CURTIS HOMOMORPHISMS AND THE INTEGRAL BERNSTEIN CENTER FOR GL_n

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Abstract. We describe two conjectures, one strictly stronger than the other, that give descriptions of the integral Bernstein center for $\mathrm{GL}_n(F)$ (that is, the center of the category of smooth $W(k)[\mathrm{GL}_n(F)]$-modules, for F a p-adic field and k an algebraically closed field of characteristic ℓ different from p) in terms of Galois theory. Moreover, we show that the weak version of the conjecture (for $m \leq n$), together with the strong version of the conjecture for $m < n$, implies the strong conjecture for GL_n. In a companion paper [HM] we show that the strong conjecture for $n-1$ implies the weak conjecture for n; thus the two papers together give an inductive proof of both conjectures. The upshot is a description of the Bernstein center in purely Galois theoretic terms; previous work of the author shows that this description implies the conjectural "local Langlands correspondence in families" of [EH].

1. Introduction

In [EH], Emerton and the author describe a conjectural "local Langlands correspondence in families" for the group $\mathrm{GL}_n(F)$, where F is a p-adic field. More precisely, we show that given a suitable coefficient ring A (in particular complete and local with residue characteristic ℓ different from p), and a family of Galois representations $\rho : G_F \to \mathrm{GL}_n(A)$, there is, up to isomorphism, at most one admissible $A[\mathrm{GL}_n(F)]$-module $\pi(\rho)$ that "interpolates the local Langlands correspondence across the family ρ" and satisfies certain technical hypotheses. (We refer the reader to [EH], Theorem 1.1.1 for the precise result.) We further conjecture that such a representation $\pi(\rho)$ exists for any ρ.

The paper [H2] gives an approach to the question of actually constructing $\pi(\rho)$ from ρ. The key new idea is the introduction of the integral Bernstein center, which is by definition the center of the category of smooth $W(k)[\mathrm{GL}_n(F)]$-modules. More prosaically, the integral Bernstein center is a ring Z that acts on every smooth $W(k)[\mathrm{GL}_n(F)]$-module, compatibly with every morphism between such modules, and is the universal such ring. The structure of Z encodes deep information about "congruences" between $W(k)[\mathrm{GL}_n(F)]$-modules (for instance, if two irreducible representations of $\mathrm{GL}_n(F)$ in characteristic zero become isomorphic modulo ℓ, the action of Z on these two representations will be via scalars that are congruent modulo ℓ.)

Morally, the problem of showing that $\pi(\rho)$ exists for all ρ amounts to showing- for a sufficiently general notion of "congruence"- that whenever there is a congruence between two representations of G_F, there is a corresponding congruence on the other side of the local Langlands correspondence. It is therefore not surprising that one can rephrase the problem of constructing $\pi(\rho)$ in terms of the structure of

2010 *Mathematics Subject Classification.* 11F33,11F70,22E50.

Z. Indeed, Theorem 7.4 of [H2] reduces the question of the existence of $\pi(\rho)$ to a conjectured relationship between the ring Z and the deformation theory of mod ℓ representations of G_F (Conjecture 7.2 of [H2].)

The primary goal of this paper, together with its companion paper [HM], is to prove a version of this conjecture, and thus establish the local Langlands correspondence in families. More precisely, we introduce a collection of finite type $W(k)$-algebras R_ν that parameterize representations of the Weil group W_F of F with fixed restriction to prime-to-ℓ inertia, and whose completion at a given maximal ideal is a close variant of a universal framed deformation ring. We then conjecture that there is a map $Z \to R_\nu$ that is "compatible with local Langlands" in a certain technical sense (see Conjecture 9.2 below for a precise statement and discussion.) This conjecture, which we will henceforth call the "Weak Conjecture", becomes Conjecture 7.2 of [H2] after one completes R_ν at a maximal ideal, and hence implies both that conjecture and the existence of $\pi(\rho)$.

If a map $Z \to R_\nu$ of the conjectured sort exists it is natural to ask what the image is. The "Strong Conjecture" (Conjecture 9.3 below) gives a description of this image (and in fact gives a description of the direct factors of Z in purely Galois-theoretic terms.) As the names suggest, the "Strong Conjecture" implies the "Weak Conjecture."

The main result of this paper is that if the weak conjecture holds for all $\mathrm{GL}_m(F)$, with m less than or equal to a fixed n, and the strong conjecture holds for $m < n$, then the strong Conjecture holds as well for the group $\mathrm{GL}_n(F)$. In the companion paper [HM], we will show that the strong conjecture for $\mathrm{GL}_{n-1}(F)$ implies the weak conjecture for $\mathrm{GL}_n(F)$. Since the case $n = 1$ is easy (it is a consequence of local class field theory), the two papers together will establish both conjectures for all n, and hence the local Langlands correspondence for GL_n in families.

Our approach relies on three main ingredients. The first is an input from finite group theory, namely the endomorphism ring of the Gelfand-Graev representation $\overline{\Gamma}$ of $\mathrm{GL}_n(\mathbb{F}_q)$. In section 2 we introduce this ring and describe some of its basic properties, following Bonnafé-Kessar [BK]. A crucial structure on this endomorphism ring is its canonical symmetrizing form, which Bonnafé-Keessar describe in terms of "Curtis homomorphisms" arising from Deligne-Lusztig restriction. In section 3 we describe the connection between this endomorphism ring and the ring Z.

The second key ingredient is the behavior of the integral Bernstein center Z with respect to parabolic induction; for a Levi M of G there are natural maps $Z \to Z_M$ compatible, in a certain sense, with parabolic induction from M to G. In section 3 we recall results of [H1] (c.f. Theorems 3.9 and 3.12, below) that say that in certain key cases the images of these maps are "large" in a certain sense, and that the failure of these maps to have image that is "as large as possible" is controlled by the endomorphism ring of a Gelfand-Graev representation.

The third key ingredient is the construction of the rings R_ν which occupies sections 4, 7, and 8. These moduli spaces admit maps between them coming from taking direct sums of representations; these maps serve a purpose analogous to the "parabolic induction" maps from Z to Z_M. The functions on such spaces also admit subalgebras $B_{q,n}$ that play a role analogous to the subalgebras of Z arising from the endomorphism ring $\overline{E}_{q,n}$ of a Gelfand-Graev representation. The strong conjecture leads us to expect that in fact $\overline{E}_{q,n}$ and $B_{q,n}$ are isomorphic, but it seems difficult to show this directly (although it is easy to show if one inverts ℓ). Instead, we make use of the symmetrizing form on $\overline{E}_{q,n}$ to show that *if* there exists a map from $\overline{E}_{q,n}$ to $B_{q,n}$ then it must be an isomorphism (c.f. sections 5 and 6.)

Once we have established this, our argument goes as follows. First we show that the strong conjecture holds after inverting ℓ; this essentially follows easily from the classical Bernstein-Deligne theory of the Bernstein center over algebraically closed fields. We then assume the strong conjecture for $m < n$, and the weak conjecture for $m \leq n$. This gives us in particular a map $\overline{E}_{q,n} \rightarrow B_{q,n}$ that is necessarily an isomorphism. Using this, and considering various parabolic restriction maps from Z to various Levi subgroups, together with the corresponding maps on the rings R_ν of representations of W_F, we show, using our "large image" results for Z, that Z must "fill out" the entire ring of invariant functions in R_ν, thus proving the strong conjecture for GL_n.

In the process of carrying out this inductive argument we prove that $\overline{E}_{q,n}$ is isomorphic to $B_{q,n}$ for all n. This is a statement purely in finite group theory that is of independent interest. We know of no more direct proof of this isomorphism than the one described here.

Throughout this paper we adopt the following conventions: F is a p-adic field with residue field $\mathbb{F}_q$, k is an algebraically closed field of characteristic $\ell \neq p$, $\mathcal{K}$ is the field of fractions of $W(k)$, and $\overline{\mathcal{K}}$ is an algebraic closure of $\mathcal{K}$. Algebraic groups over F with be denoted by uppercase mathcal letters $\mathcal{T}$, $\mathcal{G}$, etc.; for any such group the corresponding uppercase letters T, G, etc. will denote the groups of F-points of $\mathcal{T}$, $\mathcal{G}$, and so forth. In particular there is an implicit dependence of T on $\mathcal{T}$.

Acknowledgements We are grateful to Jean-Francois Dat, Robert Kurinczuk, Vincent Secherre, David Ben-Zvi, and Richard Taylor for helpful conversations and suggestions, and to Gil Moss for his comments on an earlier draft of this paper. We are also deeply indebted to Jack Shotton for noticing a serious error in an earlier version of this paper, and for bringing to our attention the argument of Proposition 7.10 of [Sh], which proved crucial to correcting this error. This research was partially supported by NSF grant DMS-1161582 and EPSRC grant EP/M029719/1.

2. Finite groups

Before beginning our study of the Bernstein center we develop some finite group theory that will be essential for our approach. Most of the ideas in this section originally appear in work of Bonnafé-Kessar [BK].

Fix distinct primes p and ℓ, and a power q of p. Let $\overline{\mathcal{G}}$ be the group GL_n over $\mathbb{F}_q$, and let $\overline{G} = \overline{\mathcal{G}}(\mathbb{F}_q)$. We will consider the representation theory of $\overline{G}$ over the Witt ring $W(k)$, where k is an algebraic closure of $\mathbb{F}_\ell$. Let $\mathcal{K}$ be the field of fractions of $W(k)$, and fix an algebraic closure $\overline{\mathcal{K}}$ of $\mathcal{K}$.

Our principal object of study in this section will be the Gelfand-Graev representation $\overline{\Gamma}$ of $\overline{G}$, with coefficients in $W(k)$. Fix a Borel $\overline{\mathcal{B}}$ in $\overline{\mathcal{G}}$, with unipotent radical $\overline{\mathcal{U}}$, and let $\overline{B}$, $\overline{U}$ denote the $\mathbb{F}_q$-points of $\overline{\mathcal{B}}$ and $\overline{\mathcal{U}}$ respectively. Also fix a generic character $\Psi : \overline{U} \rightarrow W(k)^\times$. Then, by definition, we have $\overline{\Gamma} = \text{c-Ind}_{\overline{U}}^{\overline{G}} \Psi$, where Ψ is considered as a $W(k)[\overline{U}]$-module that is free over $W(k)$ of rank one, with the appropriate action of $\overline{U}$. The module $\overline{\Gamma}$ is then independent of the choice of Ψ, up to isomorphism.

The objective of this first section is to study the endomorphism ring $\mathrm{End}_{W(k)[\overline{G}]}(\overline{\Gamma})$, which we denote by $\overline{E}_{q,n}$. Our main tool for doing so will be the Deligne-Lusztig induction and restriction functors of Bonnafé-Rouquier [BR]. Let $\overline{L}$ be the subgroup of $\overline{G}$ consisting of the $\overline{\mathbb{F}}_q$-points of a (not necessarily split) Levi subgroup $\overline{\mathcal{L}}$ of GL_n, and choose a parabolic subgroup $\overline{\mathcal{P}}$ of GL_n whose Levi subgroup is $\overline{\mathcal{L}}$. Let $\mathrm{Rep}_{W(k)}(\overline{G})$ and $\mathrm{Rep}_{W(k)}(\overline{L})$ denote the categories of $W(k)[\overline{G}]$-modules and $W(k)[\overline{L}]$-modules, respectively. Then Deligne-Lusztig induction and restriction are functors:

$$i_{\overline{\mathcal{L}}\subseteq\overline{\mathcal{P}}}^{\overline{G}} : \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{L})) \to \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{G}))$$
$$r_{\overline{G}}^{\overline{\mathcal{L}}\subseteq\overline{\mathcal{P}}} : \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{G})) \to \mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{L})).$$

We will be concerned exclusively with the case where $\overline{\mathcal{L}}$ is a maximal torus in $\overline{\mathcal{G}}$. In this case the effect of Deligne-Lusztig restriction on $\overline{\Gamma}$ has been described by Bonnafé-Rouquier when $\overline{\mathcal{L}}$ is a Coxeter torus and by Dudas [Du] in general.

Theorem 2.1 (Bonnafé-Rouquier, Dudas)**.** *When $\overline{\mathcal{L}}$ is the standard maximal torus, there is a natural isomorphism:*

$$r_{\overline{G}}^{\overline{\mathcal{L}}\subseteq\overline{\mathcal{P}}}\overline{\Gamma} \cong W(k)[\overline{L}][-\ell(w)]$$

in $\mathcal{D}^b(\mathrm{Rep}_{W(k)}(\overline{L}))$, where w is the element of the Weyl group of $\overline{\mathcal{G}}$ such that $\overline{\mathcal{P}}^w$ is the standard Borel, $\ell(w)$ is its length, and $[-\ell(w)]$ denotes a cohomological shift.

Proof. This is the main theorem of [Du]. □

An immediate consequence of this result is that, when $\overline{T}$ is the $\overline{\mathbb{F}}_q$-points of a torus in GL_n, then an endomorphism of $\overline{\Gamma}$ gives rise, by functoriality of Deligne-Lusztig restriction, to an endomorphism of $W(k)[\overline{T}]$ (or, equivalently, an element of $W(k)[\overline{T}]$.) We thus obtain homorphisms:

$$\Phi_{\overline{\mathcal{T}}} : \overline{E}_{q,n} \to W(k)[\overline{T}]$$

for each torus $\overline{\mathcal{T}}$ in $\overline{\mathcal{G}}$. These are integral versions of the classical "Curtis homomorphisms".

Over $\overline{\mathcal{K}}$, it is not difficult to describe the structure of $\overline{\Gamma} \otimes \overline{\mathcal{K}}$, its endomorphism ring, and the associated Curtis homomorphisms. Recall that an irreducible representation π of $\overline{G}$ is said to be *generic* if π contains the character Ψ, or, equivalently, if there exists a nonzero map from $\overline{\Gamma}$ to π. The irreducible generic representations of $\overline{G}$ over $\overline{\mathcal{K}}$ are indexed by semisimple conjugacy classes s in $\overline{G}'$, where $\overline{G}'$ is the group of $\overline{\mathbb{F}}_q$-points in the group $\overline{\mathcal{G}}'$ that is dual to $\overline{\mathcal{G}}$. More precisely, given such an s, there exists a unique irreducible generic representation $\overline{\mathrm{St}}_s$ in the rational series attached to s.

The association of rational series to semisimple conjugacy classes in $\overline{G}'$ depends on choices which we now recall: let $\mu^{(p)}$ denote the prime-to-p roots of unity in $\overline{\mathcal{K}}$, let $(\mathbb{Q}/\mathbb{Z})^{(p)}$ denote the elements of order prime to p in $(\mathbb{Q}/\mathbb{Z})$, and fix isomorphisms:

$$\mu^{(p)} \cong (\mathbb{Q}/\mathbb{Z})^{(p)} \cong \overline{\mathbb{F}}_q^{\times}.$$

Now let t be a semisimple element in $\overline{G}'$, let $\overline{\mathcal{T}}'$ be a maximal torus containing s, and let $\overline{\mathcal{T}}$ be the dual torus in $\overline{G}$. Let X and X' denote the character groups of $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}'$ respectively. We have isomorphisms:

$$\overline{\mathcal{T}}(\mathbb{F}_q) \cong \mathrm{Hom}(X/(\mathrm{Fr}_q - 1)X, \mathbb{G}_m)$$

$$\overline{\mathcal{T}}'(\mathbb{F}_q) \cong \mathrm{Hom}(X'/(\mathrm{Fr}_q - 1)X', \mathbb{G}_m)$$

where Fr_q is the endomorphism induced by the q-power Frobenius. We also have a natural duality $X/(\mathrm{Fr}_q - 1)X \cong \mathrm{Hom}(X'/(\mathrm{Fr}_q - 1)X', (\mathbb{Q}/\mathbb{Z})^{(p)})$. The identifications we fixed above then give rise to isomorphisms:

$$\overline{\mathcal{T}}'(\mathbb{F}_q) \cong \mathrm{Hom}(X'/(\mathrm{Fr}_q - 1)X', \mathbb{G}_m) \cong X/(\mathrm{Fr}_q - 1)X \cong \mathrm{Hom}(\overline{\mathcal{T}}(\mathbb{F}_q), \mu^{(p)}).$$

In this way we associate, to any semisimple element t of $\overline{\mathcal{G}}'(\mathbb{F}_q)$, and any $\overline{\mathcal{T}}'$ containing t, a character $\varphi_{\overline{\mathcal{T}}',t} : \overline{\mathcal{T}}(\mathbb{F}_q) \to \overline{\mathcal{K}}^{\times}$.

It is immediate (by applying the idempotent of $\overline{\mathcal{K}}[\overline{G}]$ corresponding to the rational series attached to s to Theorem 2.1) that we then have:

Proposition 2.2. *Let $\overline{\mathcal{T}}$ be a maximal torus of $\overline{\mathcal{G}}$, and let $\overline{\mathcal{B}}$ be a Borel containing $\overline{\mathcal{T}}$. Then, up to a cohomological shift depending only on $\overline{\mathcal{B}}$, we have:*

$$r_{\overline{G}}^{\overline{\mathcal{T}} \subseteq \overline{\mathcal{B}}} \overline{St}_s \cong \bigoplus_{t \sim s; t \in \overline{\mathcal{T}}'} \varphi_{\overline{\mathcal{T}}',t}.$$

Returning to $\overline{\Gamma}$, we have a direct sum decomposition:

$$\overline{\Gamma} \otimes \overline{\mathcal{K}} \cong \bigoplus_s \overline{\mathrm{St}}_s$$

It follows immediately that the endomorphism ring of $\overline{\Gamma} \otimes \overline{\mathcal{K}}$ is isomorphic to a product of copies of $\overline{\mathcal{K}}$, indexed by the semisimple conjugacy classes s in $\overline{G}'$. As the endomorphism ring $\mathrm{End}_{W(k)[\overline{G}]}(\overline{\Gamma})$ of $\overline{\Gamma}$ embeds in this product, we see immediately that $\mathrm{End}_{W(k)[\overline{G}]}(\overline{\Gamma})$ is reduced and commutative.

Indeed, it is not difficult to describe the maps $\Phi_{\overline{T}} \otimes \overline{\mathcal{K}}$. The isomorphism:

$$\overline{\Gamma} \otimes \overline{\mathcal{K}} \cong \bigoplus_s \overline{St}_s$$

where s runs over semisimple conjugacy classes in $\overline{G}'$, gives rise to an isomorphism:

$$\overline{E}_{q,n} \otimes \overline{\mathcal{K}} \cong \prod_s \overline{\mathcal{K}}.$$

On the other hand we have a direct sum decomposition:

$$\overline{\mathcal{K}}[\overline{T}] \cong \bigoplus_t \varphi_{\overline{\mathcal{T}},t}$$

of $\overline{\mathcal{K}}[\overline{T}]$-modules, and hence an algebra isomorphism:

$$\overline{\mathcal{K}}[\overline{T}] \cong \prod_t \overline{\mathcal{K}}.$$

It follows immediately from the previous paragraph that $\Phi_{\overline{T}}$ maps the factor of $\overline{\mathcal{K}}$ of $\overline{E}_{q,n} \otimes \overline{\mathcal{K}})$ corresponding to s identically to each factor of $\overline{\mathcal{K}}[\overline{T}]$ that corresponds to a t in the $\overline{G}'$-conjugacy class s, and to zero in the other factors.

Now let $\overline{\mathcal{T}}$ range over all tori in $\overline{G}$, and consider the product map:

$$\Phi : \overline{E}_{q,n} \to \prod_{\overline{\mathcal{T}}} W(k)[\overline{T}].$$

For each pair $(\overline{\mathcal{T}}, \varphi)$, where φ is a character $\overline{T} \to \overline{\mathcal{K}}^{\times}$, we have a map:

$$\xi_{\overline{\mathcal{T}},\varphi} : \prod_{\overline{T}} W(k)[\overline{T}] \to \overline{\mathcal{K}}$$

given by composing the projection onto $W(k)[\overline{T}]$ with the map: $\varphi : W(k)[\overline{T}] \to \mathcal{K}$.

