1 M3-4-5 A34 Assessed Problems # 1 Feb 2012

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 1.1. Pauli matrices

Problem statement

The Pauli matrices are given by

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

(a) *Verify the formula*

$$\sigma_a \sigma_b = \delta_{ab}\, \sigma_0 + i\epsilon_{abc}\sigma_c \quad \textit{for} \quad a, b, c = 1, 2, 3, \tag{2}$$

where ϵ_{abc} is the totally antisymmetric tensor density with $\epsilon_{123} = 1$.

(b) *Verify by antisymmetry of ϵ_{abc} the* ***commutator relation*** *for the Pauli matrices*

$$[\,\sigma_a\,,\,\sigma_b\,] := \sigma_a\sigma_b - \sigma_b\sigma_a = 2i\epsilon_{abc}\sigma_c \quad \textit{for} \quad a, b, c = 1, 2, 3, \tag{3}$$

and their ***anticommutator relation***

$$\{\,\sigma_a\,,\,\sigma_b\,\}_+ := \sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}\sigma_0 \quad \textit{for} \quad a, b = 1, 2, 3. \tag{4}$$

(c) *Verify the decomposition of a vector $\mathbf{q} \in \mathbb{R}^3$ in Pauli matrices as*

$$\mathbf{q}\,\sigma_0 = (\mathbf{q}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - i\mathbf{q}\times\boldsymbol{\sigma}\,, \tag{5}$$

where one denotes

$$\mathbf{q}\cdot\boldsymbol{\sigma} := \sum_{a=1}^{3} q_a\sigma_a \quad \textit{and} \quad (\mathbf{q}\times\boldsymbol{\sigma})_c := \sum_{a,b=1}^{3} q_a\sigma_b\epsilon_{abc}\,.$$

(d) *Verify that*

$$-\,|\mathbf{q}\times\boldsymbol{\sigma}|^2 = 2|\mathbf{q}|^2\sigma_0 = 2(\mathbf{q}\cdot\boldsymbol{\sigma})^2.$$

(e) *Verify the commutation relation*

$$[\,\mathbf{p}\cdot\boldsymbol{\sigma},\,\mathbf{q}\cdot\boldsymbol{\sigma}\,] = 2i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}$$

for three-vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$.

Answer These are all direct verifications. ▲

Exercise 1.2. Quaternions

According to Hamilton (1843), a quaternion $\mathfrak{q} = [q_0,\, \mathbf{q}] \in \mathbb{H}$ may be written as

$$\mathfrak{q} = q_0 J_0 + q_1 \mathbb{J}_1 + q_2 \mathbb{J}_2 + q_3 \mathbb{J}_3$$

where $\mathbb{J}_k^2 = -J_0 = \mathbb{J}_1\mathbb{J}_2\mathbb{J}_3$ for $k = 1, 2, 3$, and the multiplication rule for two quaternions,

$$\mathfrak{q} = [q_0,\, \mathbf{q}\,] \quad \textit{and} \quad \mathfrak{r} = [r_0, \mathbf{r}\,] \in \mathbb{H}\,,$$

may be defined in vector notation with $\mathbf{q}, \mathbf{r} \in \mathbb{R}^3$ as

$$\mathfrak{q}\mathfrak{r} = [q_0,\, \mathbf{q}\,][r_0,\, \mathbf{r}\,] = [\, q_0 r_0 - \mathbf{q}\cdot\mathbf{r}\,,\, q_0\mathbf{r} + r_0\mathbf{q} + \mathbf{q}\times\mathbf{r}\,]\;. \tag{6}$$

(a) Verify that the Pauli matrix relation (2) and the isomorphism

$$\mathfrak{q} = [q_0,\, \mathbf{q}] = q_0\sigma_0 - i\mathbf{q}\cdot\boldsymbol{\sigma}\,, \quad \textit{with} \;\; \mathbf{q}\cdot\boldsymbol{\sigma} := \sum_{a=1}^{3} q_a\sigma_a\,, \tag{7}$$

recover the multiplication rule for quaternions.

