1 M3-4-5 A34 Assessed Problems # 1 Due 11am Thurs 9 Feb 2012

Please budget your time: Many of these problems are very easy, but some of the more interesting ones may become time consuming. So work steadily through them, don't wait until the last minute.

Exercise 1.1. Pauli matrices

Problem statement
The Pauli matrices are given by

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1}$$

(a) Verify the formula

$$\sigma_a \sigma_b = \delta_{ab}\, \sigma_0 + i\epsilon_{abc}\sigma_c \quad \textit{for} \quad a, b, c = 1, 2, 3, \tag{2}$$

where ϵ_{abc} is the totally antisymmetric tensor density with $\epsilon_{123} = 1$.

(b) Verify by antisymmetry of ϵ_{abc} the ***commutator relation*** *for the Pauli matrices*

$$[\,\sigma_a\,,\,\sigma_b\,] := \sigma_a\sigma_b - \sigma_b\sigma_a = 2i\epsilon_{abc}\sigma_c \quad \textit{for} \quad a, b, c = 1, 2, 3, \tag{3}$$

and their ***anticommutator relation***

$$\{\,\sigma_a\,,\,\sigma_b\,\}_+ := \sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}\sigma_0 \quad \textit{for} \quad a, b = 1, 2, 3. \tag{4}$$

(c) Verify the decomposition of a vector $\mathbf{q} \in \mathbb{R}^3$ in Pauli matrices as

$$\mathbf{q}\,\sigma_0 = (\mathbf{q}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - i\mathbf{q}\times\boldsymbol{\sigma}\,, \tag{5}$$

where one denotes

$$\mathbf{q}\cdot\boldsymbol{\sigma} := \sum_{a=1}^{3} q_a\sigma_a \quad \textit{and} \quad (\mathbf{q}\times\boldsymbol{\sigma})_c := \sum_{a,b=1}^{3} q_a\sigma_b\epsilon_{abc}\,.$$

(d) Verify that

$$-\,|\mathbf{q}\times\boldsymbol{\sigma}|^2 = 2|\mathbf{q}|^2\sigma_0 = 2(\mathbf{q}\cdot\boldsymbol{\sigma})^2.$$

(e) Verify the commutation relation

$$[\,\mathbf{p}\cdot\boldsymbol{\sigma},\,\mathbf{q}\cdot\boldsymbol{\sigma}\,] = 2i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}$$

for three-vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$.

Exercise 1.2. Quaternions

According to Hamilton (1843), a quaternion $\mathfrak{q} = [q_0,\, \mathbf{q}] \in \mathbb{H}$ *may be written as*

$$\mathfrak{q} = q_0 J_0 + q_1 \mathbb{J}_1 + q_2 \mathbb{J}_2 + q_3 \mathbb{J}_3$$

where $\mathbb{J}_k^2 = -J_0 = \mathbb{J}_1\mathbb{J}_2\mathbb{J}_3$ *for* $k = 1, 2, 3$, *and the multiplication rule for two quaternions,*

$$\mathfrak{q} = [q_0,\, \mathbf{q}\,] \quad \textit{and} \quad \mathfrak{r} = [r_0, \mathbf{r}\,] \in \mathbb{H}\,,$$

may be defined in vector notation with $\mathbf{q}, \mathbf{r} \in \mathbb{R}^3$ *as*

$$\mathfrak{q}\mathfrak{r} = [q_0,\, \mathbf{q}\,][r_0,\, \mathbf{r}\,] = [\, q_0 r_0 - \mathbf{q}\cdot\mathbf{r}\,,\, q_0\mathbf{r} + r_0\mathbf{q} + \mathbf{q}\times\mathbf{r}\,]\,. \tag{6}$$

(a) *Verify that the Pauli matrix relation (2) and the isomorphism*

$$\mathfrak{q} = [q_0,\, \mathbf{q}] = q_0\sigma_0 - i\mathbf{q}\cdot\boldsymbol{\sigma}\,, \quad \textit{with} \;\; \mathbf{q}\cdot\boldsymbol{\sigma} := \sum_{a=1}^{3} q_a\sigma_a\,, \tag{7}$$

recover the multiplication rule for quaternions.