Define an equivalence relation on such pairs by setting $(\overline{\mathcal{T}}_1, \varphi_1) \sim (\overline{\mathcal{T}}_2, \varphi_2)$ if t_1 and t_2 are conjugate in $\overline{G}'$, where t_1 and t_2 are the elements of the dual tori $\overline{\mathcal{T}}'_1$ and $\overline{\mathcal{T}}'_2$ corresponding to φ_1 and φ_2. Then our description of each $\Phi_{\overline{T}}$ shows that, when $(\overline{\mathcal{T}}_1, \varphi_1) \sim (\overline{\mathcal{T}}_2, \varphi_2)$, one has $\xi_{\overline{\mathcal{T}}_1,\varphi_1} \circ \Phi = \xi_{\overline{\mathcal{T}}_2,\varphi_2} \circ \Phi$. Thus Φ induces a bijection between the $\overline{\mathcal{K}}$-points of $\operatorname{Spec} \overline{E}_{q,n}$ and the equivalence classes of pairs $(\overline{\mathcal{T}}, \varphi)$.

In what follows, it will be necessary for us to consider certain direct factors of $\overline{E}_{q,n}$ arising from idempotents of $W(k)[\overline{G}]$. An ℓ-regular semisimple conjugacy class s in $\overline{G}'$ gives rise, via the choices we have made above, to an idempotent e_s in $W(k)[\overline{G}]$, that acts by the identity on the rational series corresponding to those s' in $\overline{G}$ with ℓ-regular part s, and zero elsewhere. We will denote by $\overline{E}_{q,n,s}$ the direct factor $e_s \overline{E}_{q,n}$ of $\overline{E}_{q,n}$. The $\overline{\mathcal{K}}$-points of $\operatorname{Spec} \overline{E}_{q,n,s}$ are those corresponding to pairs $(\overline{\mathcal{T}}, \varphi)$ such that φ corresponds to an element t of $\overline{\mathcal{T}}'$ whose ℓ-regular part is s.

Now let $s \in \overline{G}'$ be ℓ-regular semisimple and suppose that the characteristic polynomial of s is a power of an irreducible polynomial of degree d. Then the centralizer $\overline{\mathcal{L}}'$ of s in $\overline{\mathcal{G}}'$ is a nonsplit Levi isomorphic to $\operatorname{Res}_{\mathbb{F}_{q^d}/\mathbb{F}_q} \operatorname{GL}_{\frac{n}{d}}$. Let $\overline{\mathcal{L}}$ be the Levi of $\overline{\mathcal{G}}$ dual to $\overline{\mathcal{L}}'$. By Théorème 11.8 of [BR], twisting by the character of $\overline{L}$ associated to s, followed by Deligne-Lusztig induction from $\overline{\mathcal{L}}$ to $\overline{\mathcal{G}}$, is an equivalence of categories from $e_1 \operatorname{Rep}_{W(k)}(\overline{L})$ to $e_s \operatorname{Rep}_{W(k)}(\overline{G})$. Moreover, this equivalence carries $e_1 \Gamma_{\overline{\mathcal{L}}}$ to $e_s \Gamma$. (This follows from uniqueness of projective envelopes, since the former is the projective envelope of the unique irreducible generic k-representation of $\overline{L}$ in the block corresponding to e_1, and the latter is the projective envelope of the unique irreducible generic k-representation of $\overline{G}$ in the block corresponding to e_s.) We thus have:

Proposition 2.3. *For s an ℓ-regular semisimple element of $\overline{G}'$ whose characteristic polynomial is a power of an irreducible polynomial of degree d over $\mathbb{F}_q$. Then there is a natural isomorphism:*

$$\overline{E}_{q,n,s} \cong \overline{E}_{q^d,\frac{n}{d},1}.$$

The induced map on $\overline{\mathcal{K}}$-points takes the $\overline{\mathcal{K}}$-point of $\operatorname{Spec} \overline{E}_{q^d,\frac{n}{d},1}$ corresponding to the ℓ-primary conjugacy class t of $\overline{L}'$ to the $\overline{\mathcal{K}}$-point of $\operatorname{Spec} \overline{E}_{q,n,s}$ corresponding to the conjugacy class of st in $\overline{G}'$.

Proof. The first claim is immediate from the previous paragraph. The second follows from the description of the equivalence of categories on irreducible generic $\overline{\mathcal{K}}$-representations. □

The final structure we will need to consider on $\overline{E}_{q,n}$ is a natural symmetrizing form considered by Bonnafé-Kessar ([BK], section 3.B). Define a $W(k)$-linear map $\theta : \overline{E}_{q,n} \to W(k)$ by the formula:

$$\theta(x) = \frac{1}{n!} \sum_{w \in S_n} \theta_w(\Phi_{\overline{\mathcal{T}}_w}(x)),$$

where $\overline{\mathcal{T}}_w$ is the torus of $\overline{\mathcal{G}}$ associated to the element w of the Weyl group, and $\theta_w : W(k)[\overline{T}_w] \to W(k)$ is the canonical symmetrizing form on $W(k)[\overline{T}_w]$ given by "evaluation at the identity". Note that we can extend θ to a linear map $\overline{E}_{q,n} \otimes \overline{\mathcal{K}} \to \overline{\mathcal{K}}$.

We then have:

Proposition 2.4. *Let t be a semisimple conjugacy class in $\overline{G}'$, and let e_t be the corresponding idempotent of $\overline{E}_{q,n} \otimes \overline{\mathcal{K}}$. Then*

$$\theta(e_t) = \frac{1}{n!} \sum_{w \in S_n} \frac{1}{\# \overline{T}_w} N(w,t),$$

where $N(w,t)$ is the number of elements of $\overline{T}'_w$ in the conjugacy class of t.

Proof. It is easy to see that $\Phi_{\overline{\mathcal{T}}_w}(e_t)$ is equal to the sum, over those $t' \in \overline{\mathcal{T}}'_w$ conjugate to t', of the idempotents $e_{t'}$ of $\overline{\mathcal{K}}[\overline{T}_w]$. The claim is then immediate from the formula for θ. □

3. The integral Bernstein center

We now turn to the first main object of interest in this paper: the integral Bernstein center. Let $G = \mathrm{GL}_n(F)$, and denote by $\mathrm{Rep}_{W(k)}(G)$ (resp. $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)$) the category of smooth $W(k)[G]$-modules (resp. the category of smooth $\overline{\mathcal{K}}[G]$-modules.)

By the phrase "integral Bernstein center" we mean the center of the category $\mathrm{Rep}_{W(k)}(G)$. We recall what this means:

Definition 3.1. The *center* of an Abelian category $\mathcal{A}$ is the ring of natural transformations $\mathrm{Id}_{\mathcal{A}} \to \mathrm{Id}_{\mathcal{A}}$, where $\mathrm{Id}_{\mathcal{A}}$ denotes the identity functor on $\mathcal{A}$.

By definition, if Z is the center of $\mathcal{A}$, then specifying an element of Z amounts to specifying an endomorphism of every object of $\mathcal{A}$, such that the resulting collection commutes with all arrows in $\mathcal{A}$. The center of $\mathcal{A}$ is thus a commutative ring that acts naturally on every object in $\mathcal{A}$, and this action is compatible with all morphisms in $\mathcal{A}$.

Bernstein-Deligne [BD], give a complete and explicit description of the center $\tilde{Z}$ of $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)$. We briefly summarize their results: first, define an equivalence relation on pairs $(M, \tilde{\pi})$, where M is a Levi of G and π is an irreducible supercuspidal representation of M over $\overline{\mathcal{K}}$ by declaring $(M_1, \tilde{\pi}_1)$ to be *inertially equivalent* to $(M_2, \tilde{\pi}_2)$ if $\tilde{\pi}_1$ is G-conjugate to an unramified twist of $\tilde{\pi}_2$. One then has:

Theorem 3.2 ([BD], Proposition 2.10)**.** *There is a bijection $(M, \tilde{\pi}) \mapsto e_{(M,\tilde{\pi})}$ between inertial equivalence classes of pairs $(M, \tilde{\pi})$ over $\overline{\mathcal{K}}$ and primitive idempotents of $\tilde{Z}$, such that for any irreducible smooth representation Π of G over $\overline{\mathcal{K}}$ $e_{(M,\tilde{\pi})}$ acts via the identity on Π if Π has supercuspidal support in the inertial equivalence class of $(M, \tilde{\pi})$, and by zero otherwise.*

The upshot is that $\tilde{Z}$ decomposes as an infinite product of the rings $e_{(M,\tilde{\pi})}\tilde{Z}$ as $(M, \tilde{\pi})$ runs over all inertial equivalence classes of pairs. Denote $e_{(M,\tilde{\pi})}\tilde{Z}$ by $\tilde{Z}_{(M,\tilde{\pi})}$. Then Bernstein and Deligne give a complete description of the ring structure of $\tilde{Z}_{(M,\tilde{\pi})}$ that we now explain.

Let M_0 be the smallest subgroup of M containing every compact open subgroup of M. Then M/M_0 is a free abelian group of finite rank, and $\mathrm{Spec}\, \overline{\mathcal{K}}[M/M_0]$ is a torus whose $\overline{\mathcal{K}}$-points are in bijection with the characters $M/M_0 \to \overline{\mathcal{K}}^\times$. Let H be the subgroup of these characters consisting of those characters χ such that $\tilde{\pi} \otimes \chi$ is isomorphic to $\tilde{\pi}$. Then H is a finite abelian group that acts on $\overline{\mathcal{K}}[M/M_0]$. The torus $\operatorname{Spec} \overline{\mathcal{K}}[(M/M_0)]^H$ is a quotient of $\operatorname{Spec} \overline{\mathcal{K}}[M/M_0]$; its $\overline{\mathcal{K}}$-points correspond to H-orbits of characters of M/M_0.

Now let W_M be the subgroup of the Weyl group of G consisting of w such that $wMw^{-1} = M$. Let $W_M(\tilde{\pi})$ be the subgroup of W_M consisting of w such that the representation $\tilde{\pi}^w$ of M is an unramified twist of $\tilde{\pi}$. Then we have a natural action of $W_M(\tilde{\pi})$ on $\overline{\mathcal{K}}[(M/M_0)]^H$, characterized by $\tilde{\pi} \otimes \chi^w \cong (\tilde{\pi} \otimes \chi)^w$ for characters χ of M/M_0. We then have:

Theorem 3.3 ([BD], Théorème 2.13)**.** *There is a unique natural isomorphism:*

$$\tilde{Z}_{(M,\tilde{\pi})} \cong \left(\overline{\mathcal{K}}[(M/M_0)]^H\right)^{W_M(\tilde{\pi})}$$

such that, for any irreducible representation Π over $\overline{\mathcal{K}}$ whose supercuspidal support has the form $\tilde{\pi} \otimes \chi$, $\tilde{Z}_{(M,\tilde{\pi})}$ acts on Π via the map:

$$\left(\overline{\mathcal{K}}[(M/M_0)]^H\right)^{W_M(\tilde{\pi})} \to \overline{\mathcal{K}}[M/M_0] \to \overline{\mathcal{K}}$$

corresponding to the character $\chi : M/M_0 \to \overline{\mathcal{K}}^\times$. In particular $\tilde{Z}_{(M,\tilde{\pi})}$ is a reduced, finitely generated, and normal $\overline{\mathcal{K}}$-algebra.

In particular, $\tilde{Z}$ acts on two irreducible representations Π, Π' of G via the same map $\tilde{Z} \to \overline{\mathcal{K}}$ if, and only if, Π and Π' have the same supercuspidal support. This defines, for each $(M, \tilde{\pi})$, a bijection between the $\overline{\mathcal{K}}$-points of $\operatorname{Spec} \tilde{Z}_{(M,\tilde{\pi})}$ and supercuspidal supports in the inertial equivalence class of $(M, \tilde{\pi})$; that is, unramified twists of $\tilde{\pi}$ considered up to $W_M(\tilde{\pi})$-conjugacy.

Now let L be a Levi in GL_n; then L factors as a product of L_i isomorphic to $\mathrm{GL}_{n_i}(F)$. For each i, let M_i be a Levi in L_i, and $\tilde{\pi}_i$ an irreducible supercuspidal $\overline{\mathcal{K}}$-representation of M_i. We then have isomorphisms:

$$\tilde{Z}_{M_i,\tilde{\pi}_i} \cong \left(\overline{\mathcal{K}}[(M_i/(M_i)_0)]^{H_i}\right)^{W_{M_i}(\tilde{\pi}_i)}.$$

Let M be the product of the M_i; we may regard it as a Levi of L and hence as a Levi of $\mathrm{GL}_n(F)$. Let $\tilde{\pi}$ be the tensor product of the $\tilde{\pi}_i$. The quotient M/M_0 factors naturally as a product of $M_i/(M_i)_0$, and this induces a map:

$$\left(\overline{\mathcal{K}}[(M/M_0)]^H\right)^{W_M(\tilde{\pi})} \to \bigotimes_i \left(\overline{\mathcal{K}}[(M_i/(M_i)_0)]^{H_i}\right)^{W_{M_i}(\tilde{\pi}_i)}$$

and hence a map

$$\mathrm{Ind}_{\{(M_i,\tilde{\pi}_i)\}} : \tilde{Z}_{(M,\tilde{\pi})} \to \bigotimes_i \tilde{Z}_{(M_i,\tilde{\pi}_i)}.$$

On $\overline{\mathcal{K}}$-points this takes the $\mathcal{K}$-point of the tensor product that corresponds to the collection of supercuspidal supports $\{(M_i, \tilde{\pi}_i \otimes \chi_i)\}$ to the point of $\operatorname{Spec} \tilde{Z}_{(M,\tilde{\pi})}$ corresponding to the supercuspidal support $(M, \otimes_i(\tilde{\pi}_i \otimes \chi_i))$.

We now turn to the study of $\mathrm{Rep}_{W(k)}(G)$; let Z denote the center of this category. In this setting there is an analogue of the Bernstein-Deligne characterization of the primitive idempotents of Z. By [H1], Theorem 11.8, such idempotents are parameterized by inertial equivalence classes of pairs (L, π), where π is now an irreducible supercuspidal representation of L over k.

If we let $e_{[L,\pi]}$ denote the idempotent of Z corresponding to (L,π), $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ the corresponding block, and $Z_{[L,\pi]}$ the corresponding factor of the Bernstein center, then one has the following basic structure results:

Theorem 3.4 ([H1], Theorem 12.8)**.** *The ring $Z_{[L,\pi]}$ is a finitely generated, reduced, flat $W(k)$-algebra.*

It is important to note that, in contrast to the situation over $\overline{\mathcal{K}}$, the ring $Z_{[L,\pi]}$ is in general very far from being normal.

We also have a description of $Z_{[L,\pi]} \otimes \overline{\mathcal{K}}$ in terms of $\tilde{Z}$. This can be made precise as follows: if $(M,\tilde{\pi})$ is a pair over $\overline{\mathcal{K}}$, and Π is an irreducible integral representation of G over $\overline{\mathcal{K}}$ with supercuspidal support in the inertial equivalence class of $(M,\tilde{\pi})$, then there exists a (possibly proper) Levi subgroup L of M, and an irreducible supercuspidal representation π of L, such that every irreducible subquotient of the mod ℓ reduction of Π has supercuspidal support (L,π). Moreover, the inertial equivalence class of (L,π) depends only on that of $(M,\tilde{\pi})$, and not on the particular choice of π. We say that $(M,\tilde{\pi})$ *reduces modulo* ℓ to (L,π); this defines a finite-to-one map from inertial equivalence classes over $\overline{\mathcal{K}}$ to inertial equivalence classes over k. One then has:

Theorem 3.5 ([H1], Proposition 12.1)**.** *The natural map $Z \otimes \overline{\mathcal{K}} \to \tilde{Z}$ induces an isomorphism:*

$$Z_{[L,\pi]} \otimes \overline{\mathcal{K}} \cong \prod_{(M,\tilde{\pi})} \tilde{Z}_{(M,\tilde{\pi})},$$

where the product is over all pairs $(M,\tilde{\pi})$, up to inertial equivalence, that reduce modulo ℓ to the pair (L,π).

From this and the description of the $\overline{\mathcal{K}}$-points of $\mathrm{Spec}\,\tilde{Z}_{(M,\tilde{\pi})}$ one immediately deduces:

Corollary 3.6. *The $\overline{\mathcal{K}}$-points of $\mathrm{Spec}\,Z_{[L,\pi]}$ are in bijection with the supercupidal supports of irreducible smooth $\overline{\mathcal{K}}$-representations in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$.*

We now give a more precise description of $Z_{[L,\pi]}$. We first reduce to a more easily studied special case:

Definition 3.7. A pair (L,π) is *simple* if there exist r,m such that $n = rm$, L is isomorphic to $\mathrm{GL}_m(F)^r$, and π, up to unramified twist, is of the form $(\pi')^{\otimes r}$ for an irreducible supercuspidal representation π' of $\mathrm{GL}_m(F)$.

Note that any pair (L,π) factors uniquely as a product of simple pairs (L^i,π^i), with $\pi^i \cong (\pi'_i)^{\otimes r_i}$, such that no π'_i is an unramified twist of any other. One then has:

Theorem 3.8 ([H1], Theorem 12.4)**.** *Let $\{(L^i,\pi^i)\}$ be the natural decomposition of (L,π) as a product of simple pairs. Then there is a natural isomorphism:*

$$Z_{[L,\pi]} \cong \bigotimes_i Z_{[L^i,\pi^i]}$$

such that, for any sequence $\{(M^i,\tilde{\pi}^i)\}$ reducing modulo ℓ to $\{(L^i,\tilde{\pi}^i)\}$, the diagram:

$$\begin{array}{ccc} Z_{[L,\pi]} \otimes \overline{\mathcal{K}} & \rightarrow & [\bigotimes_i Z_{[L^i,\pi^i]}] \otimes \overline{\mathcal{K}} \\ \downarrow & & \downarrow \\ \tilde{Z}_{(M,\tilde{\pi})} & \rightarrow & \bigotimes_i \tilde{Z}_{(M^i,\tilde{\pi}^i)} \end{array}$$

commutes, where $(M,\tilde{\pi})$ is the product of the $(M_i,\tilde{\pi}_i)$, and the bottom horizontal map is the map $\mathrm{Ind}_{\{(M^i,\tilde{\pi}^i)\}}$ described above.

We thus focus our attention on the case where (L,π) is simple. Fix an integer n_1 and an irreducible supercuspidal representation π' of $\mathrm{GL}_{n_1}(F)$ over k. For each $m > 0$, let L_m be a Levi of $\mathrm{GL}_{n_1 m}(F)$ isomorphic to $\mathrm{GL}_{n_1}(F)^m$, and let π_m be the representation $(\pi')^{\otimes m}$ of L_m. We can then consider the family of rings $Z_m := Z_{[L_m,\pi_m]}$ as n varies.

Section 13 of [H1] contains detailed information about the structure of the family Z_m. In particular this structure theory is closely related to the endomorphism rings of certain projective objects $\mathcal{P}_{K_m,\tau_m}$ for particular m. More precisely, consider the group of unramified characters χ of $\mathrm{GL}_{n_1}(F)$ such that $\pi' \otimes \chi$ is isomorphic to π'. This is a finite group; denote its order by f'. Then attached to the system of pairs (L_m,π_m) we have a system of projective objects $\mathcal{P}_{K_m,\tau_m}$, where m lies in the set $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \ell^2 e_{q^{f'}}, \dots\}$. (We refer the reader to Sections 7 and 9 of [H1] for a construction and structure theory of these objects.) For brevity, denote the representation $\mathcal{P}_{K_m,\tau_m}$ by $\mathcal{P}_m$.

For such m, let E_m denote the endomorphism ring of $\mathcal{P}_m$. Then, by Corollary 9.2 of [H1], E_m is a reduced, finite type, ℓ-torsion free $W(k)$-algebra. Moreover, we have a map $Z_m \to E_m$ that gives the action of Z_m on the object $\mathcal{P}_m$ of $\mathrm{Rep}_{W(k)}(\mathrm{GL}_{n_1 m}(F))_{[L_m,\pi_m]}$.