That is, verify that identifying a quaternion basis as

$$\mathbb{J}_0 = \sigma_0\,, \quad \textit{and} \;\; \mathbb{J}_a = -i\sigma_a\,, \quad \textit{where} \;\; a = 1, 2, 3,$$

recovers the basic quaternionic multiplication rules.

(b) Show that the product of a quaternion $\mathfrak{r} = [r_0, \mathbf{r}]$ with a unit quaternion $\hat{\mathfrak{q}} = [q_0, \mathbf{q}]$, whose inverse is $\hat{\mathfrak{q}}^* = [q_0, -\mathbf{q}]$ (prove that $\hat{\mathfrak{q}}\hat{\mathfrak{q}}^* = [1, 0]$), satisfies

$$\begin{aligned} \mathfrak{r}\hat{\mathfrak{q}}^* &= \big[\mathfrak{r}\cdot\hat{\mathfrak{q}},\, -r_0\mathbf{q} + q_0\mathbf{r} + \mathbf{q}\times\mathbf{r}\big]\,, \\ \hat{\mathfrak{q}}\,\mathfrak{r}\,\hat{\mathfrak{q}}^* &= \big[r_0|\hat{\mathfrak{q}}|^2, \mathbf{r} + 2q_0\mathbf{q}\times\mathbf{r} + 2\mathbf{q}\times(\mathbf{q}\times\mathbf{r})\big]\,, \end{aligned}$$

where $\mathfrak{r}\cdot\hat{\mathfrak{q}} := r_0 q_0 + \mathbf{r}\cdot\mathbf{q}$ and $|\hat{\mathfrak{q}}|^2 := \hat{\mathfrak{q}}\cdot\hat{\mathfrak{q}} = {q_0}^2 + \mathbf{q}\cdot\mathbf{q} = 1$ for products with unit quaternions.

(c) For $\mathfrak{q}^* = [q_0,\, -\mathbf{q}]$, such that $\mathfrak{q}^*\mathfrak{q} = \mathbb{J}_0|\mathfrak{q}|^2$, verify the quaternion identity

$$2\mathfrak{q}^* = -\mathbb{J}_0\mathfrak{q}\mathbb{J}_0^* + \mathbb{J}_1\mathfrak{q}\mathbb{J}_1^* + \mathbb{J}_2\mathfrak{q}\mathbb{J}_2^* + \mathbb{J}_3\mathfrak{q}\mathbb{J}_3^*\,.$$

(d) What does this identity mean geometrically? Does the complex conjugate z^* for $z \in \mathbb{C}$ satisfy such an identity? Prove it.

(e) Write De Moivre's theorem for $z \in \mathbb{C}$. Write the corresponding theorem for the quaternion $\mathfrak{q} \in \mathbb{H}$.

(f) Prove that any pure quaternion is in the conjugacy class of $\mathbb{J}_3 = [\,0,\, \hat{\mathbf{k}}\,]$ with $\hat{\mathbf{k}} = (0, 0, 1)^T$ under the Ad-action of a unit quaternion.

(g) Verify the Euler–Rodrigues formula (3.26) in the text by a direct computation using quaternionic multiplication.

(h) Compute the Cayley transform for a quaternion. Namely, for a quaternion $\mathfrak{q} = [1,\, \mathbf{q}]$, compute

$$\mathfrak{p} = [1,\, \mathbf{q}]([1,\, \mathbf{q}]^*)^{-1}$$

(i) Compute the square root of a quaternion $\mathfrak{q} = [1,\, \mathbf{q}]$.

Answer

(a) Verified by a direct calculation.

(b) Verified by a direct calculation.

(c) Verified by a direct calculation.

(d) The conjugate quaternion is the sum of the results of rotations by $\pi/2$ of the original quaternion around each of the three orthogonal axes plus a reflection of it.

