* *and* ad^* *actions. Denote elements of the dual Lie algebra as* (α,β)*.*

(c) *Compute its coadjoint motion equations as Euler-Poincaré equations.*

(d) *Legendre transform and identify the corresponding Lie-Poisson brackets*

(e) *Choose a Hamiltonian and solve its coadjoint motion equations.*

Exercise 2.2. The Clebsch Momentum Map

Consider the Clebsch constrained variational principle

$$\delta S = 0\,, \quad S = \int_a^b \ell(\xi, q) + \langle\!\langle\, p\,,\, \dot q - \Phi_\xi(q)\,\rangle\!\rangle\, dt\,,$$

where $\ell : \mathfrak{g} \times Q \to \mathbb{R}$ is the Lagrangian and $\Phi_\xi(q) \in TQ$ is the infinitesimal action of the Lie algebra $\mathfrak{g}$ on the manifold Q. The Lagrange multiplier $p \in T^*Q$ enforces the Clebsch constraint, $\dot q - \Phi_\xi(q) = 0$ at $q \in Q$, by using the pairing, $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : T^*Q \times TQ \to \mathbb{R}$ (denoted with double brackets).

Define the momentum map $J(p, q) : T^*Q \to \mathfrak{g}$ by

$$J^\xi(p, q) : \langle\, J(p, q)\,,\, \xi\,\rangle = \langle\!\langle\, p_q\,,\, \Phi_\xi(q)\,\rangle\!\rangle$$

in terms of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ (denoted with single brackets).

Problem statement:

(a) Derive the equations arising from the Clebsch approach with constrained variations of ξ, q and p in the action S above. Assume that the variation δq vanishes at the endpoints $t = a$ and $t = b$. Express your results in terms of $J(p, q)$ and $J^\xi(p, q)$.

(b) Prove the canonical (p, q) Poisson bracket relation,

$$\big\{\, J^\xi\,,\, J^\eta \big\} = J^{[\xi\,,\,\eta]}(p, q)\,,$$

where $[\xi\,,\,\eta]$ is the Lie algebra bracket.

Hint: Choose a 'Euclidean' basis for T^*Q in which it makes sense to write

$$J^\xi(p, q) = \langle\!\langle\, p_q\,,\, \Phi_\xi(q)\,\rangle\!\rangle = p_j \Phi^j_\xi(q)$$

(c) Show that the momentum map $(q, p) \to J$ is Poisson. That is, $\big\{F \circ J, H \circ J\big\} = \big\{F, H\big\} \circ J$, for smooth functions F and H.

(d) Derive the equation of motion for J from the canonical Hamiltonian equations for $J^\xi(q, p)$ given Hamiltonian $H(J(q, p))$ and express the $\dot J$ equation in Lie-Poisson Hamiltonian form. That is, derive the $\dot J$ equation expressed in terms of a Lie-Poisson bracket.

(e) Use the Clebsch constrained variational principle to compute the momentum map when:

(i) $\Phi_\xi(q) = \xi q$ for a skew-symmetric 3×3 matrix $\xi \in \mathfrak{so}(3)$ and a vector $q \in \mathbb{R}^3$.

(ii) $\Phi_\xi(q) = [\,\xi,\, q\,]$ for skew-Hermitian $n \times n$ matrix $\xi \in \mathfrak{su}(n)$ and q an $n \times n$ Hermitian matrix.

(iii) $\Phi_\xi(q) = -\mathcal{L}_\xi q$ for q in manifold Q and $\mathcal{L}_\xi q$ the Lie derivative of $q \in Q$ with respect to $\xi \in \mathfrak{g}$.

Exercise 2.3. (Clebsch approach for motion on $T^*(G \times V)$) versus EP for $T^*(G \circledS V)$)

In the first part of this problem, one assumes an action $G \times V \to V$ of the Lie group G on the vector space V that may represent a feature of the potential energy of the system. The Lagrangian then takes the form $L : TG \times V \to \mathbb{R}$. We assume that the Lagrangian is left invariant under the isotropy subgroup G_V that leaves invariant the elements of V.

One then computes the variations of the Clebsch-constrained action integral

$$S(\xi, q, \dot{q}, p) = \int_a^b \Big[l(\xi, q) + \Big\langle\!\Big\langle \, p \,,\, \dot{q} + \pounds_\xi q \,\Big\rangle\!\Big\rangle \Big] \, dt$$

for the reduced left-invariant Lagrangian $l(\xi, q) : \mathfrak{g} \times V \to \mathbb{R}$. The point is to determine the effects of the presence of $q \in V$ in the Euler–Poincaré equations for motion on $T^*(G \times V)$) versus $T^*(G \circledS V)$).

These steps will lead you through the problem.

(a) *Start by showing that stationarity of S implies the following set of equations:*

$$\frac{\delta l}{\delta \xi} = p \diamond q \,, \quad \dot{q} = - \, \pounds_\xi q \,, \quad \dot{p} = \pounds_\xi^T p + \frac{\delta l}{\delta q} \,.$$

(b) *Transform to the variable $\delta l / \delta \xi = p \diamond q$ and derive its equation of motion on the space $\mathfrak{g}^* \times V$,*

(c) *Perform the Legendre transformation to derive the Lie–Poisson Hamiltonian formulation corresponding to $l(\xi, q)$.*

(d) *Compute the Euler–Poincaré equations on the space $\mathfrak{g}^* \times V^*$ for the semidirect product group $S = G \circledS V$ and discuss the differences between those equations and the Euler–Poincaré equations in the previous part.*