If m is arbitrary, the relationship between the rings Z_m and E_m is more complicated. For a partition ν of m, we will say that ν is q-relevant if each ν_i belongs to the set $\{1, e_q, \ell e_q, \ell^2 e_q, \dots\}$, where e_q is the multiplicative order of q modulo ℓ (relevant partitions were called admissible in [H1]). Let ν be the maximal $q^{f'}$-relevant partition of m. Let M_ν and P_ν be the standard Levi and (upper triangular) parabolic subgroups of $\mathrm{GL}_{n_1 m}$ attached to $n_1\nu$, so that M_ν is a product of $\mathrm{GL}_{n_1\nu_i}(F)$, and consider the representation $\bigotimes_i \mathcal{P}_{\nu_i}$ of M_ν. Then Z_m acts on the parabolic induction $i_{P_\nu}^{\mathrm{GL}_{n_1 m}(F)} \bigotimes_i \mathcal{P}_{\nu_i}$, and we have:

Theorem 3.9 ([H1], Theorem 13.7)**.** *The action of Z_m on $i_{P_\nu}^{\mathrm{GL}_{n_1 m}(F)} \bigotimes_i \mathcal{P}_{\nu_i}$ factors through the action of $\bigotimes_i E_{\nu_i}$ on $\bigotimes_i \mathcal{P}_{\nu_i}$. Moreover, the resulting map:*

$$Z_m \to \bigotimes_i E_{\nu_i}$$

is injective with saturated image, and is an isomorphism if m lies in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$. (Note that in this case ν is the one-element partition $\{m\}$ of m.)

For m in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$ we thus have a natural identification of Z_m with E_m. For arbitrary m, we can regard the map $Z_m \to \bigotimes_i E_{\nu_i}$ as a map $Z_m \to \bigotimes_i Z_{\nu_i}$. Denote this map by Ind_ν. It is injective with saturated image.

For m in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$, the results of Sections 7 and 9 of [H1] give very precise information about E_m, and hence Z_m. In particular there is an integer f dividing f', and a a cuspidal k-representation σ_m of $\mathrm{GL}_{\frac{mf'}{f}}(\mathbb{F}_{q^f})$ (attached to an ℓ-regular conjugacy class $(s_1')^m$ with s_1' irreducible of degree f' over $\mathbb{F}_{q^f}$), such that the projective $\mathcal{P}_m$ is a compact induction $\text{c-Ind}_{K_m}^{\mathrm{GL}_{n_1 m}(F)} \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m}$, where $\tilde{\kappa}_m$ comes from type theory and $\mathcal{P}_{\sigma_m}$ is the projective envelope of σ_m, inflated to a representation of K_m via a natural map $K_m \to \mathrm{GL}_{\frac{mf'}{f}}(\mathbb{F}_{q^f})$.

Section 5 of [H1] shows that $\mathcal{P}_{\sigma_m}$ is the projection of the Gelfand-Graev representation of $\mathrm{GL}_{\frac{mf'}{f}}(\mathbb{F}_{q^f})$ to the block containing σ_m. In particular, the results of section 2 identify the endomorphisms of $\mathcal{P}_{\sigma_m}$ with $\overline{E}_{q^f,md,s}$, where we have written $s = (s_1)^m$ and $d = \frac{f'}{f}$. By Proposition 2.3 we may identify $\overline{E}_{q^f,md,s}$ with $\overline{E}_{q^{f'},m,1}$.

We thus obtain an embedding of $\overline{E}_{q^{f'},m,1}$ in E_m for such m. Furthermore, section 9 of [H1] constructs an invertible element $\Theta_{m,m}$ of E_m. We thus obtain a map

$$\overline{E}_{q^{f'},m,1}[T, T^{-1}] \to E_m$$

taking T to $\Theta_{m,m}$. It follows easily from the description of E_m as a Hecke algebra in section 9 of [H1] that the image of this map consists of the elements of E_m supported on double cosets of the form $K_m z_{m,m}^r K_m$ for various r. (In particular, this image is saturated in E_m.)

The image of $\overline{E}_{q^{f'},m,1}$ in Z_m is easy to describe. Indeed, we have:

Proposition 3.10. *Let m lie in $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$, and let x be an element of $\overline{E}_{q^{f'},m,1}$, where the latter is considered as a subalgebra of Z_m. Then for any irreducible $\overline{\mathcal{K}}$-representations Π, Π' of $\mathrm{GL}_{n_1 m}(F)$ in the same block of $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_{n_1 m}(F))$, the action of x on Π and Π' is via the same scalar. Conversely, any element of Z_m with this property lies in $\overline{E}_{q^{f'},m,1}$.*

Proof. The ring Z_m annihilates both Π and Π' unless Π and Π' belong to a block of the form $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_{n_1 m}(F))_{(M_s,\pi_s)}$ for a suitable s, in the notation of [H1], section 9. In this case the action of Z_m on Π and Π' factors through the action of Z_m on the summand $\text{c-Ind}_{K_m}^{\mathrm{GL}_{n_1 m}(F)} \tilde{\kappa}_m \otimes \mathrm{St}_s$ of $\text{c-Ind}_{K_m}^{\mathrm{GL}_{n_1 m}(F)} \tilde{\kappa}_m \otimes \mathcal{P}_{\sigma_m} \otimes \overline{\mathcal{K}}$. In particular the action of x on Π and Π' factors through the action of x on St_s, which is by a scalar.

Since $\overline{E}_{q^{f'},m,1}$ is saturated in Z_m, it suffices to prove the converse over $\overline{\mathcal{K}}$. But it follows easily from our factorization of Z_m in characteristic zero that every idempotent of $Z_m \otimes \overline{\mathcal{K}}$ is contained in $\overline{E}_{q^{f'},m,1}$; since these idempotents correspond to the blocks of $\mathrm{Rep}_{\overline{\mathcal{K}}}(\mathrm{GL}_{n_1 m}(F))_{M,\pi}$ the claim follows. □

We also make the following observation about the action of $\Theta_{m,m} \in Z_m$:

Proposition 3.11. *Let P be a parabolic subgroup of $\mathrm{GL}_{n_1 m}(F)$, with Levi subgroup M, and let π be an irreducible cuspidal $\overline{\mathcal{K}}$-representation of M such that $i_P^G \pi$ lies in the block corresponding to L_m, π_m. Suppose that M decomposes as a product of groups $M_i = \mathrm{GL}_{n_1 m_i}(F)$, and let χ be an unramified character of M, of the form $\otimes_i (\chi_i \circ \det)$, where we regard $(\chi_i \circ \det)$ as a character of M_i.*

Let $x \in \overline{\mathcal{K}}^\times$ be the scalar by which $\Theta_{m,m}$ acts on $i_P^G \pi$. Then $\Theta_{m,m}$ acts on $i_P^G \pi \otimes \chi$ via $x \prod_i \chi_i^{f'}(\varpi_F)$.

Proof. For some s, the pair (M, π) is conjugate to an unramified twist of one of the pairs (M_s, π_s) described in section 9 of [H1]. Thus, by Theorem 9.4 of [H1], the action of $\Theta_{m,m}$ on π is via the element $\theta_{m,s}$ of Z_{M_s,π_s} defined in section 9 of [H1], and the claim is immediate from the definition of $\theta_{m,s}$ in that section. □

Finally, let m' and m be two consecutive elements of $\{1, e_{q^{f'}}, \ell e_{q^{f'}}, \dots\}$, and set $j = \frac{m}{m'}$. Theorem 13.5 of [H1] then provides a map:

$$\mathrm{Ind}_{m',m} : Z_m \to Z_{m'}^{\otimes j},$$

that is compatible with parabolic induction, in the sense that the action of x in Z_m on $i_P^{\mathrm{GL}_{n_1 m}(F)}\pi$ (where $P = MU$ is a parabolic such that M is isomorphic to $\mathrm{GL}_{n_1 m'}(F)^j$) is induced by the action of $\mathrm{Ind}_{m',m}(x)$ on π. The image of this map is not saturated but we have:

Theorem 3.12 ([H1], Theorem 13.6)**.** *Let y be an element of $Z_{m'}^{\otimes j}$ such that, for some a, $\ell^a y$ lies in the image of $\mathrm{Ind}_{m',m}$. Then there exists an element $\tilde{y}$ of Z_m, an element x of $\overline{E}_{q^{f'},m,1}[T^{\pm 1}]$, and an integer $b > 0$ such that $\mathrm{Ind}_{m',m}(x) = \ell^b(y - \mathrm{Ind}_{m',m}(\tilde{y}))$.*

The map $\mathrm{Ind}_{m',m}$ is not injective, but its kernel has a rather simple structure:

Proposition 3.13. *There exists an ideal $I_{m',m}$ of $\overline{E}_{q^{f'},m,1}$ such that the kernel of $\mathrm{Ind}_{m',m}$ is equal to $I_{m',m}[\Theta_{m,m}^{\pm 1}]$.*

Proof. Since $\overline{E}_{q^{f'},m,1}[\Theta_{m,m}^{\pm 1}]$ is saturated in Z_m we can prove this after tensoring with $\overline{\mathcal{K}}$. We have a decomposition

$$Z_m \otimes \overline{\mathcal{K}} \cong \prod_i \tilde{Z}_{(M_i,\tilde{\pi}_i)}$$

where $(M_i, \tilde{\pi}_i)$ run over the $\overline{\mathcal{K}}$-inertial equivalence classes in the block corresponding to $[L_m, \pi_m]$. In particular the partitions corresponding to the M_i are all $q^{f'}$-relevant. Fix a factor in this product corresponding to a pair $(M_i, \tilde{\pi}_i)$. On this factor, we can descrive the map $\mathrm{Ind}_{m',m}$ in the following way: let $(M_{ij}, \tilde{\pi}_{ij})$ run over the set of M_ν-intertial equivalence classes of pairs that are $\mathrm{GL}_{n_1 m}$-inertially equivalent to $(M_i, \tilde{\pi}_i)$, where ν is the partition $(m', \dots, m')$ of m and M_ν is the corresponding Levi of $\mathrm{GL}_{n_1 m}$. Since M_{ij} is a Levi contained in M_ν the pair $(M_{ij}, \tilde{\pi}_{ij})$ breaks up as a product of $\frac{m}{m'}$ pairs $(M_{ijk}, \tilde{\pi}_{ijk})$ in $\mathrm{GL}_{n_1 m'}$. On the factor $\tilde{Z}_{(M_i,\tilde{\pi}_i)}$ of $Z_m \otimes \overline{\mathcal{K}}$, $\mathrm{Ind}_{m,n}$ is the sum of the maps:

$$\mathrm{Ind}_{(M_{ij},\tilde{\pi}_{ij})} : \tilde{Z}_{(M_i,\tilde{\pi}_i)} \to \bigotimes_k \tilde{Z}_{(M_{ijk},\tilde{\pi}_{ijk})}.$$

In particular $\mathrm{Ind}_{m',m}$ is injective on the factor $\tilde{Z}_{(M_i,\tilde{\pi}_i)}$ if M_i is a proper Levi subgroup and zero otherwise. When M_i is not a proper Levi then the pair $(M_i, \tilde{\pi}_i)$ gives a cuspidal inertial equivalence class, so $\tilde{Z}_{(M_i,\tilde{\pi}_i)}$ is isomorphic to $\overline{\mathcal{K}}[\Theta_{m,m}^{\pm 1}]$. Thus the kernel of $\mathrm{Ind}_{m',m} \otimes \overline{\mathcal{K}}$ is equal to $\tilde{I}_{m',m}[\Theta_{m,m}^{\pm 1}]$, where $\tilde{I}_{m,m}$ is the ideal of $\overline{E}_{q^{f'},m,1} \otimes \overline{\mathcal{K}}$ generated by the idempotents of the latter that correspond to cuspidal inertial equivalence classes $(M_i, \tilde{\pi}_i)$. ∎

4. The ring $R_{q,n}$

We now turn to the second principal object of study of this paper, which is a moduli space of representations of W_F. We begin by studying spaces of tame representations. Let $X_{q,n}$ be the affine $W(k)$-scheme paramaterizing pairs of invertible n by n matrices (Fr, σ) such that $\mathrm{Fr}\,\sigma\,\mathrm{Fr}^{-1} = \sigma^q$, and let $X_{q,n}^0$ be the connected component of $X_{q,n}$ containing the k-point $\mathrm{Fr} = \sigma = \mathrm{Id}_n$. Let $S_{q,n}$ (resp. $R_{q,n}$) be the ring of functions on $X_{q,n}$ (resp. $X_{q,n}^0$), so that $X_{q,n} = \mathrm{Spec}\, S_{q,n}$ and $X_{q,n}^0 = \mathrm{Spec}\, R_{q,n}$.

Lemma 4.1. *Let L be an algebraically closed field that is a $W(k)$-algebra and x be an L-point of $X_{q,n}$ corresponding to a pair $(\mathrm{Fr}_x, \sigma_x)$ of elements of $\mathrm{GL}_n(L)$. Then x lies in $X^0_{q,n}$ if, and only if, the eigenvalues of σ_x are ℓ-power roots of unity.*

Proof. Consider the map $X_{q,n} \to \mathbb{A}^n_{W(k)}$ that takes a point x to the coefficients of the characteristic polynomial of σ_x. Let Y be the image of this map. For all L and x, σ_x is an element of $\mathrm{GL}_n(L)$ conjugate to its qth power, so its image in $Y(L)$ is a polynomial of degree n whose roots, counted with multiplicities, are stable under the qth power map. That is, every point of $Y(L)$ corresponds to the characteristic polynomial of a diagonal matrix that is conjugate to its qth power. Conversely, given such a matrix σ it is easy to construct an L-point x of $X_{q,n}$ with $\sigma_x = \sigma$.

Let $\tilde{Y} \subset \mathbb{A}^n_{W(k)}$ be the space of diagonal matrices that are conjugate to their qth powers; we then have a map $\tilde{Y} \to \mathbb{A}^n_{W(k)}$ that sends such a matrix to the coefficients of its characteristic polynomial. The argument of the previous paragraph shows that the (set-theoretic) image of $\tilde{Y}$ is equal to Y. On the other hand, $\tilde{Y}(\overline{\mathcal{K}})$ is a finite collection of points; indeed, the entries of any diagonal matrix that is conjugate to its qth power are roots of unity of order bounded in terms of q and n. Thus the "coordinates" of each $\overline{\mathcal{K}}$-point of $\tilde{Y}$ are integral over $W(k)$, and every point of $\tilde{Y}(k)$ is in the closure of some point of $\tilde{Y}(\overline{\mathcal{K}})$. It follows that the same is true for Y; in particular Y is the closure of a finite set of $\overline{\mathcal{K}}$-points, and the closure of any $\overline{\mathcal{K}}$-point of Y meets the special fiber of Y. Therefore, the connected component Y^0 of Y containing the image of $X^0_{q,n}$ is the closure of the set of $\overline{\mathcal{K}}$-points of Y that "specialize" mod ℓ to the characteristic polynomial $(X-1)^n$ of the identity matrix. The only k-point of this component arises from the characteristic polynomial of the identity matrix, and the $\overline{\mathcal{K}}$-points of this component correspond to characteristic polynomials of elements of $\tilde{Y}(\overline{\mathcal{K}})$ whose roots reduce to 1 modulo ℓ. The roots of such a polynomial are ℓ-power roots of unity. Therefore, for x in $X^0_{q,n}(L)$ the roots of the characteristic polynomial of σ_x are ℓ-power roots of unity, as required.

Conversely, let x be an L-point of $X_{q,n}$, and suppose that the eigenvalues of σ_x are ℓ-power roots of unity. Note that $\mathrm{GL}_n(L)$ acts on $X_{q,n}(L)$, by conjugation on both F and σ, and this action preserves the connected components. We may thus assume σ_x is in Jordan normal form; in particular its entries lie in k or an integral extension $\mathcal{O}$ of $W(k)$. Moreover, for a fixed σ_x, the set of Fr_x such that $\mathrm{Fr}_x \sigma_x = \sigma_x^q \mathrm{Fr}_x$ is a linear space; there is thus an invertible Fr'_x whose entries lie in k or $W(k)$, such that $\mathrm{Fr}'_x \sigma_x = \sigma_x^q \mathrm{Fr}'_x$ and $(\mathrm{Fr}'_x, \sigma_x)$ lies on the same connected component as x.

If L has characteristic ℓ, the above construction yields a k-point of $X_{q,n}$ in the same connected component as x. If L has characteristic zero, the closure of the point (Fr', σ) constructed above contains a k-point (Fr'', σ') of $X_{q,n}$ in the same connected component as x. Moreover, σ' is unipotent and in Jordan normal form. Thus in the closure of orbit of (Fr'', σ') under conjugation by diagonal matrices there is a point where σ is the identity. It is clear that such a point lies in the connected component of the k-point x where $\mathrm{Fr}_x = \sigma_x = \mathrm{Id}_n$. □

The ring $R_{q,n}$ is rather well-behaved from an algebraic standpoint. In particular, one has:

Proposition 4.2. *The ring $R_{q,n}$ is reduced and locally a complete intersection. Moreover, $R_{q,n}$ is flat as a $W(k)$-algebra.*

Proof. This argument is a slight elaboration of an argument due to Choi [Ch]. We give a sketch here.

First note that $X_{q,n}$ is given by n^2 relations in a space of dimension $2n^2+1$. Consider the map $X_{q,n} \to \mathbb{A}^{n^2}_{W(k)}$ that sends a point x to the matrix σ_x. Let L be an algebraically closed field that is a $W(k)$-algebra, and let x be an L-point of $X_{q,n}$.

The group $\mathrm{GL}_n(L)$ acts on the set of L-points of $X_{q,n}$ by conjugation. Consider the locally closed subset U_{σ_x} of $\operatorname{Spec} \mathbb{A}^{n^2}_L$ consisting of those σ' conjugate to σ_x. For any L-point σ' of U_{σ_x}, the fiber of $X_{q,n} \times_{W(k)} L$ over σ' consists of pairs $(\mathrm{Fr}'\, h, \sigma')$, where Fr' is a fixed element of GL_n such that $\mathrm{Fr}'\, \sigma' (\mathrm{Fr}')^{-1} = (\sigma')^q$ and h commutes with σ'

In particular, the dimension of the preimage of U_σ in $X_{q,n} \times_{W(k)} L$ is equal to the dimension of U_σ plus the dimension of the stabilizer of σ under conjugation; this is clearly n^2. As σ varies over a finite list of conjugacy classes, the preimages of the U_σ cover $X_{q,n} \times_{W(k)} L$; thus $X_{q,n} \times_{W(k)} L$ is equidimensional of dimension n^2. On the other hand the dimension of $X_{q,n}$ is at least n^2+1. It follows that the Zariski closures of the preimages of sets U_σ are irreducible components of $X_{q,n}$, and that no irreducible component of $X_{q,n}$ is contained in the special fiber (as it would then be a component of $X_{q,n} \times_{W(k)} k$ of dimension at most n^2). It also follows that every irreducible component of $X_{q,n}$ has dimension n^2+1, because if we had a component of larger dimension then its base change to $\overline{\mathcal{K}}$ would have dimension greater than n^2. In particular $X_{q,n}$ is a complete intersection. It follows that $R_{q,n}$ is a local complete intersection.

An argument of Choi ([Ch], Theorem 3.0.13) shows that for any maximal ideal m of $R_{q,n}$, $(\operatorname{Spec} R_{q,n})_m[\frac{1}{\ell}]$ is generically smooth; in particular $X^0_{q,n}$ is generically reduced. By the unmixedness theorem the local complete intersection $X^0_{q,n}$ has no embedded points, so $R_{q,n}$ is reduced. As the generic points of $\operatorname{Spec} R_{q,n}$ all have characteristic zero, we may conclude that $R_{q,n}$ is flat over $W(k)$. □

We have a universal pair of matrices (Fr, σ) in $\mathrm{GL}_n(R_{q,n})$. The above result immediately implies:

Corollary 4.3. *There exists a power ℓ^a of ℓ such that σ^{ℓ^a} is unipotent in $\mathrm{GL}_n(R_{q,n})$.*

Proof. Since $R_{q,n}$ is reduced and flat over $W(k)$, it suffices to check that σ^{ℓ^a} is unipotent for some a at each of the generic points of $\operatorname{Spec} R_{q,n}$, all of which lie in characteristic zero. This is an immediate consequence of Lemma 4.1. □

Let L be a finite extension of $\mathcal{K}$. We call an L-point of $X^0_{q,n}$ *integral* if the corresponding map $R_{q,n} \to L$ factors through the ring of integers $\mathcal{O}_L$.