The complex conjugate z^* for $z \in \mathbb{C}$ is not an analytic function of z, let alone an algebraic function of it.

(e) De Moivre's theorem for $z \in \mathbb{C}$ and M an integer is

$$(\cos\theta + i\sin\theta)^M = (\cos M\theta + i\sin M\theta)$$

The corresponding theorem for a unit quaternion in CK parameters is

$$\Big(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\,\hat{\mathbf{n}}\Big)^{2M} = (\cos M\theta, \sin M\theta\,\hat{\mathbf{n}})$$

(f) This may be shown immediately by using the isomorphism of quaternions with $SU(2)$ and computing the $SU(2)$ multiplication for $|a_1|^2 + |a_2|^2 = 1$, to find

$$\begin{bmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} a_1^* & a_2^* \\ -a_2 & a_1 \end{bmatrix} = \begin{bmatrix} -in_3 & -in_1 + n_2 \\ -in_1 - n_2 & in_3 \end{bmatrix},$$

where (n_1, n_2, n_3) are the components of the Hopf fibration.
In other words,

$$-ig\sigma_3 g^\dagger = -i\mathbf{n}\cdot\boldsymbol{\sigma},$$

in which $g^\dagger = g^{-1} \in SU(2)$ and $|\mathbf{n}|^2 = 1$. That is, any pure ***unit*** quaternion is in this conjugacy class. Hence, conjugating the pure unit quaternion along the z-axis $[0,\,\hat{\mathbf{k}}]$ by the other unit quaternions yields the entire unit two-sphere S^2.

(g) Verified by a direct calculation.

(h) Cayley transform

$$\begin{aligned}\mathfrak{p} = [1,\,\mathbf{q}]([1,\,\mathbf{q}]^*)^{-1} &= \frac{[1,\,\mathbf{q}]^2}{1+|\mathbf{q}|^2} = \frac{[\,1-|\mathbf{q}|^2,\,2\mathbf{q}\,]}{1+|\mathbf{q}|^2} = \frac{[1,\,\mathbf{q}]^2}{|\,[1,\,\mathbf{q}]\,|^2} \\ &= [\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\,\hat{\mathbf{n}}]^2 = [\cos^2(\theta/2) - \sin^2(\theta/2),\, 2\cos(\theta/2)\sin(\theta/2)\hat{\mathbf{n}}] = [\cos\theta, \sin\theta\,\hat{\mathbf{n}}]\end{aligned}$$

for $[1,\mathbf{q}] = (1+|\mathbf{q}|^2)^{1/2}(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\,\hat{\mathbf{n}})$. So the Cayley transform of a quaternion is the square of its associated unit quaternion. The same is true about complex numbers. That is, the square root of a unit quaternion is given by

$$[\cos\theta,\,\sin\theta\hat{\mathbf{n}}]^{1/2} = \pm[\cos(\theta/2),\,\sin(\theta/2)\hat{\mathbf{n}}]$$

(i) Write the CK form of the quaternion

$$[1,\,\mathbf{q}] = (1+|\mathbf{q}|^2)^{1/2}\Big(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\,\hat{\mathbf{n}}\Big)$$

Its square root is

$$[1,\,\mathbf{q}]^{1/2} = \pm(1+|\mathbf{q}|^2)^{1/4}\Big(\cos\frac{\theta}{4}, \sin\frac{\theta}{4}\,\hat{\mathbf{n}}\Big)$$

▲

Exercise 1.3. ***Rigid body motion (and EP equation) in quaternions***

(a) Compute the adjoint and coadjoint actions AD, Ad, ad, Ad and ad* for $SU(2)$ using quaternions.*

(b) Formulate rigid body dynamics as an EP problem in quaternions. (Use the Rodrigues formula). For this, state and prove Hamilton's principle for the rigid body in quaternionic form.