That is, verify that identifying a quaternion basis as

$$\mathbb{J}_0 = \sigma_0\,, \quad \textit{and} \;\; \mathbb{J}_a = -i\sigma_a\,, \quad \textit{where} \;\; a = 1, 2, 3,$$

recovers the basic quaternionic multiplication rules.

(b) *Show that the product of a quaternion* $\mathfrak{r} = [r_0, \mathbf{r}]$ *with a unit quaternion* $\hat{\mathfrak{q}} = [q_0, \mathbf{q}]$, *whose inverse is* $\hat{\mathfrak{q}}^* = [q_0, -\mathbf{q}]$ *(prove that* $\hat{\mathfrak{q}}\hat{\mathfrak{q}}^* = [1, 0]$*), satisfies*

$$\begin{aligned} \mathfrak{r}\hat{\mathfrak{q}}^* &= \big[\mathfrak{r}\cdot\hat{\mathfrak{q}}, -r_0\mathbf{q} + q_0\mathbf{r} + \mathbf{q}\times\mathbf{r}\big]\,, \\ \hat{\mathfrak{q}}\,\mathfrak{r}\,\hat{\mathfrak{q}}^* &= \big[r_0|\hat{\mathfrak{q}}|^2, \mathbf{r} + 2q_0\mathbf{q}\times\mathbf{r} + 2\mathbf{q}\times(\mathbf{q}\times\mathbf{r})\big]\,, \end{aligned}$$

where $\mathfrak{r}\cdot\hat{\mathfrak{q}} := r_0 q_0 + \mathbf{r}\cdot\mathbf{q}$ *and* $|\hat{\mathfrak{q}}|^2 := \hat{\mathfrak{q}}\cdot\hat{\mathfrak{q}} = {q_0}^2 + \mathbf{q}\cdot\mathbf{q} = 1$ *for products with unit quaternions.*

(c) *For* $\mathfrak{q}^* = [q_0,\, -\mathbf{q}]$, *such that* $\mathfrak{q}^*\mathfrak{q} = \mathbb{J}_0|\mathfrak{q}|^2$, *verify the quaternion identity*

$$2\mathfrak{q}^* = -\mathbb{J}_0\mathfrak{q}\mathbb{J}_0^* + \mathbb{J}_1\mathfrak{q}\mathbb{J}_1^* + \mathbb{J}_2\mathfrak{q}\mathbb{J}_2^* + \mathbb{J}_3\mathfrak{q}\mathbb{J}_3^*\,.$$

(d) *What does this identity mean geometrically? Does the complex conjugate* z^* *for* $z \in \mathbb{C}$ *satisfy such an identity? Prove it.*

(e) *Write De Moivre's theorem for* $z \in \mathbb{C}$. *Write the corresponding theorem for the quaternion* $\mathfrak{q} \in \mathbb{H}$.

(f) *Prove that any pure quaternion is in the conjugacy class of* $\mathbb{J}_3 = [\,0,\, \hat{\mathbf{k}}\,]$ *with* $\hat{\mathbf{k}} = (0, 0, 1)^T$ *under the Ad-action of a unit quaternion.*

(g) *Verify the Euler–Rodrigues formula (3.26) in the text by a direct computation using quaternionic multiplication.*

(h) *Compute the Cayley transform for a quaternion. Namely, for a quaternion* $\mathfrak{q} = [1,\, \mathbf{q}]$, *compute*

$$\mathfrak{p} = [1,\, \mathbf{q}]([1,\, \mathbf{q}]^*)^{-1}$$

(i) *Compute the square root of a quaternion* $\mathfrak{q} = [1,\, \mathbf{q}]$.

Exercise 1.3. ***Rigid body motion (and EP equation) in quaternions***

(a) Compute the the adjoint and coadjoint actions AD, Ad, ad, Ad and ad* for $SU(2)$ using quaternions.*

(b) Formulate rigid body dynamics as an EP problem in quaternions. (Use the Rodrigues formula). For this, state and prove Hamilton's principle for the rigid body in quaternionic form.