Lemma 4.4. *Let x be an L-point of $X^0_{q,n}$, and suppose that the eigenvalues of Fr_x lie in $\mathcal{O}^\times_{L'}$ for some finite extension L' of L. Then there is an integral point of $X^0_{q,n}$ in the GL_n-orbit of x.*

Proof. Extending L if necessary, we may assume that the eigenvalues of σ_x are in L, and hence $\mathcal{O}_L$. Then (for instance, by putting σ_x in Jordan normal form) we can find an $\mathcal{O}_L$-sublattice M of L^n preserved by σ_x. Using $\mathrm{Fr}_x\, \sigma_x\, \mathrm{Fr}_x^{-1} = \sigma_x^q$, we find that $\mathrm{Fr}_x M$, $\mathrm{Fr}_x^2 M$, etc. are also preserved by σ_x. Consider the lattice M' given by $M + \mathrm{Fr}_x M + \cdots + \mathrm{Fr}_x^{n-1} M$; it is clearly preserved by σ_x. On the other hand, since Fr_x is annihilated by a polynomial with integral coefficients, $\mathrm{Fr}_x^n M$ is contained in

M', and hence $\mathrm{Fr}_x M'$ is contained in M'. Since Fr_x has unit determinant we must have $\mathrm{Fr}_x M' = M'$. Thus M' is stable under both Fr_x and σ_x. Choosing a basis for M', we find an integral point of $X^0_{q,n}$ in the same GL_n-orbit as x. □

Lemma 4.5. *For any positive integer m, and any element λ of $\mathcal{O}_L^\times$, there is an element $g_{m,\lambda}$ of $\mathrm{GL}_m(L)$, with unit eigenvalues, such that $g_{m,\lambda} J_{m,\lambda^q} g_{m,\lambda}^{-1} = J_{m,\lambda}^q$, where $J_{m,\lambda}$ is the unipotent Jordan block of size m.*

Proof. The matrices J_{m,λ^q} and $J^q_{m,\lambda}$ are regular with the same eigenvalues, hence conjugate by some $g' \in \mathrm{GL}_m(L)$. Since J_{m,λ^q} is contained in a unique Borel subgroup of GL_m (namely, the standard one), the same is true of J_{m,λ^q}. Thus g' normalizes the standard Borel, so g' is upper triangular. The eigenvalues of g' are thus given by its diagonal entries $g'_1, \dots, g'_m$. Comparing the $(i, i+1)$ entries of $J^q_{m,\lambda}$ and J_{m,λ^q} we find that that $\frac{g'_{i+1}}{g'_i} = \lambda^{q-1} q$. In particular, multiplying g' by a suitable scalar we may assume g' has integral eigenvalues, as desired. □

Proposition 4.6. *The images of the integral points of $X^0_{q,n}$ are dense in $X^0_{q,n}$.*

Proof. Fix a point $(\mathrm{Fr}_x, \sigma_x)$ of $X^0_{q,n}$. After conjugating σ_x appropriately we may assume that σ_x is in Jordan normal form (and thus in particular has integral entries, since we have shown that the eigenvalues of σ_x are roots of unity). Moreover, since σ_x is conjugate to its qth power, for any eigenvalue λ of σ there is a size-preserving bijection between the Jordan blocks of σ_x of eigenvalue λ and those of eigenvalue λ^q. Let (m_i, λ_i) denote the size and eigenvalue of the ith Jordan block of σ_x. Then we can find a permutation matrix w such that $w\sigma_x w^{-1}$ is also in Jordan normal form, but where the ith Jordan block is of size m_i with eigenvalue λ_i^q. Let g be the block diagonal matrix whose ith block is the matrix g_{m_i,λ_i} from the above lemma. Then $gw\sigma_x(gw)^{-1} = \sigma_x^q$. Moreover gw has unit eigenvalues, as some power of gw is block diagonal with blocks given by powers of the matrices g_{m_i,λ_i}. Thus by Lemma 4.4 we can find an integral point $(\mathrm{Fr}'_x, \sigma'_x)$ of $X^0_{q,n}$ in the GL_n-orbit of the point (gw, σ_x).

Now consider the condition $g'\sigma'_x = \sigma'_x g'$, for arbitrary matrices g'. This is a linear condition on g' with coefficients in $\mathcal{O}_L$. The scheme parameterizing such g' is not quite a vector space scheme over $\mathcal{O}_L$ (it need not be flat over $\mathcal{O}_L$), but the closure of its general fiber is such a scheme. Let U be the open subscheme of this closure consisting of invertible g'. Then U contains the identity in particular, so its special fiber is nonempty. However, in an open subset of a vector space scheme over $\mathcal{O}_L$ whose special fiber is nonempty, the $\mathcal{O}_L$-points form a dense subset. Thus integral points are dense in U.

On the other hand, the points $(\mathrm{Fr}'_x u, \sigma'_x)$, as u runs over the integral points of U, are all integral points of $X^0_{q,n}$, and (since integral points of U are dense in U) their closure is the set of all points $(\mathrm{Fr}'_y, \sigma'_x)$ in $X^0_{q,n}$. Conjugating by integral points of GL_n, which are clearly dense in GL_n, we find that the closure of the integral points contains the entire locus of points $(\mathrm{Fr}''_x, \sigma''_x)$ with σ''_x conjugate to σ_x. Since σ_x was chosen arbitrarily the result follows. □

Corollary 4.7. *The ring $R_{q,n}$ is ℓ-adically separated; that is, the intersection of the ideals $\ell^i R_{q,n}$ is zero.*

Proof. Let f be an element of $R_{q,n}$ that is divisible by ℓ^i for all i. Then, for any integral point $x : R_{q,n} \to \mathcal{O}_L$, the image $x(f)$ is divisible by ℓ^i for all i and is therefore zero. In other words, f vanishes on a dense subset of $X^0_{q,n}$. Since $X^0_{q,n}$ is reduced, f is zero. □

Now fix a Frobenius element $\tilde{\mathrm{Fr}}$ in W_F, and a topological generator $\tilde{\sigma}$ of the quotient $I_F/I_F^{(\ell)}$. Let t_ℓ be the isomorphism of $I_F/I_F^{(\ell)}$ with the additive group of $\mathbb{Z}_\ell$ that takes $\tilde{\sigma}$ to 1. By Corollary 4.3, for some positive integer a the matrix σ^{ℓ^a} in $\mathrm{GL}_n(R_{q,n})$ is unipotent; that is, its characteristic polynomial is $(X-1)^n$. The following lemma allows us to make sense of $(\sigma^{\ell^a})^b$ for any $b \in \mathbb{Z}_\ell$:

Lemma 4.8. *Let R be a flat, ℓ-adically separated $\mathbb{Z}_\ell$-algebra, and $M \in \mathrm{GL}_n(R)$ such that $(M-1)^n = 0$. Then there exists a unique ℓ-adically continuous homomorphism $\phi_M : \mathbb{Z}_\ell \to \mathrm{GL}_n(R)$ such that for all $b \in \mathbb{Z}$, $\phi_M(b) = M^b$.*

Proof. Consider the power series $\exp t \log(1+X)$ in $\mathbb{Q}[t][[X]]$, and let $p_i(t)$ be the coefficient of X^i in this power series. For any i, and any integer b, Let N_i be the $(i+1)$ by $(i+1)$ Jordan block with eigenvalue zero; then $p_i(b)$ is the upper right entry of $(1+N_i)^b$, and is thus an integer. In particular each p_i is a $\mathbb{Z}_\ell$-valued function on $\mathbb{Z}_\ell$. Given M as above, and $t \in \mathbb{Z}_\ell$, we may thus define ϕ_M by

$$\phi_M(t) = 1 + p_1(t)(M-1) + \cdots + p_{n-1}(t)(M-1)^{n-1},$$

and it is clear that this has the claimed properties. □

(Recall that for an ℓ-adically separated ring A, and a locally profinite group H, a representation $\rho : H \to \mathrm{GL}_n(A)$ is ℓ-adically continuous if, for all positive integers i, the preimage of the subgroup $\mathrm{Id} + \ell^i M_n(A)$ of $\mathrm{GL}_n(A)$ is open in H.)

We will henceforth write $(\sigma^{\ell^a})^b$ for $\phi_{(\sigma^{\ell^a})}(b)$, given $b \in \mathbb{Z}_\ell$.

We thus have an ℓ-adically continous representation $\rho_{F,n} : W_F \to \mathrm{GL}_n(R_{q,n})$ defined by

$$\rho_{F,n}(\tilde{\mathrm{Fr}}^i w) = \mathrm{Fr}^i \sigma^j (\sigma^{\ell^a})^b,$$

for any $w \in I_F$ and any $j \in \mathbb{Z}$, $b \in \mathbb{Z}_\ell$ such that $j + \ell^a b = t_\ell(w)$. Note that, by the above lemma, this is the *unique* ℓ-adically continuous representation that takes $\tilde{\mathrm{Fr}}$ to Fr and $\tilde{\sigma}$ to σ.

The pair $(R_{q,n}, \rho_{F,n})$ has the following universal property, which is easily seen to characterize the pair up to isomorphism:

Proposition 4.9. *For any finitely generated, ℓ-adically separated $W(k)$-algebra A, and any framed, ℓ-adically continuous representation $\rho : W_F/I_F^{(\ell)} \to \mathrm{GL}_n(A)$, there is a unique map: $R_{q,n} \to A$ such that ρ is the base change of $\rho_{F,n}$.*

Proof. Given ρ, we have a pair of matrices $(\rho(\tilde{\mathrm{Fr}}), \rho(\tilde{\sigma}))$ in $\mathrm{GL}_n(A)$, satisfying

$$\rho(\tilde{\mathrm{Fr}})\rho(\tilde{\sigma})\rho(\tilde{\mathrm{Fr}})^{-1} = \rho(\tilde{\sigma})^q,$$

and hence a map $S_{q,n} \to A$. Moreover, since the restriction of ρ to I_F factors through $I_F/I_F^{(\ell)}$ and is ℓ-adically continuous, the eigenvalues of $\rho(\tilde{\sigma})$ are ℓ-power roots of unity. Thus the map $S_{q,n} \to A$ factors through $R_{q,n}$ and the result follows. □

If we regard the $\overline{\mathcal{K}}$-points of $X^0_{q,n}$ as framed representations of $W_F/I_F^{(\ell)}$, then one can show:

Proposition 4.10. *Let x be a $\overline{\mathcal{K}}$-point of $X^0_{q,n}$. Then there is a point y in the closure of the GL_n-orbit of x such that the representation ρ_y is semisimple.*

Proof. Replacing x with a point in the same GL_n-orbit, we may assume that the framing on ρ_x is such that ρ_x is block upper triangular, with block sizes $n_1, \dots n_r$, and that for $1 \leq i \leq r$, the restriction ρ_i of ρ_x to the ith diagonal block is irreducible. Let M be the block diagonal matrix whose ith block is given by t^i times the n_i by n_i identity matrix, for some parameter t. Then the limit, as t approaches zero, of $M\rho_x M^{-1}$ exists and is semisimple. □

We will later need the following observation about the representation $\rho_{F,n}$.

Proposition 4.11. *As x varies over the $\overline{\mathcal{K}}$-points of $X^0_{q,n}$, the restriction of ρ_x^{ss} to I_F is constant on connected components of $X^0_{q,n} \times_{W(k)} \overline{\mathcal{K}}$.*

Proof. The restriction of ρ_x^{ss} to I_F is determined by the characteristic polyomial of σ_x; since the eigenvalues of σ_x have bounded ℓ-power order there are only finitely possible characteristic polynomials of σ_x. □

5. The inertial subalgebra of $S_{q,n}$

Our next goal is to study the finite rank $W(k)$-subalgebra of $S_{q,n}$ generated by the coefficients of the characteristic polynomial of σ. Consider the map:

$$W(k)[r_1, \dots, r_n, r_n^{-1}] \to S_{q,n}$$

that takes r_i to the coefficient of X^{n-i} in this characteristic polynomial.

By the theory of symmetric functions, for $1 \leq i \leq n$ there are unique polynomials $P_{i,q}$ in the variables $r_1, \dots, r_n$ with the following property: for all $t_1, \dots, t_n \in \overline{\mathcal{K}}$, define $r_1, \dots, r_n \in \overline{\mathcal{K}}$ by the identity:

$$(X - t_1) \cdots (X - t_n) = X^n + r_1 X^{n-1} + r_2 X^{n-2} + \cdots + r_n.$$

Then the $P_{i,q}$ are the unique polynomials satisfying:

$$(X - t_1^q) \cdots (X - t_n^q) = X^n + P_{1,q}(r_1, \dots, r_n) X^{n-1} + \cdots + P_{n,q}(r_1, \dots, r_n).$$

Since σ is conjugate to its qth power, for $1 \leq i \leq n$ the element $P_{i,q}(r_1, \dots, r_n) - r_i$ lies in the kernel of the map $W(k)[r_1, \dots, r_n, r_n^{-1}] \to S_{q,n}$. Let $I_{q,n}$ denote the ideal of $W(k)[r_1, \dots, r_n, r_n^{-1}]$ generated by the $P_{i,q}(r_1, \dots, r_n) - r_i$, and let $B_{q,n}$ denote the quotient $W(k)[r_1, \dots, r_n, r_n^{-1}]/I_{q,n}$. We will show that in fact the map $B_{q,n} \to S_{q,n}$ is injective, and that moreover its image in $S_{q,n}$ is saturated.

We will now realize $B_{q,n}$ as a quotient of $S_{q,n}$ in a natural way. We are grateful to Jack Shotton for making us aware of the following construction, which is adapted from Proposition 7.10 in [Sh]. (Shotton uses a slightly different form for the matrix σ, that is less convenient for our purposes, but the arguments are otherwise exactly analogous.)

Let $Y \subseteq \operatorname{Spec} S_{q,n}$ denote the locus on which σ has the form:

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -r_n \\ 1 & 0 & 0 & \dots & 0 & -r_{n-1} \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & -r_1 \end{pmatrix}.$$

(that is, on which σ is the "companion matrix" of the polynomial $X^n + r_1 X^{n-1} + \cdots + r_n$.) We may embed Y as an open subscheme of the scheme Y' parameterizing pairs of matrices (Fr, σ) such that σ is invertible of the above form, the characteristic polynomial of σ is equal to that of σ^q, and $\mathrm{Fr}\,\sigma = \sigma^q\,\mathrm{Fr}$. Then Y is simply the open subscheme of Y' on which Fr is invertible. The scheme Y' then maps to $\operatorname{Spec} B_{q,n}$ via the map that takes (Fr, σ) to the tuple $(r_1, \dots, r_n)$.

We have a map: $Y' \to \operatorname{Spec} B_{q,n} \times_{W(k)} \mathbb{A}^n_{W(k)}$ that takes (Fr, σ) to the point $(r_1, \dots, r_n, \mathrm{Fr}(e_1))$, where $e_1, \dots, e_n$ is the standard basis for $W(k)^n$. In fact, one then has:

Proposition 5.1. *The map $Y' \to \operatorname{Spec} B_{q,n} \times_{W(k)} \mathbb{A}^n_{W(k)}$ is an isomorphism.*

Proof. We describe an inverse map. Given $(r_1, \dots, r_n, v)$ in $\operatorname{Spec} B_{q,n} \times_{W(k)} \mathbb{A}^n_{W(k)}$ we associate the pair (Fr, σ), where σ has the above form with $-r_n, \dots, -r_1$ in the right column, and Fr is defined by: $\mathrm{Fr}\, e_i = \sigma^{(i-1)q} v$ for $1 \leq i \leq n$. One verifies easily that for $1 \leq i \leq n-1$, we have $\mathrm{Fr}\,\sigma(e_i) = \sigma^q\,\mathrm{Fr}(e_i)$. On the other hand, we have:

$$\sigma^q\,\mathrm{Fr}\, e_n - \mathrm{Fr}\,\sigma e_n = (\sigma^q)^n + r_1(\sigma^q)^{n-1} + \cdots + r_n)v = P_\sigma(\sigma^q)v,$$

where P_σ is the characteristic polynomial of σ. The relations on $\tilde{B}_{q,n}$ guarantee that $P_\sigma = P_{\sigma^q}$, so $P_\sigma(\sigma^q)v = 0$ by Cayley-Hamilton.

We thus have a well-defined map that is clearly a right inverse to the map constructed above. To see that it is also a left inverse, note that if $\mathrm{Fr}\,\sigma = \sigma^q\,\mathrm{Fr}$, and $\mathrm{Fr}(e_1) = v$, then we must have

$$\mathrm{Fr}\, e_i = \mathrm{Fr}\,\sigma(e_{i-1}) = \sigma^q\,\mathrm{Fr}\, e_{i-1}$$

so by induction Fr is determined by $\mathrm{Fr}(e_1)$. □

Lemma 5.2. *Let B be a finite rank $W(k)$-algebra, and V an open subset of $\operatorname{Spec} B \times_{W(k)} \mathbb{A}^n_{W(k)}$ such that the projection $V \to \operatorname{Spec} B$ is surjective. Then the map from B to $\mathcal{O}_V$ induced by the projection of V onto $\operatorname{Spec} B$ is injective. If moreover, B is flat over $W(k)$, then the image of B in $\mathcal{O}_V$ is saturated.*

Proof. For each closed point x of $\operatorname{Spec} B$, there exists an element $\overline{a}_x$ of k^n such that $(x, \overline{a}_x)$ lies in V. Lift $\overline{a}_x$ to a $W(k)$-point a_x of $\mathbb{A}^n_{W(k)}$, and let $V_x = V \cap (\operatorname{Spec} B \times_{W(k)} a_x)$. Then the projection of V to $\operatorname{Spec} V$ identifies V_x with an open subset of $\operatorname{Spec} B$, and as x varies, the V_x cover $\operatorname{Spec} B$. If b is an element of B that maps to zero in $\mathcal{O}_V$, then it vanishes in particular on each V_x and hence on $\operatorname{Spec} B$, so injectivity is clear.

Now consider an element b of $B/\ell B$, and suppose B maps to zero in $\mathcal{O}_V/\ell\mathcal{O}_V$. Then b maps to zero in $\mathcal{O}_{V_x}/\ell\mathcal{O}_{V_x}$ for all x, but since the V_x are an open cover of $\operatorname{Spec} B$ this means b is zero in $B/\ell B$. □

We can now show:

Proposition 5.3. *The map $B_{q,n} \to S_{q,n}$ is injective with saturated image.*

Proof. We first show that the projection map from Y to $\operatorname{Spec} B$ is surjective. Indeed, for any algebraically closed field L that is a $W(k)$-algebra, and any L-point $(r_1, \dots, r_n)$ of $\operatorname{Spec} B$, the corresponding σ is a regular element of L whose characteristic polynomial is equal to that of σ^q. In particular the eigenvalues of σ are roots of unity of order prime to q. It is then clear, by considering the Jordan normal form of σ, that σ^q is also regular. Over L any two regular matrices with the same characteristic polynomial are conjugate, so there exists an element Fr of $\mathrm{GL}_n(L)$ that conjugates σ to σ^q. Then (Fr, σ) is an L-point of T mapping to $(r_1, \dots, r_n)$.

The lemma now shows that the map from $B_{q,n}$ to $\mathcal{O}_Y$ is injective; since this map factors through $S_{q,n}$ we see that $B_{q,n}$ embeds in $S_{q,n}$. Thus $B_{q,n}$ is flat over $W(k)$, and the lemma then shows that its image in $\mathcal{O}_Y$ is saturated. Once again using that the map from $B_{q,n}$ to $\mathcal{O}_Y$ factors through $S_{q,n}$ we see that $B_{q,n}$ is also saturated in $S_{q,n}$. □

The map $B_{q,n} \to S_{q,n}$ induces a map $B_{q,n,1} \to R_{q,n}$, where $B_{q,n,1}$ is the direct factor of $B_{q,n}$ whose $\overline{\mathcal{K}}$-points correspond to conjugacy classes whose reduction modulo ℓ is the identity. Proposition 4.11, together with Proposition 5.3, shows that $B_{q,n,1}$ is precisely the subalgebra of $R_{q,n}$ consisting of elements whose value at a $\overline{\mathcal{K}}$-point x of $\operatorname{Spec} R_{q,n}$ depends only on the semisimplification of the restriction of ρ_x to I_F.