Answer

(a) One associates the matrix $Q \in SU(2)$ with the unit quaternion $\hat{\mathfrak{q}} = [q_0, \mathbf{q}]$ ($|\hat{\mathfrak{q}}|^2 = q_0^2 + \mathbf{q}\cdot\mathbf{q} = 1$) by using the Pauli matrix basis, for which

$$
\begin{aligned}
\hat{\mathfrak{q}} &= q_0\sigma_0 - i\mathbf{q}\cdot\boldsymbol{\sigma} \\
&= \begin{bmatrix} q_0 - iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & q_0 + iq_3 \end{bmatrix} = Q \in SU(2).
\end{aligned}
\tag{8}
$$

$Q \in SU(2)$ is a unitary 2×2 matrix ($QQ^\dagger = Id$) with unit determinant ($\det Q = 1$). Likewise, one associates the skew Hermitian matrix $\tilde{\boldsymbol{q}} \in \mathfrak{su}(2)$ with the pure quaternion $\mathfrak{q} = [0,\, \mathbf{q}]$ via the tilde map

$$
\begin{aligned}
\mathbf{q} \in \mathbb{R}^3 \mapsto [0, \mathbf{q}] = -i\,\mathbf{q}\cdot\boldsymbol{\sigma} &= -i\sum_{j=1}^{3} q_j\sigma_j \\
&= \begin{bmatrix} -iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & iq_3 \end{bmatrix} =: \tilde{\boldsymbol{q}} \in \mathfrak{su}(2)\,.
\end{aligned}
\tag{9}
$$

The tilde map (9) is a Lie algebra isomorphism between $\mathbb{R}^3$ with the cross product of vectors and the Lie algebra $su(2)$ of 2×2 skew-Hermitian traceless matrices.

- AD (conjugacy of quaternions),

$$\mathrm{AD}_{\hat{\mathfrak{q}}}\hat{\mathfrak{r}} := \hat{\mathfrak{q}}\,\hat{\mathfrak{r}}\,\hat{\mathfrak{q}}^* = QRQ^\dagger = \mathrm{AD}_Q R\,,$$

- Ad (conjugacy of angular velocities),

$$\mathrm{Ad}_{\hat{\mathfrak{q}}}\boldsymbol{\Omega} = \hat{\mathfrak{q}}\,\boldsymbol{\Omega}\,\hat{\mathfrak{q}}^* = Q\tilde{\boldsymbol{\Omega}}Q^\dagger = \mathrm{Ad}_Q\tilde{\boldsymbol{\Omega}}\,,$$

- ad (commutator of angular velocities),

$$\mathrm{ad}_{\boldsymbol{\Omega}}\boldsymbol{\Xi} = \mathrm{Im}(\boldsymbol{\Omega}\,\boldsymbol{\Xi}) := \tfrac{1}{2}(\boldsymbol{\Omega}\,\boldsymbol{\Xi} - \boldsymbol{\Xi}\,\boldsymbol{\Omega}) = \tfrac{1}{2}(\tilde{\boldsymbol{\Omega}}\,\tilde{\boldsymbol{\Xi}} - \tilde{\boldsymbol{\Xi}}\,\tilde{\boldsymbol{\Omega}}) = \tfrac{1}{2}\mathrm{ad}_{\tilde{\boldsymbol{\Omega}}}\tilde{\boldsymbol{\Xi}}\,.$$

The pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathbb{H}\times\mathbb{H} \mapsto \mathbb{R}$

$$\langle\,\hat{\mathfrak{p}}\,,\,\hat{\mathfrak{q}}\,\rangle = \mathrm{Re}(\hat{\mathfrak{p}}\hat{\mathfrak{q}}^*) = \tfrac{1}{2}(\hat{\mathfrak{p}}\hat{\mathfrak{q}}^* + \hat{\mathfrak{q}}\hat{\mathfrak{p}}^*) = \tfrac{1}{4}\mathrm{tr}(PQ^\dagger + QP^\dagger) = \tfrac{1}{2}\mathrm{tr}(PQ^\dagger)\,. \tag{10}$$