6. The symmetrizing form on $B_{q,n}$

We now relate $B_{q,n}$ with the endomorphism ring $\overline{E}_{q,n}$ of the Gelfand-Graev representation. We first work over $\overline{\mathcal{K}}$; since both $B_{q,n}$ and $\overline{E}_{q,n}$ are reduced, constructing an isomorphism of $B_{q,n} \otimes \overline{\mathcal{K}}$ with $\overline{E}_{q,n} \otimes \overline{\mathcal{K}}$ amounts to constructing a bijection on their $\overline{\mathcal{K}}$-points.

Recall that the $\overline{\mathcal{K}}$-points of $\operatorname{Spec} \overline{E}_{q,n}$ are in bijection with the isomorphism classes of irreducible generic representations of $\overline{G}$ and therefore (via Deligne-Lusztig restriction) with the equivalence classes of pairs (w, φ) where w is an element of the Weyl group of $\overline{G}$ and $\varphi : \overline{T}_w \to \overline{\mathcal{K}}^\times$ is a character. On the other hand, a $\overline{\mathcal{K}}$-point of $\operatorname{Spec} B_{q,n}$ is represented by an invertible diagonal matrix, with entries in $\overline{\mathcal{K}}$, that is conjugate to its q-th power; that is, it is an invertible diagonal matrix t such that there exists a permutation matrix w with $t^w = t^q$.

In order to construct a natural bijection between these two sets we must fix some choices. First, we identify $\mathrm{GL}_n(\overline{\mathcal{K}})$ with the Langlands dual group $\hat{G}$ of G, with (diagonal) maximal torus $\hat{T}$. Second, we choose a topological generator $\tilde{\sigma}$ of the tame inertia group I_F/P_F of F. Local class field theory gives an isomorphism:

$$I_F/P_F \cong \varprojlim \mathbb{F}_{q^n}^\times \cong \varprojlim \operatorname{Hom}(\frac{1}{q^n-1}\mathbb{Z}/\mathbb{Z}, \overline{\mathbb{F}}_q^\times),$$

where the first limit is over the norm maps, and the transition maps in the second limit, for m dividing n, are given by "multiplication by $\frac{q^n-1}{q^m-1}$".

On the other hand we have a chain of natural isomorphisms:

$$\operatorname{Hom}((\mathbb{Q}/\mathbb{Z})^{(p)}, \overline{\mathbb{F}}_q^\times) = \operatorname{Hom}(\varinjlim(\frac{1}{q^n-1}\mathbb{Z}/\mathbb{Z}), \overline{\mathbb{F}}_q^\times) \cong I_F/P_F,$$

so our choice of $\tilde{\sigma}$ gives us a natural map $(\mathbb{Q}/\mathbb{Z})^{(p)} \to \overline{\mathbb{F}}_q^\times$ that is easily seen to be an isomorphism.

Now fix a w in the Weyl group $W(\overline{G})$; we identify $W(\overline{G})$ with the group of permutation matrices in $\mathrm{GL}_n(\overline{\mathcal{K}})$. Let X be the character group of the torus $\overline{\mathcal{T}}_w$ of $\overline{\mathcal{G}}$; then X is dual to the character group X' of the group of diagonal matrices in $\mathrm{GL}_n(\overline{\mathcal{K}})$. We have an isomorphism $\overline{\mathcal{T}}_w(\mathbb{F}_q) \cong \operatorname{Hom}(X/(\operatorname{Fr}_q - 1)X, \overline{\mathbb{F}}_q^\times)$, where Fr_q is the q-power Frobenius. If we denote by $\mu^{(p)}$ the prime-to-p roots of unity in $\overline{\mathcal{K}}^\times$, then we have an isomorphism:

$$\operatorname{Hom}(\overline{T}_w, \mu^{(p)}) \cong X/(\operatorname{Fr}_q - 1)X \otimes \operatorname{Hom}(\overline{\mathbb{F}}_q^\times, \mu^{(p)}).$$

Noting that Fr_q acts on X by qw, and applying the duality isomorphism:

$$X/(qw-1)X \cong \mathrm{Hom}(X'/(qw-1)X', (\mathbb{Q}/\mathbb{Z})^{(p)})$$

as well as our isomorphism of $(\mathbb{Q}/\mathbb{Z})^{(p)}$ with $\overline{\mathbb{F}}_q^{\times}$ arising from our choice of s, we see that $\mathrm{Hom}(\overline{T}_w, \mu^{(p)})$ is naturally isomorphic to $\mathrm{Hom}(X'/(qw-1)X', \mu^{(p)})$. An element of the latter is precisely a diagonal matrix t, with entries in $\overline{\mathcal{K}}$, such that $(t^w)^q = t$. We let $T_q^{w^{-1}}$ denote the set of such matrices.

This construction associates to every w, and every character $\varphi : \overline{T}_w \to \overline{\mathcal{K}}^{\times}$, an element of $T_q^{w^{-1}}$. One easily verifies that it sends equivalent pairs $(\overline{\mathcal{T}}_w, \varphi)$ and $(\overline{\mathcal{T}}_{w'}, \varphi')$ to conjugate diagonal matrices, and further induces a bijection between $\overline{\mathcal{K}}$-points of $\mathrm{Spec}\, \overline{E}_{q,n}$ and those of $\mathrm{Spec}\, B_{q,n}$. We thus obtain an isomorphism of $\overline{E}_{q,n} \otimes \overline{\mathcal{K}}$ with $B_{q,n} \otimes \overline{\mathcal{K}}$. This isomorphism is $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$-equivariant and thus descends to an isomorphism of $\overline{E}_{q,n}[\frac{1}{\ell}]$ with $B_{q,n}[\frac{1}{\ell}]$.

Remark 6.1. The choices made in defining the bijection above means that this bijection is compatible with local Langlands in the following sense: let π be an irreducible depth zero generic representation of G over $\overline{\mathcal{K}}$, and let ρ be its Langlands parameter. If K_1 denotes the kernel of the map $\mathcal{G}(\mathcal{O}_G) \to \overline{\mathcal{G}}(\mathbb{F}_q)$, then π^{K_1} is an irreducible generic $\overline{\mathcal{K}}$-representation of $\overline{G}$, and hence gives rise to a $\overline{\mathcal{K}}$-point of $\mathrm{Spec}\, \overline{E}_{q,n}$. On the other hand, the conjugacy class of the semisimplification of $\rho(\sigma)$ gives a $\overline{\mathcal{K}}$-point of $\mathrm{Spec}\, B_{q,n}$. The bijection constructed above identifies these two points for every choice of π and ρ.

Since $B_{q,n}$ and $\overline{E}_{q,n}$ are ℓ-torsion free, we may regard them as $W(k)$-lattices in $B_{q,n}[\frac{1}{\ell}] \cong \overline{E}_{q,n}[\frac{1}{\ell}]$. A priori it is not clear that either lattice is contained in the other. We will show later that in fact these lattices coincide, but this is quite difficult- it will emerge from the same inductive argument that proves both the weak and strong conjecture in section 10. For the moment, it will suffice to prove something much weaker.

Recall that one has a symmetrizing form $\theta : \overline{E}_{q,n} \to W(k)$; the inclusion $B_{q,n} \to \overline{E}_{q,n}[\frac{1}{\ell}]$ allows us to regard θ as a map from $B_{q,n}$ to $\overline{\mathcal{K}}$. The goal of the remainder of this section is to prove:

Theorem 6.2. *The map $\theta : B_{q,n} \to \overline{\mathcal{K}}$ takes values in $W(k)$.*

As a corollary, we immediately deduce

Corollary 6.3. *Suppose that the isomorphism $B_{q,n}[\frac{1}{\ell}] \cong \overline{E}_{q,n}[\frac{1}{\ell}]$ identifies $\overline{E}_{q,n}$ with a subring of $B_{q,n}$. Then this isomorphism identifies $\overline{E}_{q,n}$ with $B_{q,n}$.*

Proof. (c.f. Lemma 3.8 of [BK]) If $\overline{E}_{q,n}$ is contained in $B_{q,n}$, then $\theta(be)$ lies in $W(k)$ for all $b \in B_{q,n}$, $e \in \overline{E}_{q,n}$; thus $B_{q,n}$ is contained in the dual lattice to $\overline{E}_{q,n}$ with respect to θ. But since θ is a symmetrizing form on $\overline{E}_{q,n}$, this dual lattice is $\overline{E}_{q,n}$. Thus $B_{q,n}$ and $\overline{E}_{q,n}$ must coincide inside $\overline{E}_{q,n}[\frac{1}{\ell}]$. □

In order to prove Theorem 6.2 we compute the values of θ on a $W(k)$-spanning set for $B_{q,n}$. By definitition we have a surjection:

$$W(k)[X']^{S_n} = W(k)[r_1, \dots, r_n, r_n^{-1}]^{S_n} \to B_{q,n}$$

with kernel $I_{q,n}$. For each character $\lambda \in X'$, let N_λ denote the subgroup of S_n normalizing λ. Then the elements

$$r_\lambda = \frac{1}{\# N_\lambda} \sum_{w \in S_n} \lambda^w$$

form a $W(k)$-basis of $W(k)[X']^{S_n}$, as λ runs over the elements of X', so their images in $B_{q,n}$ (which we also, slightly abusively, denote by r_λ) span $B_{q,n}$ over $W(k)$.

Lemma 6.4. *For $\lambda \in X'$, let M_λ denote the number of $w \in S_n$ such that the restriction of λ to the subgroup T_q^w of $\mathrm{Hom}(X', \overline{\mathcal{K}}^\times)$ is trivial. Then we have:*

$$\theta(r_\lambda) = \frac{M_\lambda}{\# N_\lambda}.$$

Proof. Let x be a $\overline{\mathcal{K}}$-point of $\mathrm{Spec}\, B_{q,n}$, and let e_x denote the element of $B_{q,n} \otimes \overline{\mathcal{K}}$ that takes the value 1 at x and zero at all other $\overline{\mathcal{K}}$-points of $\mathrm{Spec}\, B_{q,n}$. Our construction of the isomorphism $\overline{E}_{q,n} \otimes \overline{\mathcal{K}} \cong B_{q,n} \otimes \overline{\mathcal{K}}$, together with Proposition 2.4, shows that

$$\theta(e_x) = \frac{1}{n!} \sum_{w \in S_n} \frac{N(w^{-1}, x)}{\# \overline{T}_w(\mathbb{F}_q)} = \frac{1}{n!} \sum_{w \in S_n} \frac{N'(w,x)}{\# T_q^{w^{-1}}},$$

where $N'(w,x)$ denotes the number of elements of T_q^w in the equivalence class corresponding to x. It follows that we have:

$$\theta(r_\lambda) = \frac{1}{n!} \sum_x r_\lambda(x) \sum_{w \in S_n} \frac{N'(w,x)}{\# T_q^{w^{-1}}}.$$

Since $r_\lambda(t)$ depends only on the equivalence class of $t \in T_q^w$, we can rewrite this as:

$$\theta(r_\lambda) = \frac{1}{n!} \sum_{w \in S_n} \frac{1}{\# T_q^{w^{-1}}} \sum_{t \in T_q^{w^{-1}}} \frac{1}{\# N_\lambda} \sum_{v \in S_n} \lambda^v(t).$$

Changing the order of the summation, we obtain:

$$\theta(r_\lambda) = \frac{1}{n! \# N_\lambda} \sum_{v \in S_n} \sum_{w \in S_n} \frac{1}{\# T_q^{w^{-1}}} \sum_{t \in T_q^{w^{-1}}} \lambda^v(t),$$

and the innermost sum is equal to 0 if λ^v is nontrivial on $T_q^{w^{-1}}$ and equal to $\# T_q^{w^{-1}}$ otherwise. Thus the sum over w is equal to M_{λ^v} which is equal to M_λ. We thus have $\theta(r_\lambda) = \frac{M_\lambda}{\# N_\lambda}$ as claimed. □

In light of this result, the proof of Theorem 6.2 is reduced to the following result:

Lemma 6.5. *For any $\lambda \in X'$, the order of N_λ divides M_λ.*

It is clear that the set of w such that λ is trivial on T_w^q is stable under conjugation by elements of N_λ, but of course this action is not faithful, so the divisibility is not immediate.

We begin by observing that N_λ is the Weyl group of the Levi subgroup of GL_n centralizing λ. This Levi corresponds to a partition of the $\{1, 2, \ldots, n\}$ into subsets, and N_λ is then the subgroup of S_n that preserves this partition. In particular if w lies in N_λ, then any cycle occuring in the cycle decomposition of w also lies in N_λ.

Now let $N_{\lambda,w}$ denote the centralizer of w in N_λ. Let $O(w)$ be the partition of the set $\{1, \ldots, n\}$ into orbits under the action of w; then conjugation by $N_{\lambda,w}$ permutes the orbits of w, yielding a map $N_{\lambda,w} \to \mathrm{Aut}(O(w))$, where $\mathrm{Aut}(O(w))$ is the group of permutations of $O(w)$.

Definition 6.6. We will say that w is N_λ-minimal if the map $N_{\lambda,w} \to \mathrm{Aut}(O(w))$ is injective.

Note that the property of being N_λ-minimal is stable under N_λ-conjugacy. Given an arbitrary N_λ-conjugacy class in S_n, we will associate an N_λ-minimal conjugacy class in a natural way. On the level of specific permutations w this construction will depend not just on w but on a particular choice of cycle representation for w. Here by a "cycle representation" of w we mean an unordered collection of expressions of the form $(x_1 \dots x_r)$, with $x_1, \dots, x_r$ distinct elements of $\{1, \dots, n\}$, that correspond to a disjoint set of cycles whose product is w. To give a cycle representation of w is equivalent to specifying, for each orbit x of w on $\{1, \dots, n\}$, a distinguished element x_1 of the orbit x.

Now fix $w \in S_n$, along with a cycle representation of w, and let K be the kernel of the map from $N_{\lambda,w}$ to $\mathrm{Aut}(O(w))$. Then K acts on each orbit $O(w)$; such an orbit x comes from a cycle $(x_1 \dots x_r)$ in our chosen cycle representation of w. Since K centralizes w, it must "cyclically permute" the elements of this orbit; that is, the action of K factors through a map $K \to \mathbb{Z}/r\mathbb{Z}$, where $s \in \mathbb{Z}/r\mathbb{Z}$ acts by sending each x_i to x_{i+s}, and the indices are considered modulo r. Let m be the order of the image of the map $K \to \mathbb{Z}/r\mathbb{Z}$, and set $s = \frac{r}{m}$. Let $x^{\min}$ be the permutation given by the product of the m disjoint cycles $(x_1 \dots x_s)(x_{s+1} \dots x_{2s}) \dots (x_{r-s+1} \dots x_r)$. We then define $w^{\min}$ to be the product, over all cycles $x \in O(w)$, of the permutations $x^{\min}$. It is clear from the construction that if w is N_λ-minimal then $w^{\min} = w$.

This construction depends on our choice of cycle representation of w; in particular if we represented the cycle $x = (x_1 \dots x_r)$ as $(x_{t+1} \dots x_{t+r})$ instead then we would obtain the product of cycles

$$(x_{t+1} \dots x_{t+s})(x_{t+s+1} \dots x_{t+2s}) \dots (x_{r+t-s+1} \dots x_{r+t})$$

instead of the product:

$$(x_1 \dots x_s)(x_{s+1} \dots x_{2s}) \dots (x_{r-s+1} \dots x_r).$$

Note that the former is N_λ-conjugate to the latter, via the permutation that, for each $0 \leq a < \frac{r}{s}$ and each $1 \leq b \leq s$, takes x_{as+b} to x_{t+as+c}, where c is the unique integer between 1 and s such that $as + b$ is congruent to $as + t + c$ modulo s. However, the two permutations are of course not equal. Thus changing the cycle representation of w conjugates $w^{\min}$ by an element of N_λ. In particular the N_λ-conjugacy class $[w^{\min}]$ depends only on w and not its cycle representation.

On the other hand, if we fix a $v \in N_\lambda$, and a cycle representation of w, then conjugating this cycle representation by v gives a cycle representation of vwv^{-1}. Then if we compute $w^{\min}$ and $(vwv^{-1})^{\min}$ using these cycle representations it is easy to see that $(vwv^{-1})^{\min} = vw^{\min}v^{-1}$. In particular $[w^{\min}]$ depends only on the N_λ-conjugacy class of w.

Lemma 6.7. *For any $w \in S_n$, $w^{\min}$ is N_λ-minimal.*

Proof. Suppose for a contradiction that $w^{\min}$ is not N_λ-minimal, and let K be the kernel of the map $N_{\lambda,w^{\min}} \to \mathrm{Aut}(O(w^{\min}))$. Choose an element k of K other than the identity. By definition k preserves every orbit in $O(w)$ and acts nontrivially on at least one such orbit $x = (x_1 \dots x_r)$; we have an s such that $kx_i = x_{i+s}$ for all i. Let k' denote the permutation that sends x_i to x_{i+s} for all i and fixes all other elements. Then k' lies in N_λ, since k does and k' is a product of cycles of k. Moreover it is clear that k' commutes with $w^{\min}$.

Our construction of $w^{\min}$ from w implies that the $w^{\min}$-cycle x is contained in a w-cycle x' of the form $(x_1 \dots x_{r'})$ for some multiple r' of r, and that the cycles $(x_{r+1} \dots x_{2r})$, etc. are cycles of $w^{\min}$. Let k'' be the permutation that takes x_i to x_{i+s} for all $1 \leq i \leq r'$; then it is clear that k'' centralizes w. We will show that in fact k'' lies in N_λ; this gives a contradiction as then we have an element of $N_{\lambda,w}$ that acts by a shift of length s on the cycle x', meaning that in passing from w to $w^{\min}$ the cycle x' should decompose into cycles of length dividing s, and not cycles of length r as we have supposed.

To show that k'' lies in N_λ it suffices to show that for all i, x_{i+s} and x_i lie in the same N_λ-orbit. For $1 \leq i \leq r-s$ this is clear since k' lies in N_λ. On the other hand, since x' decomposes into cycles of length r in the cycle decomposition of $w^{\min}$, there is an element of N_λ that carries x_i to x_{i+r} for all i. The claim follows. □

The association $w \mapsto [w^{\min}]$ defines an equivalence relation $\sim$ on S_n, such that $w \sim v$ if, and only if, $[w^{\min}] = [v^{\min}]$. It is clear that each equivalence class for $\sim$ is a union of N_λ-orbits. We will show that in fact each equivalence class has cardinality equal to $\# N_\lambda$. We begin by fixing an N_λ-minimal w. Then we have an injection $N_{\lambda,w} \to \mathrm{Aut}(O(w))$. We will say two orbits x, x' in $O(w)$ are $N_{\lambda,w}$-equivalent if there is an element of $N_{\lambda,w}$ that takes x to x'. We then have:

Lemma 6.8. *Suppose w is N_λ-minimal, and let v be a permutation of $O(w)$ such that for all $x \in O(w)$, vx is $N_{\lambda,w}$-equivalent to x. Then there is a unique element $\tilde{v}$ of $N_{\lambda,w}$ whose image in $\mathrm{Aut}(O(w))$ is v. In particular, $N_{\lambda,w}$ is a product of symmetric groups.*

Proof. Uniqueness is clear from the definition of N_λ-minimality. For existence, fix an orbit $x \in O(w)$. Then there is an element v'_x of $N_{\lambda,w}$ that takes x to vx. We can then define $\tilde{v}$ to be the bijection on $\{1, 2, \dots, n\}$ that agrees with v'_x on x for all orbits x. Note that for all $1 \leq i \leq n$, we have $\tilde{v}(i) = v'_x(i)$ for x the w-orbit containing i; since v'_x is in N_λ we have $\lambda_i = \lambda_{v'_x(i)} = \lambda_{\tilde{v}(i)}$, so $\tilde{v}$ lies in N_λ. □

We now fix a particular N_λ-minimal w, and a particular cycle representation of w. Since w is N_λ-minimal we may (and do) choose this cycle representation so that it is preserved by the action of $N_{\lambda,w}$. Then given any $v \in N_{\lambda,w}$, define $\tilde{w}(v)$ to be the permutation constructed as follows: for orbit of v on $O(w)$, choose an x representing that orbit. The orbit x then corresponds to a term $(x_1 \dots x_r)$ in our chosen cycle representation of w. Let $\tilde{w}(v)_x$ be the permutation $(x_1 \dots x_r \, vx_1 \dots vx_r \dots v^{d-1}x_1 \dots v^{d-1}x_r)$, where d is the order of the v-orbit of x. Let $\tilde{w}(v)$ be the product, over a set of representatives x for the orbits of v on $O(w)$, of $\tilde{w}(v)_x$. Note that as a permutation, $\tilde{w}(v)$ is independent of our choices of representatives x but does depend on our choice of cycle representation of w. On the other hand, our initial choice of cycle representation of w, together with the choices of representatives x, gives rise to a cycle representation of $\tilde{w}(v)$.

Lemma 6.9. *Let u be an element of N_λ. Then u conjugates $\tilde{w}(v)$ to $\tilde{w}(v')$ if, and only if, u normalizes w and conjugates v to v'. Moreover, we have $\tilde{w}(v)^{\min} = w$.*

Proof. First assume that u normalizes w. Then u actually fixes our chosen cycle representation of w, since w is N_λ-minimal. It is then easy to see from the construction that $\tilde{w}(uvu^{-1}) = u\tilde{w}(v)u^{-1}$. .

Conversely, assume u conjugates $\tilde{w}(v)$ to $\tilde{w}(v')$. Let $x = (x_1 \dots x_r)$ be a cycle in our chosen representation of w, such that the induced cycle of $\tilde{w}(v)$ is $(x_1 \dots x_r\, vx_1 \dots vx_r \dots v^{d-1}x_1 \dots v^{d-1}x_r)$. Since u conjugates $\tilde{w}(v)$ to $\tilde{w}(v')$the cycle $(ux_1 \dots ux_r\, uvx_1 \dots uvx_r \dots uv^{d-1}x_1 \dots uv^{d-1}x_r)$ is a cycle of $\tilde{w}(v')$. This cycle contains a cycle $(y_1 \dots y_{r'})$ of our chosen representation of w. Thus, by construction of $\tilde{w}(v')$, there is an $s \in \mathbb{Z}/dr\mathbb{Z}$ such that the sequence:

$$ux_1, \dots ux_r, uvx_1, \dots uvx_r, \dots uv^{d-1}x_1, \dots uv^{d-1}x_r$$

coincides with the cyclic shift by s of the sequence:

$$y_1, \dots y_{r'}, vy_1, \dots vy_{r'}, \dots v^{d'-1}y_1, \dots v^{d'-1}y_{r'},$$

where $dr = d'r'$.

Since u and v both lie in N_λ, it follows that for all $1 \leq i \leq r$, and all integers j, x_i lies in the same N_λ-orbit as $y_{i+s+jr'}$, where the indices are taken modulo r. Let $a = (r, r')$. Then for all i, x_i lies in the same N_λ-orbit as x_{i+a}. Thus the permutation that takes x_i to x_{i+a} for all i and fixes all other elements lies in N_λ. This permutation clearly normalizes w and fixes all orbits of w, so must be the identity since w is N_λ-minimal. Thus $a = r$, so r divides r'. Similar reasoning shows that r' divides r, so in fact r equals r'.

Now for all $1 \leq i \leq r$, x_i is in the same N_λ-orbit as y_{i+s}; there is thus an element of $N_{\lambda,w}$ that carries the cycle $(x_1 \dots x_r)$ of w to the cycle $(y_1 \dots y_r)$. Since we chose our cycle representation of w to be $N_{\lambda,w}$-stable, there is also an element of $N_{\lambda,w}$ that takes x_i to y_i for all i. There is thus an element of $N_{\lambda,w}$ that takes x_i to x_{i+s} for all i, and fixes all other elements of $\{1, \dots, n\}$. Since w is minimal, this is impossible unless r divides s.

We have thus established that u takes the cycle $x = (x_1 \dots x_r)$ of w to the cycle $(v^e y_1, \dots, v^e y_r)$ for some e, which is also a cycle of w. Since x was arbitrary, u preserves the cycles of w and thus normalizes w. But now we have $\tilde{w}(uvu^{-1}) = u\tilde{w}(v)u^{-1} = \tilde{w}(v')$, and it is easy to see that this implies that $uvu^{-1} = v'$.

For the final claim, let $x = (x_1 \dots x_r)$ be a cycle in our chosen representation of w, contained in the cycle $(x_1 \dots x_r vx_1 \dots vx_r \dots v^{d-1}x_1 \dots v^{d-1}x_r)$ of $\tilde{w}(v)$. The subgroup of $N_{\lambda,w}$ preserving the latter cycle acts on it by cyclic shifts, and minimality of w implies that r divides the length of any of these shifts. On the other hand it is clear that the permutation that agrees with v on the set $\{x_1, \dots, x_r, vx_1, \dots, vx_r, \dots, v^{d-1}x_1, \dots v^{d-1}x_r\}$ and is the identity elsewhere induces a shift of length r on this cycle. Our construction of $\tilde{w}(v)^{\min}$ thus demands that we break this cycle of $\tilde{w}(v)$ into cycles of length r. Doing this for all cycles of $\tilde{w}(v)$ recovers w. □

We now show:

Lemma 6.10. *Suppose w is N_λ-minimal and $w' \sim w$. Then there exists $v \in N_\lambda(w)$ such that w' is N_λ-conjugate to $\tilde{w}(v)$.*

Proof. We first construct a cycle representation of w' such that the induced cycle representation of $(w')^{\min}$ is $N_{\lambda,(w')^{\min}}$-invariant. To do this, first fix any orbit of w' and choose a representation of the corresponding cycle; we then obtain representations of one or more cycles in $(w')^{\min}$, all of which are N_λ-conjugate. We then proceed inductively: for each orbit x of w', choose a cycle representation arbitrarily and consider the resulting cycles of $(w')^{\min}$. If these cycles are not N_λ-conjugate to other cycles of $(w')^{\min}$ that have already been constructed, there is nothing further to do and we may proceed to the next orbit of w'. If they are conjugate to cycles we have already constructed, it need not be the case that the corresponding cycle *representations* are N_λ-conjugate to those already extant (they may differ by a cyclic shift). However, adjusting our choice of cycle representation of x by a suitable shift we may arrange that this holds. Proceeding inductively we arrive at a $(w')^{\min}$ and an $N_{\lambda,(w')^{\min}}$-invariant cycle representation of it.

Now for each cycle x of w', our chosen decompositions give $x = (x_1 \dots x_{rs})$ in w', for some integers r, s such that the corresponding cycles of $(w')^{\min}$ are $(x_1 \dots x_r)$, $(x_{r+1} \dots x_{2r})$, etc. Let v'_x be the permutation that takes x_i to x_{i+r} for all i (indices modulo rs); then v'_x lies in N_λ. Taking v' to be the product over the orbits x of the v'_x we obtain an element of $N_{\lambda,(w')^{\min}}$ such that $w' = \widetilde{(w')^{\min}}(v')$. Now if $w' \sim w$ then there exists a $u \in N_\lambda$ such that $u(w')^{\min}u^{-1} = w$; taking $v = uv'u^{-1}$ we find that $uw'u^{-1} = \tilde{w}(v)$. □

Corollary 6.11. *Suppose w is N_λ-minimal. The number of w' such that $w' \sim w$ is equal to the order of N_λ.*

Proof. The previous lemmas show that the set of such w' is the union of the N_λ-conjugacy classes of $\tilde{w}(v)$, as v runs over a set of representatives for the conjugacy classes in $N_{\lambda,w}$. For each such v the size of its N_λ-conjugacy class is equal to $\frac{\#N_\lambda}{\#N_{\lambda,v}}$. For each v, the index of $N_{\lambda,w}$ in $N_{\lambda,v}$ is equal to the size of the $N_{\lambda,w}$-conjugacy class C_v of v. Thus the total number of such w' is the sum:

$$\#N_\lambda \sum_v \frac{\#C_v}{\#N_{\lambda,w}}$$

which is clearly equal to $\#N_\lambda$. □

We now relate the equivalence $\sim$ to M_λ. Specifically, we observe:

Proposition 6.12. *Suppose that $w \sim w'$. Then λ is trivial on T_q^w if, and only if, λ is trivial on $T_q^{w'}$.*

Proof. It suffices to show this in the case where $w' = w^{\min}$ (for some chosen cycle representation of w), as we can deduce any other case from this one and N_λ-conjugacy.

Let S_λ be the set of N_λ-orbits on $\{1, \dots, n\}$, and $f : \{1, \dots, n\} \to S_\lambda$ the map that sends an element to its N_λ-orbit. There exists a map $g : S_\lambda \to \mathbb{Z}$ such that on the diagonal matrix t with entries $t_1, \dots, t_n$, we have $\lambda(t) = \prod_i t_i^{g(f(i))}$.

An element of T_q^w is a diagonal matrix whose entries t_i satisfy $t_{w(i)} = t_i^q$ for all i. In particular, for each i, t_i is a $q^{d_i} - 1$st root of unity, where d_i is the size of the w-orbit of i. In particular, λ is trivial on T_q^w if, and only if, for all i the sum:

$$\Sigma_i = \sum_{j=0}^{d_i - 1} q^j g(f(w^j(i)))$$

is divisible by $q^{d_i} - 1$.

In $w^{\min}$ the w-orbit of i breaks up as a union of N_λ-conjugate orbits, each of size r. In particular for each j, the elements $w^j(i)$ and $w^{j+r}(i)$ lie in the same N_λ-orbit, so $g(w^j(i)) = g(w^{j+r}(i))$. This means that the sum Σ_i can be rewritten as:

$$\Sigma_i = (1 + q^r + \cdots + q^{d_i - r}) \sum_{j=0}^{r-1} q^j g(f(w^j(i))).$$

In particular Σ_i is divisible by $q^{d_i} - 1$ if, and only if, the sum:

$$\sum_{j=0}^{r-1} q^j g(f(w^j(i)))$$

is divisible by $q^r - 1$. But this is precisely the condition for λ to be trivial on $w^{\min}$. □

From this it follows that the quotient $\frac{M_\lambda}{\# N_\lambda}$ counts the number of N_λ-minimal orbits of w in S_n such that λ is trivial on T_q^w. In particular this quotient is an integer. This completes the proof of Lemma 6.5 and hence of Theorem 6.2.

7. Deformation theory

In this section we examine the local deformation theory of a representation $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$. As in previous sections, let $I_F^{(\ell)}$ denote the prime to ℓ part of the inertia group of F, and fix a topological generator $\tilde{\sigma}$ of $I_F/I_F^{(\ell)}$ and a Frobenius element $\tilde{\mathrm{Fr}}$ in $W_F/I_F^{(\ell)}$.

We first recall some results of Clozel-Harris-Taylor:

Proposition 7.1 ([CHT], Lemmas 2.4.11-2.4.13)**.** *Let $\overline{\tau}$ be an irreducible representation of $I_F^{(\ell)}$ over k, and let $G_{\overline{\tau}}$ be the subgroup of G_F that preserves $\overline{\tau}$ under conjugation. Then:*

1. *$\overline{\tau}$ lifts uniquely to a representation τ of $I_F^{(\ell)}$ over $W(k)$.*
2. *τ extends uniquely to a representation of $I_F \cap G_{\overline{\tau}}$ of determinant prime to ℓ.*
3. *τ extends (non-uniquely) to a representation of $G_{\overline{\tau}}$.*

If we fix a representation τ of $G_{\overline{\tau}}$ as in part (3), we obtain an action of $G_{\overline{\tau}}/I_F^{(\ell)}$ on $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho)$ for any G_F-module ρ. Moreover, we have a direct sum decomposition of G_F-modules:

$$\rho \cong \bigoplus_{[\overline{\tau}]} \mathrm{Ind}_{G_{\overline{\tau}}}^{G_F} [\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho) \otimes \tau],$$

where $\overline{\tau}$ runs over G_F-conjugacy classes of irreducible representations of $I_F^{(\ell)}$ over k.

Fix, for each G_F-conjugacy class of $\overline{\tau}$, a τ as in the proposition. Suppose we are given a representation $\rho_A : G_F \to \mathrm{GL}_n(A)$. We then obtain a direct sum decomposition:

$$\rho_A = \bigoplus_{[\overline{\tau}]} \mathrm{Ind}_{G_{\overline{\tau}}}^{G_F} [\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A) \otimes \tau].$$

It is clear that $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A)$ is a free A-module for all τ, and that the collection of $G_{\overline{\tau}}$-representations $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho)_A$ determines the representation ρ_A up to isomorphism.

Definition 7.2. A *pseudo-framing* of a continuous representation $\rho_A : G_F \to \mathrm{GL}_n(A)$ is a choice, for each $\overline{\tau}$, of basis for each $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_A)$. A *pseudo-framed deformation* of a continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$ (together with a chosen pseudo-framing) is a lift $\rho_A : G_F \to \mathrm{GL}_n(A)$ of $\overline{\rho}$, together with a pseudo-framing of ρ_A that lifts the chosen pseudo-framing of ρ.

Fix a $\overline{\rho}$ and a pseudo-framing of $\overline{\rho}$, and, for each $\overline{\tau}$, let $\overline{\rho}_{\overline{\tau}}$ be the $G_{\overline{\tau}}$-representation $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \overline{\rho})$. Let $R^{\diamond}_{\overline{\rho}}$ be the completed tensor product

$$\hat{\bigotimes}_{[\overline{\tau}]} R^{\square}_{\overline{\rho}_{\overline{\tau}}}$$

of the universal framed deformation rings of the $\overline{\rho}_{\overline{\tau}}$. Over each such ring we have the universal framed deformation $\rho^{\square}_{\overline{\tau}}$ of $\overline{\rho}_{\overline{\tau}}$.

Using these, we construct a representation:

$$\rho^{\diamond} := \bigoplus_{[\overline{\tau}]} \mathrm{Ind}_{G_{\overline{\tau}}}^{G_F} [\rho^{\square}_{\overline{\tau}} \otimes \tau]$$

that has a natural pseudo-framing induced by the universal framings of the representations $\rho^{\square}_{\overline{\tau}}$. One easily verifies that the pair $R^{\diamond}_{\rho}, \rho^{\diamond}$ is a universal object for pseudo-framed deformations of ρ.

For each $\overline{\tau}$, the formal group $\mathcal{G}^{\square}_{\overline{\rho}_{\overline{\tau}}}$ acts on $\mathrm{Spf}\, R^{\square}_{\rho_{\overline{\tau}}}$ by "change of frame". Let $\mathcal{G}^{\diamond}_{\overline{\rho}}$ be the product of the $\mathcal{G}^{\square}_{\overline{\rho}_{\overline{\tau}}}$. Then $\mathcal{G}^{\diamond}_{\overline{\rho}}$ acts on $\mathrm{Spf}\, R^{\diamond}_{\overline{\rho}}$ by "change of pseudo-framing".

For computational purposes it is often easier to work with $R^{\diamond}_{\overline{\rho}}$ rather than $R^{\square}_{\overline{\rho}}$, as $R^{\diamond}_{\overline{\rho}}$ can be made quite explicit. The two rings are related in a natural way: one has a ring $R^{\square,\diamond}_{\overline{\rho}}$ that is universal for triples consisting of a deformation ρ of $\overline{\rho}$, a framing of ρ lifting that of $\overline{\rho}$, and a pseudo-framing of ρ lifting that of $\overline{\rho}$. Then $\mathrm{Spf}\, R^{\square,\diamond}_{\overline{\rho}}$ is a (split) $\mathcal{G}^{\diamond}_{\overline{\rho}}$-torsor over $\mathrm{Spf}\, R^{\square}_{\overline{\rho}}$ and a (split) $\mathcal{G}^{\square}_{\overline{\rho}}$-torsor over $\mathrm{Spf}\, R^{\diamond}_{\overline{\rho}}$.

We immediately deduce:

Corollary 7.3. *The ring $R^{\square}_{\overline{\rho}}$ is a reduced, ℓ-torsion free local complete intersection.*

Proof. The construction above shows that it suffices to prove the same claim with $R^{\square}_{\overline{\rho}}$ replaced by $R^{\diamond}_{\overline{\rho}}$. But the latter is a completed tensor product of rings of the form $R^{\square}_{\overline{\rho}_{\overline{\tau}}}$, and each of these is isomorphic to the completion of a ring of the form $R_{q,n}$ (with q and n depending on $\overline{\tau}$) at a maximal ideal. The result thus follows from the results of section 4. □

Moreover, we may canonically identify both the $\mathcal{G}^{\square}_{\overline{\rho}}$-invariant elements of $R^{\square}_{\overline{\rho}}$ and the $\mathcal{G}^{\diamond}_{\overline{\rho}}$-invariant elements of $R^{\diamond}_{\overline{\rho}}$ with the $\mathcal{G}^{\square}_{\overline{\rho}} \times \mathcal{G}^{\diamond}_{\overline{\rho}}$-invariant elements of $R^{\square,\diamond}_{\overline{\rho}}$. In particular these spaces of invariants are naturally isomorphic.

Given a choice of framing of $\rho^{\diamond}$, we get a map $R^{\square}_{\overline{\rho}} \to R^{\diamond}_{\overline{\rho}}$. When restricted to $\mathcal{G}^{\square}_{\overline{\rho}}$-invariants this map is the isomorphism of $\left(R^{\square}_{\overline{\rho}}\right)^{\mathcal{G}^{\square}_{\overline{\rho}}}$ with $\left(R^{\diamond}_{\overline{\rho}}\right)^{\mathcal{G}^{\diamond}_{\overline{\rho}}}$ constructed above. Summarizing, we have:

Lemma 7.4. *For any choice of framing of $\rho^\diamond$, the induced map: $R^\square_{\overline{\rho}} \to R^\diamond_{\overline{\rho}}$ identifies the $\mathcal{G}^\square_{\overline{\rho}}$-invariant elements of $R^\square_{\overline{\rho}}$ with the $\mathcal{G}^\diamond_{\overline{\rho}}$-invariant elements of $R^\diamond_{\overline{\rho}}$. (In particular the image of this set of invariant elements is saturated in $R^\diamond_{\overline{\rho}}$.)*

8. The rings R_ν

Let $\overline{\rho} : W_F/I_F^{(\ell)} \to \mathrm{GL}_n(k)$ be a representation. Then we have a corresponding map $x : R_{q,n} \to k$, with kernel $\mathfrak{m}$. It follows easily from the universal property of the pair $(R_{q,n}, \rho_{F,n})$ that the completion $(R_{q,n})_\mathfrak{m}$ is isomorphic to $R^\diamond_{\overline{\rho}}$, and that this isomorphism is induced by the base change of $\rho_{F,n}$ to $(R_{q,n})_\mathfrak{m}$. In other words, $R_{q,n}$ is a global object that interpolates the formal deformation rings $R^\diamond_{\overline{\rho}}$ for $\overline{\rho}$ trivial on $I_F^{(\ell)}$.

We would like to construct similar objects for $\overline{\rho}$ whose restriction to $I_F^{(\ell)}$ is nontrivial. Let us define:

Definition 8.1. An ℓ-inertial type is a representation ν of $I_F^{(\ell)}$ over k that extends to a representation of W_F.

Note that (as $I_F^{(\ell)}$ is a profinite group of pro-order prime to ℓ), such a representation lifts uniquely to a representation of $I_F^{(\ell)}$ over $W(k)$, and this lift also extends to a representation of W_F. We will thus consider an ℓ-inertial type ν as a representation over $W(k)$ rather than over k whenever it is convenient to do so.

Now fix an ℓ-inertial type ν, and for each irreducible representation $\overline{\tau}$ of $I_F^{(\ell)}$ over k, let $n_{\overline{\tau}}$ be the multiplicity of $\overline{\tau}$ in ν (note that $n_{\overline{\tau}}$ depends only on the W_F-conjugacy class of $\overline{\tau}$.) Let $W_{\overline{\tau}}$ be the subgroup of W_F that fixes $\overline{\tau}$ under conjugation, let $F_{\overline{\tau}}$ be the fixed field of $W_{\overline{\tau}}$, and let $q_{\overline{\tau}}$ denote the cardinality of the residue field of $F_{\overline{\tau}}$.