also allows one to define the corresponding dual operations. These are

- coAD:

$$
\begin{aligned}
\langle\,\mathrm{AD}^*_{\hat{\mathfrak{q}}}\,\hat{\mathfrak{s}}\,,\hat{\mathfrak{r}}\,\rangle &= \langle\,\hat{\mathfrak{s}}\,,\,\mathrm{AD}_{\hat{\mathfrak{q}}}\hat{\mathfrak{r}}\,\rangle \\
&= \tfrac{1}{2}\mathrm{tr}(S^\dagger QRQ^\dagger) = \tfrac{1}{2}\mathrm{tr}((Q^\dagger SQ)^\dagger R) = \tfrac{1}{2}\mathrm{tr}((\mathrm{AD}^*_Q S)^\dagger R)
\end{aligned}
$$

  Thus, $\mathrm{AD}^*_Q S = Q^\dagger SQ$ for $Q, S \in SU(2)$

- coAd:

$$
\begin{aligned}
\langle\,\mathrm{Ad}^*_{\hat{\mathfrak{q}}}\,\boldsymbol{\Xi}\,,\boldsymbol{\Omega}\,\rangle = \langle\,\boldsymbol{\Xi}\,,\,\mathrm{Ad}_{\hat{\mathfrak{q}}}\,\boldsymbol{\Omega}\,\rangle &= \langle\,\breve{\boldsymbol{\Xi}}\,,\,\mathrm{Ad}_Q\tilde{\boldsymbol{\Omega}}\,\rangle \\
&= \tfrac{1}{2}\mathrm{tr}(\breve{\boldsymbol{\Xi}}^\dagger Q\tilde{\boldsymbol{\Omega}}Q^\dagger) = \tfrac{1}{2}\mathrm{tr}((Q^\dagger\breve{\boldsymbol{\Xi}}Q)^\dagger\tilde{\boldsymbol{\Omega}}) = \tfrac{1}{2}\mathrm{tr}((\mathrm{Ad}^*_Q\breve{\boldsymbol{\Xi}})^\dagger\tilde{\boldsymbol{\Omega}})
\end{aligned}
$$

  Thus, $\mathrm{Ad}^*_Q\breve{\boldsymbol{\Xi}} = Q^\dagger\breve{\boldsymbol{\Xi}}Q$ for $Q \in SU(2)$ and $\breve{\boldsymbol{\Xi}} \in \mathfrak{su}(2)^*$.

- coad:

$$\begin{aligned}\langle\, \mathrm{ad}^*_{\boldsymbol{\Omega}} \boldsymbol{\Upsilon}\,,\, \boldsymbol{\Xi}\,\rangle &= \langle\, \boldsymbol{\Upsilon}\,,\, \mathrm{ad}_{\boldsymbol{\Omega}}\, \boldsymbol{\Xi}\,\rangle = \tfrac{1}{2}\mathrm{tr}(\breve{\boldsymbol{\Upsilon}}^\dagger \mathrm{ad}_{\widetilde{\boldsymbol{\Omega}}} \widetilde{\boldsymbol{\Xi}}) \\ &= \tfrac{1}{2}\mathrm{tr}(\breve{\boldsymbol{\Upsilon}}^\dagger (\widetilde{\boldsymbol{\Omega}}\, \widetilde{\boldsymbol{\Xi}} - \widetilde{\boldsymbol{\Xi}}\, \widetilde{\boldsymbol{\Omega}})) = \tfrac{1}{2}\mathrm{tr}([\widetilde{\boldsymbol{\Omega}}^\dagger,\, \breve{\boldsymbol{\Upsilon}}]^\dagger \widetilde{\boldsymbol{\Xi}}) = \tfrac{1}{2}\mathrm{tr}((\mathrm{ad}^*_{\widetilde{\boldsymbol{\Omega}}} \breve{\boldsymbol{\Upsilon}})^\dagger \widetilde{\boldsymbol{\Xi}})\end{aligned}$$