We define R_ν to be the tensor product:

$$R_\nu := \bigotimes_{\overline{\tau}} R_{q_{\overline{\tau}}, n_{\overline{\tau}}}$$

where $\overline{\tau}$ runs over a set of representatives for the W_F-conjugacy classes of irreducible representations appearing in ν. For each $\overline{\tau}$ we have a representation $\rho_{F_{\overline{\tau}}, n_{\overline{\tau}}}$ over $R_{q_{\overline{\tau}}, n_{\overline{\tau}}}$, which we regard as a representation over R_ν in the obvious way.

Define the representation $\rho_\nu : W_F \to \mathrm{GL}_n(R_\nu)$ as follows:

$$\rho_\nu := \bigoplus_{\overline{\tau}} \mathrm{Ind}_{W_{\overline{\tau}}}^{W_F} \rho_{F_{\overline{\tau}}, n_{\overline{\tau}}} \otimes \tau,$$

where $\overline{\tau}$ runs over a set of representative for the W_F-conjugacy classes of irreducible representations appearing in ν, and for each such $\overline{\tau}$, we have chosen an extension τ of $\overline{\tau}$ to a representation $W_F \to \mathrm{GL}_n(W(k))$ as in Proposition 7.1. Note that ρ_ν inherits a pseudo-framing from the natural framings of the $\rho_{F_{\overline{\tau}}, n_{\overline{\tau}}}$, and that the restriction of ρ_ν to $I_F^{(\ell)}$ is given by ν.

For a map $x : R_\nu \to k$, the specialization $(\rho_\nu)_x$ is a pseudo-framed representation $W_F \to \mathrm{GL}_n(k)$, whose restriction to $I_F^{(\ell)}$ is given by ν. This defines a bijection between k-points of $\operatorname{Spec} R_\nu$ and such pseudo-framed representations. Moreover, it follows directly from the constructions of R_ν and $R^\diamond_{(\rho_\nu)_x}$ that the completion of R_ν at the maximal ideal corresponding to x is naturally isomorphic to $R^\diamond_{(\rho_\nu)_x}$, in a manner compatible with the universal family on the latter.

Moreover, the universal property for each $R_{q_{\overline{\tau}},n_{\overline{\tau}}}$ immediately yields:

Proposition 8.2. *For any finitely generated, ℓ-adically separated $W(k)$-algebra A, and any pseudo-framed, ℓ-adically continuous representation $\rho : W_F \to \mathrm{GL}_n(A)$ whose restriction to $I_F^{(\ell)}$ is isomorphic to ν, there is a unique map: $R_\nu \to A$ such that ρ is the base change of ρ_ν.*

For each $\overline{\tau}$, the group $\mathrm{GL}_{n_{\overline{\tau}}}$ acts on $R_{q_{\overline{\tau}},n_{\overline{\tau}}}$. Let $\mathcal{G}_\nu$ be the product of the $\mathrm{GL}_{n_{\overline{\tau}}}$; then $\mathcal{G}_\nu$ acts on $\operatorname{Spec} R_\nu$ by "changing the pseudo-frame".

9. Maps from $Z_{[L,\pi]}$ to R_ν

Now fix a pair (L,π), where L is a Levi subgroup of $\mathrm{GL}_n(F)$ and π is an irreducible supercuspidal k-representation of L. The mod ℓ semisimple local Langlands correspondence of Vigneras [Vi] attaches to π a semisimple k-representation ρ of W_F. Let $\overline{\nu}$ be the restriction of ρ to $I_F^{(\ell)}$. Then $\overline{\nu}$ lifts uniquely to a $W(k)$-representation ν of $I_F^{(\ell)}$, and we have:

Proposition 9.1. *The irreducible $\overline{\mathcal{K}}$-representations of $\mathrm{GL}_n(F)$ that are objects of $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))_{[L,\pi]}$ correspond, via local Langlands, to the $\overline{\mathcal{K}}$-representations of W_F whose restriction to $I_F^{(\ell)}$ is isomorphic to ν.*

Proof. This is an easy consequence of the compatibility of Vigneras' mod ℓ correspondence with reduction mod ℓ. □

This proposition shows that for any $\overline{\mathcal{K}}$-point x of $\operatorname{Spec} R_\nu$, the representation ρ_x corresponds, via local Langlands (and Frobenius semsimplification if necessary) to an irreducible $\overline{\mathcal{K}}$-representation Π_x in $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))_{[L,\pi]}$, and hence to a $\overline{\mathcal{K}}$-point of $\operatorname{Spec} Z_{[L,\pi]}$. It is a natural question to ask whether this map is induced by a map $Z_{[L,\pi]} \to R_\nu$. Indeed, we conjecture:

Conjecture 9.2 (Weak local Langlands in families)**.** *There is a map $Z_{[L,\pi]} \to R_\nu$ such that the induced map on $\overline{\mathcal{K}}$-points takes a point x of $\operatorname{Spec} R_\nu$ to the $\overline{\mathcal{K}}$-point of $Z_{[L,\pi]}$ that gives the action of $Z_{[L,\pi]}$ on the representation Π_x corresponding to ρ_x by local Langlands. (We will say such a map is* compatibile with local Langlands*.)*

Since R_ν is reduced and ℓ-torsion free, such a map is unique if it exists. Note also that the image of any element of $Z_{[L,\pi]}$ under such a map is invariant under the action of $\mathcal{G}_\nu$, and so any such map must factor through the subalgebra R_ν^{inv} of $\mathcal{G}_\nu$-invariant elements of R_ν. We further conjecture:

Conjecture 9.3 (Strong local Langlands in families)**.** *There is an isomorphism $Z_{[L,\pi]} \cong R_\nu^{\mathrm{inv}}$ such that the composition*

$$Z_{[L,\pi]} \to R_\nu^{\mathrm{inv}} \to R_\nu$$

is compatible with local Langlands.

If one completes at a maximal ideal of R_ν, corresponding to a representation $\overline{\rho}$ of W_F over k, and uses Lemma 7.4 to relate the invariant elements of $R_{\overline{\rho}}^{\square}$ and $R_{\overline{\rho}}^{\diamond}$, one recovers Conjectures 7.5 and 7.6 of [H2]. In particular (c.f. Theorem 7.9 of [H2]), Conjecture 9.2 above implies the "local Langlands in families" conjecture of Emerton-Helm (conjecture 1.1.3 of [EH]).

These conjectures should be viewed as relating "congruences" between admissible representations (which are in some sense encoded in the structure of $Z_{[L,\pi]}$) with "congruences" between representations of W_F (encoded in R_ν). Since inverting ℓ destroys information about such congruences, one expects such conjectures to be relatively straightforward with ℓ inverted. We will show that this is indeed the case.

First, note that any map:

$$Z_{[L,\pi]} \otimes \overline{\mathcal{K}} \to R_\nu \otimes \overline{\mathcal{K}}$$

that is compatible with local Langlands is Galois equivariant, and hence descends to a map

$$Z_{[L,\pi]}[\frac{1}{\ell}] \to R_\nu[\frac{1}{\ell}]$$

compatible with local Langlands. It thus suffices to show:

Theorem 9.4. *There is a map $Z_{[L,\pi]} \otimes \overline{\mathcal{K}} \to R_\nu \otimes \overline{\mathcal{K}}$ compatible with local Langlands (and therefore a corresponding map over $\mathcal{K}$.) Moreover, the image of this map is $R_\nu^{\mathrm{inv}} \otimes \overline{\mathcal{K}}$.*

To prove this, we first work on the level of connected components. We have an isomorphism:

$$Z_{[L,\pi]} \otimes \overline{\mathcal{K}} \cong \prod_{M,\tilde{\pi}} \tilde{Z}_{(M,\tilde{\pi})},$$

by Theorem 3.5, where $(M, \tilde{\pi})$ varies over the inertial equivalence classes of pairs that reduce modulo ℓ to (L, π). Thus the connected components of $\mathrm{Spec}\, Z_{[L,\pi]} \otimes \overline{\mathcal{K}}$ are in bijection with the pairs $(M, \tilde{\pi})$. Via local Langlands, these correspond to representations of I_F. More precisely, let Π be an admissible representation of G, let $\rho : W_F \to \mathrm{GL}_n(\overline{\mathcal{K}})$ correspond to Π via local Langlands, and let $\tilde{\rho} : W_F \to \mathrm{GL}_n(\overline{\mathcal{K}})$ be the representation of W_F corresponding to $\tilde{\pi}$ via local Langlands. Then Π belongs to the block corresponding to $(M, \tilde{\pi})$ if and only if the restriction of ρ^{ss} to I_F coincides with the restriction of $\tilde{\rho}$ to I_F.

On the other hand, it is an easy consequence of Proposition 4.11 that as x varies over $\overline{\mathcal{K}}$-points of $\mathrm{Spec}\, R_\nu$, the restriction of $\rho^{\mathrm{ss}}_{\nu,x}$ to I_F is constant on connected components of $\mathrm{Spec}\, R_\nu \otimes \overline{\mathcal{K}}$. We can thus let $R_\nu^{\tilde{\rho}}$ be the direct factor of $R_\nu \otimes \overline{\mathcal{K}}$ corresponding to the union of the connected components of $\mathrm{Spec}\, R_\nu \otimes \overline{\mathcal{K}}$ on which the restriction of $\rho^{\mathrm{ss}}_{\nu,x}$ to I_F is isomorphic to the restriction of $\tilde{\rho}$ to I_F. We will see later that $\mathrm{Spec}\, R_\nu^{\tilde{\rho}}$ is in fact connected.

It then suffices to construct, for each $(M, \tilde{\pi})$, an isomorphism:

$$\tilde{Z}_{(M,\tilde{\pi})} \to (R_\nu^{\tilde{\rho}})^{\mathrm{inv}}$$

compatible with local Langlands. Since $(M, \tilde{\pi})$ is only well-defined up to inertial equivalence, we may assume that $\tilde{\pi}$ has the form:

$$\tilde{\pi} \cong \bigotimes_i \tilde{\pi}_i^{\otimes r_i},$$

where the $\tilde{\pi}_i$ are pairwise inertially inequivalent representations of $\mathrm{GL}_{n_i}(F)$. Unwinding the Bernstein-Deligne description of $\tilde{Z}_{(M,\tilde{\pi})}$, we obtain an isomorphism:

$$\tilde{Z}_{(M,\tilde{\pi})} \cong \bigotimes_i \overline{\mathcal{K}}[X_{i,1}^{\pm 1}, \dots, X_{i,r_i}^{\pm 1}]^{S_{r_i}},$$

where the symmetric group S_{r_i} acts by permuting the elements $X_{i,1}, \dots, X_{i,r_i}$.

For each i, and any $\alpha \in \overline{\mathcal{K}}$, let $\chi_{i,\alpha}$ denote the unramified character of $\mathrm{GL}_{n_i}(F)$ that takes the value α on any element of $\mathrm{GL}_{n_i}(F)$ with determinant ϖ_F. An irreducible Π in $\mathrm{Rep}_{\overline{\mathcal{K}}}(M, \tilde{\pi})$ has supercuspidal support $(M, \tilde{\pi}')$ for some $\tilde{\pi}'$ of the form:

$$\tilde{\pi}' \cong \bigotimes_i \bigotimes_{j=1}^{r_i} \tilde{\pi}_i \otimes \chi_{i,\alpha_{i,j}}$$

for suitable $\alpha_{i,j}$. Then the dth elementary symmetric function in $X_{i,1}, \dots, X_{i,r_i}$, considered as an element of $\tilde{Z}_{(M,\tilde{\pi})}$, acts on Π via the dth elementary symmetric function in the $\alpha_{i,1}^{f_i'}, \dots, \alpha_{i,r_i}^{f_i'}$, where f_i' is the order of the group of unramified characters χ such that $\tilde{\pi}_i \otimes \chi$ is isomorphic to $\tilde{\pi}_i$.

For each i, the irreducible representation $\tilde{\rho}_i$ of W_F corresponding to $\tilde{\pi}_i$ via local Langlands decomposes, when restricted to I_F, as a direct sum of distinct irreducible representations of I_F, all of which are W_F-conjugate. Fix an irreducible representation $\tilde{\tau}_i$ of I_F contained in $\tilde{\rho}_i$, and let W_i be the normalizer of $\tilde{\tau}_i$ in W_F. Then there is a unique way of extending $\tilde{\tau}_i$ to a representation of W_i such that the induction of the resulting extension to W_F is isomorphic to $\tilde{\rho}_i$. (Note that this implies that W_i has index f_i' in W_F.)

This choice of extension of $\tilde{\tau}_i$ to W_i gives rise to an action of W_i on the space $\mathrm{Hom}_{I_F}(\tilde{\tau}_i, \rho_\nu)$. The quotient of this space that lives over $R_\nu^{\tilde{\rho}}$ is a free $R_\nu^{\tilde{\rho}}$-module of rank r_i, with an unramified action of W_i.

Let $\tilde{\mathrm{Fr}}_i$ be a Frobenius element of W_i, and let $P_i(x) = \sum_{j=0}^{r_i} a_{i,j} X^j$ be the characteristic polyomial of $\tilde{\mathrm{Fr}}_i$ on $\mathrm{Hom}_{I_F}(\tilde{\tau}_i, \rho_\nu)$ (over $R_\nu^{\tilde{\rho}}$). Consider the map $\tilde{Z}_{(M,\tilde{\pi})} \to R_\nu^{\tilde{\rho}}$ that sends the dth elementary symmetric function in $X_{i,1}, \dots, X_{i,r_i}$ to the element $(-1)^d a_{i,r_i-d}$ of $R_\nu^{\tilde{\rho}}$. One verifies easily that this map is compatible with local Langlands.

It remains to show that $(R_\nu^{\tilde{\rho}})^{\mathrm{inv}}$ is generated by the images of these elements. Given a polynomial P_i of degree r_i, with coefficients in a ring R, we can associate to it the unramified R-representation $M_i(P_i)$ of W_i on which $\tilde{\mathrm{Fr}}_i$ acts via the companion matrix of P_i. The representation $\rho(\{P_i\})$ given by:

$$\rho(\{P_i\}) = \bigoplus_i \mathrm{Ind}_{W_i}^{W_F} M_i(P_i) \otimes \tilde{\tau}_i$$

is then an R-point of $\mathrm{Spec}\, R_\nu^{\tilde{\rho}}$. In this way we obtain a natural map:

$$R_\nu^{\tilde{\rho}} \to \bigotimes_i \overline{\mathcal{K}}[Y_{i,1}, \dots, Y_{i,r_i}]$$

that in particular takes the element $(-1)^d a_{i,r_i-d}$ of $R_\nu^{\tilde{\rho}}$ to $Y_{i,d}$. On the other hand, it is easy to see that for every y in $(\mathrm{Spec}\, R_\nu^{\tilde{\rho}})(\overline{\mathcal{K}})$, there is a point x in $(\mathrm{Spec}\, R_\nu^{\tilde{\rho}})(\overline{\mathcal{K}})$ arising from a collection of polynomials $\{P_i(x)\}$ such that y is in the closure of the G_ν-orbit of x. It follows that the map:

$$R_\nu^{\tilde{\rho}} \to \bigotimes_i \overline{\mathcal{K}}[Y_{i,1}, \dots, Y_{i,r_i}]$$

is injective on $(R_\nu^{\tilde{\rho}})^{\mathrm{inv}}$. Therefore $(R_\nu)^{\tilde{\rho}})^{\mathrm{inv}}$ is generated by the elements $a_{i,r_{i-d}}$, completing the proof.

It is not hard to go slightly further, and show:

Theorem 9.5. *The image of $Z_{[L,\pi]}$ in $R_\nu[\frac{1}{\ell}]$ under the map of Theorem 9.4 lies in the normalization of R_ν.*

Proof. Fix an element x of $Z_{[L,\pi]}$, and let y be its image in $R_\nu[\frac{1}{\ell}]$. Let A be a discrete valuation ring that is a $W(k)$-algebra, with field of fractions K of characteristic zero, and fix a map $R_\nu \to A$. This corresponds to a pseudo-framed representation ρ_A of W_F. Let Π_K denote the admissible K-representation corresponding to $\rho_A \otimes_A K$ via local Langlands. Since $\rho_A \otimes_A K$ admits an A-lattice, so does Π_K. In particular the action of x on Π_K is via an element of A, so y maps to an element of A under the map $R_\nu[\frac{1}{\ell}] \to K$. Since this is true for every A and every map $R_\nu \to A$, y lives in the normalization of R_ν as claimed. □

10. Main results

The main objective of this section (and, indeed, the paper) is to show the following:

Theorem 10.1. *Suppose that Conjecture 9.2 holds for all $\mathrm{GL}_m(F)$, $m \leq n$, and Conjecture 9.3 holds for $m < n$. Then:*

1. *The map $\overline{E}_{q,n}[\frac{1}{\ell}] \to B_{q,n}[\frac{1}{\ell}]$ of section 6 induces an isomorphism of $\overline{E}_{q,n}$ with $B_{q,n}$, and*
2. *Conjecture 9.3 holds for $\mathrm{GL}_n(F)$.*

We begin by proving the first claim, using the weak conjecture for GL_n in depth zero. Let Z_n^0 be the product of the depth zero blocks of $\mathrm{Rep}_{W(K)}(G)$. The weak conjecture then gives rise to a map $Z_n^0 \to S_{q,n}$ compatible with the local Langlands correspondence. The subalgebra of Z_n^0 consisting of elements that are constant on inertial equivalence classes is isomorphic to $\overline{E}_{q,n}$, by Proposition 3.10. By compatibility with local Langlands together with Proposition 4.11 and Proposition 5.3 the image of $\overline{E}_{q,n}$ in $S_{q,n}$ is contained in $B_{q,n}$, and the induced map: $\overline{E}_{q,n}[\frac{1}{\ell}] \to B_{q,n}[\frac{1}{\ell}]$ is the map considered in section 6. It thus follows from Corollary 6.3 that the map $\overline{E}_{q,n} \to B_{q,n}$ is an isomorphism.

We now turn to the second claim. Fix a mod ℓ supercuspidal inertial equivalence class $[L,\pi]$, corresponding to an ℓ-inertial type ν, and note that we have tensor factorizations:

$$Z_{[L,\pi]} \cong \bigotimes_i Z_{[L_i,\pi_i]}$$

$$R_\nu \cong \bigotimes_{\overline{\tau}} R_{q_{\overline{\tau}},n_{\overline{\tau}}}$$

where the $[L_i,\pi_i]$ are simple blocks. The former factorization is compatible with parabolic induction and the latter arises from the direct sum decomposition:

$$\rho_\nu = \bigoplus_{\overline{\tau}} \mathrm{Ind}_{W_{\overline{\tau}}}^{W_F} \rho_{F_{\overline{\tau}},n_{\overline{\tau}}} \otimes \tau.$$

Since simple blocks correspond to types ν with only one $n_{\overline{\tau}}$ nonzero, these factorizations are compatible, in the sense that if we have maps $Z_{[L_i,\pi_i]} \to R_{\nu_i}$ for each i that are compatible with local Langlands, then their tensor product gives a map $Z_{[L,\pi]} \to R_\nu$ compatible with local Langlands. Thus both Conjecture 9.2 and Conjecture 9.3 reduce to the corresponding conjectures on simple blocks. We thus henceforth assume that $[L,\pi]$ is of the form $[L_n,\pi_n]$ with $\pi_n \cong \pi_1^{\otimes n}$ for a supercuspidal representation π_1. Following section 3 we set $Z_n = Z_{[L_n,\pi_n]}$. The corresponding R_{ν_n} is then isomorphic to $R_{q_{\overline{\tau}},n}$ for some fixed $\overline{\tau}$.

We first consider the case in which n is not $q_{\overline{\tau}}$-relevant. Let ν be the maximal $q_{\overline{\tau}}$-relevant partition of n. We have a commutative diagram:

$$\begin{array}{ccc} Z_n & \rightarrow & R^{\mathrm{inv}}_{q_{\overline{\tau}},n} \\ \downarrow & & \downarrow \\ \otimes_i Z_{\nu_i} & \rightarrow & \otimes_i R^{\mathrm{inv}}_{q_{\overline{\tau}},\nu_i} \end{array}$$

in which the horizontal maps are those arising from the weak conjecture, the left-hand vertical map is Ind_ν, and the right-hand vertical map is induced by the map $\mathrm{Spec} \otimes_i R_{q_{\overline{\tau}},\nu_i} \to R_{q_{\overline{\tau}},n}$ that takes a collection $(\mathrm{Fr}_i, \sigma_i)$ of matrices with $\mathrm{Fr}_i\, \sigma_i\, \mathrm{Fr}_i^{-1} = \sigma_i^{q_{\overline{\tau}}}$ to the pair $(\oplus_i \mathrm{Fr}_i, \oplus_i \sigma_i)$.

The horizontal maps are isomorphisms after inverting ℓ, and our hypotheses imply that the lower horizontal map is an isomorphism integrally. Moreover the left-hand vertical map is injective with saturated image by Theorem 3.9 and the discussion in the paragraph following it. It follows immediately that the top horizontal map must also be an isomorphism.

We now assume that n is $q_{\overline{\tau}}$-relevant (that is, it lies in $\{1, e_{q_{\overline{\tau}}}, \ell e_{q_{\overline{\tau}}}, \dots\}$) let m be the largest element of this set that is strictly less than n. Set $j = \frac{n}{m}$.

We have a subalgebra $\overline{E}_{q^{f'},n,1}$ of Z_m and compatibility with local Langlands shows that $q^{f'} = q_{\overline{\tau}}$. Thus the map $Z_n \to R_{q_{\overline{\tau}},n}$ induces a map $\overline{E}_{q_{\overline{\tau}},n,1} \to R_{q_{\overline{\tau}},n}$. Reasoning as in the depth zero setting we see that the image of this map is contained in $B_{q_{\overline{\tau}},n,1}$. It seems likely that the resulting map $\overline{E}_{q_{\overline{\tau}},n,1} \to B_{q_{\overline{\tau}},n,1}$ is the one considered in section 6, but we do not prove this here. Instead we use the fact that we have shown these two rings to be abstractly isomorphic, together with the following lemma:

Lemma 10.2. *Let $\overline{E}$ be a finite rank, reduced, ℓ-torsion free $W(k)$-algebra, and let $f : \overline{E} \to \overline{E}$ be an injection. Then f is an isomorphism.*

Proof. Clearly f is an isomorphism after inverting ℓ. On the other hand, the hypotheses guarantee that $\overline{E}[\frac{1}{\ell}]$ is a product of finite extensions of $\mathcal{K}$, and f is a $\mathcal{K}$-linear automorphism of this product. In particular there is some power of f that is the identity. □

We thus conclude that the map $Z_n \to R_{q_{\overline{\tau}},n}$ coming from the weak conjecture induces an isomorphism of $\overline{E}_{q_{\overline{\tau}},n,1}$ with $B_{q_{\overline{\tau}},n,1}$.

Now consider the commutative diagram:

$$\begin{array}{ccc} K & \rightarrow & K' \\ \downarrow & & \downarrow \\ Z_n & \rightarrow & R^{\mathrm{inv}}_{q_{\overline{\tau}},n} \\ \downarrow & & \downarrow \\ Z_m^{\otimes j} & \rightarrow & (R^{\mathrm{inv}}_{q_{\overline{\tau}},m})^{\otimes j} \end{array}$$

in which the horizontal maps are induced by the weak conjecture, the lower left vertical map is $\mathrm{Ind}_{m,n}$, the lower right vertical map is the one taking a collection of pairs $(\mathrm{Fr}_i, \sigma_i)$ to their direct sum, and K and K' are the kernels of the lower left and lower right vertical maps, respectively. As in the previous case, all horizontal maps become isomorphisms after inverting ℓ and the bottom horizontal map is an isomorphism integrally.

By Proposition 3.13 K is contained in the subalgebra $\overline{E}_{q_{\mp},n,1}[\Theta_{n,n}^{\pm 1}]$ of Z_n, and the image of this subalgebra in $R_{q_{\mp},n}$ is saturated. It follows that the map from K to K' is an isomorphism: if x is an element of K', then for some a, the product $\ell^a x$ is in the image of K. But then $\ell^a x$ is in the image of $\overline{E}_{q_{\mp},n,1}[\Theta_{n,n}^{\pm 1}]$, so x is as well. On the other hand, the image of K in $\overline{E}_{q_{\mp},n,1}$ is saturated (as K is the kernel of a map of rings that have no ℓ-torsion), so x must lie in the image of K.

Let r be an element of $R^{\mathrm{inv}}_{q_{\mp},n}$, and let r' be its image in $(R^{\mathrm{inv}}_{q_{\mp},m})^{\otimes j}$. There is then an element y of $Z_m^{\otimes j}$ whose image under the bottom horizontal map is r'. Since the map $Z_n \to R^{\mathrm{inv}}_{q_{\mp},n}$ is an isomorphism after inverting ℓ, there exists a such that $\ell^a y$ is in the image of $\mathrm{Ind}_{m,n}$.

By Theorem 3.12, there exist $\tilde{y}$ in Z_n and x in $\overline{E}_{q_{\mp},n,1}[\Theta_{n,n}^{\pm 1}]$ such that $\mathrm{Ind}_{m,n}(x) = \ell^b(\mathrm{Ind}_{m,n}(\tilde{y}) - y)$. Let s be the image of $\tilde{y}$ in $R^{\mathrm{inv}}_{q_{\mp},n}$. The image of $\ell^b(s-r)$ in $(R^{\mathrm{inv}}_{q_{\mp},m})^{\otimes j}$ coincides with the image of $\mathrm{Ind}_{m,n}(x)$. Thus $\ell^b(s-r)$ lies in the image of $\overline{E}_{q_{\mp},n,1}[\Theta_{n,n}^{\pm 1}]$. Since this image is saturated, the element $s-r$ also lives in this image. Thus the map $Z_n \to R^{\mathrm{inv}}_{q_{\mp},n}$ is surjective, so it is an isomorphism.

We have thus completed the proof of Theorem 10.1. In [HM] we show that the strong conjecture for GL_{n-1} implies the weak conjecture for GL_n. Together with Theorem 10.1 and the fact that the strong conjecutre for GL_1 is an easy consequence of local class field theory, we obtain an unconditional proof both of the strong conjecture, and of the existence of an isomorphism $\overline{E}_{q,n} \cong B_{q,n}$. We refer the reader to the final section of [HM] for the details.

Remark 10.3. The isomorphism of $\overline{E}_{q,n}$ with $B_{q,n}$ is an interesting result in finite group theory in its own right. We are aware of no proof other than the one presented here; it is an interesting question to find a purely group-theoretic proof of this result.

11. Affine Curtis Homomorphisms

Having established both Conjectures 9.2 and 9.3 we now turn to an interesting consequence of Conjecture 9.2. Fix a w in S_n (which we identify with the Weyl group of $\mathcal{G}$). The conjugacy class of w gives rise to a conjugacy class of nonsplit, unramified tori in $\mathcal{G}$; we let $\mathcal{T}_w$ denote a representative of this conjugacy class. In particular we have $\mathcal{T}_w \cong \prod_{w_i} \mathrm{Res}_{F_i/F}\, \mathbb{G}_m$, where the product is over the cycles w_i of w and F_i/F is unramified of degree equal to the length of w_i. Let d be the order of w in S_n.

Let X be the character group of $\mathcal{T}_w$, and let $\mathcal{T}_w^L$ denote the algebraic group $\mathrm{Hom}(X', \mathbb{G}_m) \rtimes \mathbb{Z}/d\mathbb{Z}$ (regarded as an algebraic group over $W(k)$), where the action of $1 \in \mathbb{Z}/d\mathbb{Z}$ on X' is via w^{-1}. Then $\mathcal{T}_w^L$ is the L-group of $\mathcal{T}_w$. Moreover, if we identify GL_n (over $W(k)$) with the L-group of $\mathcal{G}$ in such a way that X' becomes identified with the character group of the diagonal torus in GL_n, then we have a natural L-homomorphism from $\mathcal{T}_w^L$ to GL_n that takes $\mathrm{Hom}(X', \mathbb{G}_m)$ to the diagonal torus and takes $1 \in \mathbb{Z}/n\mathbb{Z}$ to w^{-1}. This allows us to transfer a Langlands parameter $\rho_w : W_F \to \mathcal{T}_w^L(\overline{\mathcal{K}})$ for $\mathcal{T}_w$ to a Langlands parameter $\rho : W_F \to \mathrm{GL}_n(\overline{\mathcal{K}})$ for $\mathcal{G}$.

It will be useful to understand the interaction between this transfer and the block decompositions for $\mathrm{Rep}_{W(k)}(T_w)$ and $\mathrm{Rep}_{W(k)}(G)$. Note that

$$T_w = \mathcal{T}_w(F) = \mathrm{Hom}(X, (F^{\mathrm{ur}})^{\times})^{\tilde{\mathrm{Fr}}},$$

where $\tilde{\mathrm{Fr}}$ is a fixed Frobenius element of W_F, and its action on X is via w. Let $T_w^{(\ell)}$ denote the subgroup $\mathrm{Hom}(X, (\mathcal{O}_{F^{\mathrm{ur}}}^{\times})^{(\ell)})^{\tilde{\mathrm{Fr}}}$ of T_w, where $(\mathcal{O}_{F^{\mathrm{ur}}}^{\times})^{(\ell)}$ denotes the elements of pro-order prime to ℓ in $\mathcal{O}_{F^{\mathrm{ur}}}^\times$. Then $T_w^{(\ell)}$ is profinite, of pro-order prime to ℓ, and the quotient $T_w/T_w^{(\ell)}$ is a discrete group. Indeed, explicitly, one has:

$$T_w/T_w^{(\ell)} \cong \prod_{w_i} F_i^\times/(\mathcal{O}_{F_i}^\times)^{(\ell)} \cong \prod_{w_i} (\mathbb{Z} \cdot \varpi \times \mathbb{F}_{q^i}^\times),$$

where ϖ is a uniformizer of F (hence also of F_i.)

The blocks of $\mathrm{Rep}_{W(k)}(T_w)$ are thus given by characters $\chi^{(\ell)} : T_w^{(\ell)} \to W(k)^\times$. Choose an extension χ of $\chi^{(\ell)}$ to a character $T_w \to W(k)^\times$. Then "twisting by χ" induces an equivalence of categories between the block of $\mathrm{Rep}_{W(k)}(T_w)$ corresponding to the trivial character of $T_w^{(\ell)}$ and the block corresponding to $\chi^{(\ell)}$. Denote the centers of these blocks by $Z_{w,1}$ and $Z_{w,\chi^{(\ell)}}$, respectively; our choice of χ then gives an isomorphism of $Z_{w,1}$ with $Z_{w,\chi^{(\ell)}}$.

On the other side of the Langlands correspondence, the local Langlands correspondence for tori associates to χ a Langlands parameter $\tilde{\nu}_w : W_F \to \mathcal{T}_w^L(\overline{\mathcal{K}})$; the restriction ν_w of $\tilde{\nu}_w$ to $I_F^{(\ell)}$ depends only on $\chi^{(\ell)}$. Consider the functor that associates to a $W(k)$-algebra R the set of parameters $W_F \to \mathcal{T}_w^L(R)$ whose restriction to $I_F^{(\ell)}$ is equal to ν_w. This functor is easily seen to be representable by a finite type affine scheme $\mathrm{Spec}\, R_\nu^w$, and there is a universal Langlands parameter $\rho_{w,\nu} : W_F \to \mathcal{T}_w^L(R_\nu^w)$. Note that the torus $\mathrm{Hom}(X', \mathbb{G}_m) \subseteq \mathcal{T}_w^L$ acts on $\mathrm{Spec}\, R_\nu^w$ by conjugation; let $(R_\nu^w)^{\mathrm{inv}}$ be the subring of R_ν^w invariant under this action.

We then have the following Proposition, which can be seen as an analogue of Conjecture 9.3 for the nonsplit torus T_w:

Proposition 11.1. *There is a unique isomorphism*

$$\mathbb{L}_w : Z_{w,\chi^{(\ell)}} \to (R_\nu^w)^{\mathrm{inv}}$$

which is compatible with the local Langlands correspondence for tori, in the sense that for any Langlands parameter $\rho : W_F \to \mathcal{T}_w^L(\overline{\mathcal{K}})$, corresponding to a character χ_ρ of T_w, and any $z \in Z_{w,\chi^{(\ell)}}$, the value of χ_ρ at z is equal to the value of $\mathbb{L}_w$ at the point of $\mathrm{Spec}(R_\nu^w)$ corresponding to ρ.

Proof. Any parameter $W_F \to \mathcal{T}_w^L(R)$ of type ν_w differs from $\tilde{\nu}_w$ by a parameter $W_F \to \mathcal{T}_w^L(R)$ that is trivial on $I_F^{(\ell)}$. Thus "twisting by ν_w" induces an isomorphism of $\mathrm{Spec}\, R_\nu^w$ with $\mathrm{Spec}\, R_1^w$, where 1 is the trivial character of $I_F^{(\ell)}$. On $\overline{\mathcal{K}}$-points, this isomorphism is compatible with the local Langlands correspondence for tori and the "twisting by χ" isomorphism of $Z_{w,1}$ with $Z_{w,\chi^{(\ell)}}$. We can thus reduce to the case where $\chi^{(\ell)}$ and ν_w are the trivial character.

In this case we can be very explicit: on the one hand, we have isomorphisms:

$$Z_{w,1} = W(k)[T_w/T_w^{(\ell)}] \cong W(k)[\mathrm{Hom}(X, (F^{\mathrm{ur}})^\times/(\mathcal{O}_{F^{\mathrm{ur}}}^\times)^{(\ell)})^{\tilde{\mathrm{Fr}}}],$$

where $\tilde{\mathrm{Fr}}$ acts on X via w and on $\mathcal{O}_{F^{\mathrm{ur}}}^\times$ in the usual way. Let ϖ be a uniformizer of F corresponding to our Frobenius element $\tilde{\mathrm{Fr}}$. We then have a canonical isomorphism:

$$(F^{\mathrm{ur}})^\times/(\mathcal{O}_{F^{\mathrm{ur}}}^\times)^{(\ell)} \cong \mathbb{Z} \cdot \varpi \times \overline{\mathbb{F}}_q^\times,$$

where $\tilde{\mathrm{Fr}}$ acts trivially on the first factor and by qth powers on the second. We thus obtain an isomorphism:

$$Z_{w,1} \cong W(k)[\mathrm{Hom}(X, \mathbb{Z})^w] \otimes W(k)[\mathrm{Hom}(X/(qw-1)X, \overline{\mathbb{F}}_q^\times)].$$

On the other side of the Langlands correspondence, fix a generator $\tilde{\sigma}$ of I_F/P_F. Then a Langlands parameter $W_F \rightarrow {}^L\mathcal{T}_w$ trivial on $I_F^{(\ell)}$ is determined by the images of $\tilde{\mathrm{Fr}}$ and $\tilde{\sigma}$; these form a pair of diagonal matrices $\mathcal{F}$ and σ such that $\mathcal{F}w^{-1}\sigma(\mathcal{F}w^{-1})^{-1} = \sigma^q$. Since $\mathcal{F}$ and σ commute, this condition is equivalent to the condition $\sigma^{w^{-1}} = \sigma^q$. Thus $\operatorname{Spec} R_1^w$ decomposes as a product:

$$\operatorname{Spec} R_1^w \cong \operatorname{Spec} W(k)[X'] \times \operatorname{Spec} W(k)[X'/(q-w)X'],$$

where the first factor parameterizes $\mathcal{F}$ and the second parameterizes σ. The conjugation action of $t \in \operatorname{Hom}(X', \mathbb{G}_m)$ on this product fixes the second factor and acts by multiplication by $t^{w^{-1}-1}$ on the first. We thus obtain a product decomposition:

$$\operatorname{Spec}(R_1^w)^{\mathrm{inv}} \cong \operatorname{Spec} W(k)[X'/(1-w)X'] \times \operatorname{Spec} W(k)[X'/(q-w)X'].$$

On the first factor, the isomorphism of $Z_{w,1}$ with $(R_1^w)^{\mathrm{inv}}$ is induced by the isomorphism $\operatorname{Hom}(X,\mathbb{Z})^w \cong X'/(w-1)X'$. On the second factor we have to work a bit harder. Note that $qw - 1$ divides $q^r - 1$, where r is a multiple of the order of w. Thus $\operatorname{Hom}(X/(qw-1)X, \overline{\mathbb{F}}_q^{\times})$ is isomorphic to $\operatorname{Hom}(X/(qw-1)X, \mathbb{F}_{q^r}^{\times})$. Our choice of s gives rise to a system of generators for $\mathbb{F}_{q^r}^{\times}$ for all r, compatible with respect to norm maps; we can thus identify $\operatorname{Hom}(X/(qw-1)X, \mathbb{F}_{q^r}^{\times})$ with the kernel of $qw^{-1} - 1$ on $X'/(q^r-1)X'$, via the isomorphism

$$X'/(q^r-1)X' \cong \operatorname{Hom}(X/(q^r-1)X, \mathbb{Z}/(q^r-1)\mathbb{Z}).$$

Finally, multiplication by $1 + qw^{-1} + \cdots + q^{r-1}w^{1-r}$ identifies this kernel with $X'/(qw^{-1}-1)X'$. The resulting isomorphism of $\operatorname{Hom}(X/(qw-1)X, \overline{\mathbb{F}}_q^{\times})$ with $X'/(qw^{-1}-1)X'$ is independent of r, and gives the desired map from the second factor of $Z_{w,1}$ to the second factor of $(R_1^w)^{\mathrm{inv}}$. One checks easily that the resulting isomorphism is compatible with local Langlands. □

The L-homomorphism of ${}^L\mathcal{T}_w$ into GL_n takes Langlands parameters for T_w to Langlands parameters for G. If the former has type ν_w, then so does the latter (where we regard ν_w as an ℓ-inertial type by embedding it in $\mathrm{GL}_n(W(k))$ by identifying $\operatorname{Hom}(X', \mathbb{G}_m)$ with the diagonal matrices.) Thus this L-homomorphism induces a map $R^\nu \rightarrow R_\nu^w$ that takes R_ν^{inv} to $(R_\nu^w)^{\mathrm{inv}}$. Combining this with Proposition 11.1 and Conjecture 9.2, we obtain a map:

$$eZ_n \rightarrow Z_{w,\chi^{(\ell)}},$$

where e is the idempotent of Z_n corresponding to the ℓ-inertial type ν. On $\overline{\mathcal{K}}$-points this map takes a point of $\operatorname{Spec} Z_{w,\chi^{(\ell)}}$ corresponding to a character with Langlands parameter ρ to the point of $\operatorname{Spec} eZ_n$ corresponding to the Langlands parameter obtained by composing ρ with the L-homomorphism of ${}^L\mathcal{T}_w$ into GL_n.

On the other hand, if we fix a generic character Ψ of the unipotent radical U of G, and let Γ be the module $\text{c-Ind}_U^G \Psi$, then it follows from results in [H2] that the natural map $eZ_n \rightarrow \operatorname{End}_{W(k)[G]}(\Gamma)$ is an isomorphism. We can thus view the map $eZ_n \rightarrow Z_{w,\chi^{(\ell)}}$ as the affine group analogue of a Curtis homomorphism. Since the Curtis homomorphisms have such a nice interpretation via Deligne-Lusztig theory, it is natural to ask if a similar phenomenon is at play here:

Question 11.2. Does there exist an adjoint pair of functors:

$$i_w : \mathcal{D}^b(\operatorname{Rep}_{W(k)}(T_w)) \rightarrow \mathcal{D}^b(\operatorname{Rep}_{W(k)}(G))$$

$$r_w : \mathcal{D}^b(\mathrm{Rep}_{W(k)}(G)) \to \mathcal{D}^b(\mathrm{Rep}_{W(k)}(F))$$

such that $r_w(\Gamma)$ is a shift of the induction $\text{c-Ind}_e^{T_w} 1$, and the induced homomorphism:

$$Z_n \to Z_w$$

is the product over suitable idempotents of the "affine Curtis homomorphisms" constructed above? Moreover, is there a natural geometric construction of such an adjoint pair?

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