Thus, for $\breve{\boldsymbol{\Upsilon}} \in \mathfrak{su}(2)^*$ and $\widetilde{\boldsymbol{\Omega}} \in \mathfrak{su}(2)$, we have $\mathrm{ad}^*_{\widetilde{\boldsymbol{\Omega}}} \breve{\boldsymbol{\Upsilon}} = [\widetilde{\boldsymbol{\Omega}}^\dagger,\, \breve{\boldsymbol{\Upsilon}}] = -\,[\widetilde{\boldsymbol{\Omega}},\, \breve{\boldsymbol{\Upsilon}}]$, the last change of sign because $\widetilde{\boldsymbol{\Omega}}^\dagger = -\,\widetilde{\boldsymbol{\Omega}}$.

(b) The body angular velocity is defined as $\boldsymbol{\Omega} = 2\hat{\mathfrak{q}}^*\dot{\hat{\mathfrak{q}}}$ for a pure quaternion $\mathfrak{q} = [0, \mathbf{q}]$. According to the Cayley–Klein variational formula in the text (page 115), the variation of the pure quaternion $\boldsymbol{\Omega} = 2\hat{\mathfrak{q}}^*\dot{\hat{\mathfrak{q}}}$ corresponding to body angular velocity in Cayley–Klein parameters satisfies the identity

$$\boldsymbol{\Omega}' - \dot{\boldsymbol{\Xi}} = \tfrac{1}{2}(\boldsymbol{\Omega}\,\boldsymbol{\Xi} - \boldsymbol{\Xi}\,\boldsymbol{\Omega}) = \mathrm{Im}(\boldsymbol{\Omega}\,\boldsymbol{\Xi})\,, \tag{11}$$

where $\boldsymbol{\Xi} := 2\hat{\mathfrak{q}}^*\hat{\mathfrak{q}}'$ and $(\,\cdot\,)'$ denotes variation.

Using the quaternionic pairing (10), we define the Lagrangian as $\ell(\boldsymbol{\Omega}) = \frac{1}{2}\langle \boldsymbol{\Omega}, \mathbb{A}\boldsymbol{\Omega}\rangle$ for a symmetric operator $\mathbb{A}$.

Hamilton's principle is then $\delta S = 0$ with $S = \int_a^b \ell(\boldsymbol{\Omega})\, dt$, and with homogeneous endpoint conditions.

One denotes the quantity $\mathbf{M} := \mathbb{A}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbb{A}$ and computes

$$\begin{aligned}\delta S &= \frac{1}{2}\int_a^b \langle \delta\boldsymbol{\Omega}, \mathbb{A}\boldsymbol{\Omega}\rangle + \langle \boldsymbol{\Omega}, \mathbb{A}\delta\boldsymbol{\Omega}\rangle\, dt \\ &= \frac{1}{2}\int_a^b \langle \delta\boldsymbol{\Omega}, \mathbf{M}\rangle\, dt = \frac{1}{2}\int_a^b \langle \dot{\boldsymbol{\Xi}} + \tfrac{1}{2}(\boldsymbol{\Omega}\,\boldsymbol{\Xi} - \boldsymbol{\Xi}\,\boldsymbol{\Omega}), \mathbf{M}\rangle\, dt \\ &= \frac{1}{2}\int_a^b \langle \boldsymbol{\Xi},\, -\dot{\mathbf{M}} + \tfrac{1}{2}(\mathbf{M}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{M})\rangle\, dt + \langle \boldsymbol{\Xi},\, \mathbf{M}\rangle\Big|_a^b .\end{aligned}$$

Upon invoking the homogeneous endpoint conditions one obtains the rigid body equations for quaternions,

$$\dot{\mathbf{M}} = \tfrac{1}{2}(\mathbf{M}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{M}) = \mathrm{Im}(\mathbf{M}\boldsymbol{\Omega})\,.$$

▲