COHERENT SPRINGER THEORY AND THE CATEGORICAL DELIGNE-LANGLANDS CORRESPONDENCE

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Abstract. Kazhdan and Lusztig identified the affine Hecke algebra $\mathcal{H}$ with an equivariant K-group of the Steinberg variety, and applied this to prove the Deligne-Langlands conjecture, i.e., the local Langlands parametrization of irreducible representations of reductive groups over nonarchimedean local fields F with an Iwahori-fixed vector. We apply techniques from derived algebraic geometry to pass from K-theory to Hochschild homology and thereby identify $\mathcal{H}$ with the endomorphisms of a coherent sheaf on the stack of unipotent Langlands parameters, the *coherent Springer sheaf*. As a result the derived category of $\mathcal{H}$-modules is realized as a full subcategory of coherent sheaves on this stack, confirming expectations from strong forms of the local Langlands correspondence (including recent conjectures of Fargues-Scholze, Hellmann and Zhu). We explain how this refines the more familiar description of representations, one central character at a time, in terms of categories of perverse sheaves (as previously observed in local Langlands over $\mathbb{R}$).

In the case of the general linear group our result allows us to lift the local Langlands classification of irreducible representations to a categorical statement: we construct a full embedding of the derived category of smooth representations of $\mathrm{GL}_n(F)$ into coherent sheaves on the stack of Langlands parameters.

Contents

1. Introduction

Our goals in this paper are to provide a spectral description of the category of representations of the affine Hecke algebra and deduce applications to the local Langlands correspondence. We begin with a quick review of Springer theory and then discuss our main results starting in Section 1.3.

We will work in the setting of derived algebraic geometry over a field k of characteristic zero, as presented in [GR17]. In particular all operations, sheaves, categories etc will be derived unless otherwise noted.

1.1. **Springer theory and Hecke algebras.** We first review some key points of Springer theory, largely following the perspective of [CG97, GB98]. Let G denote a complex reductive group with Lie algebra $\mathfrak{g}$ and Borel $B \subset G$. We denote by $\mathcal{B} \simeq G/B$ the flag variety, $\mathcal{N}$ the nilpotent cone, $\mu : \widetilde{\mathcal{N}} = T^*\mathcal{B} \to \mathcal{N}$ the Springer resolution, and $\mathcal{Z} = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ the Steinberg variety.

The Springer correspondence provides a geometric realization of representations of the Weyl group W of G. The Weyl group is in bijection with the Bruhat double cosets $B\backslash G/B = G\backslash(\mathcal{B} \times \mathcal{B})$, and hence with the conormals to the Schubert varieties, which form the irreducible components of the Steinberg variety $\mathcal{Z}$. In fact the group algebra of the Weyl group can be identified with the top Borel-Moore homology of $\mathcal{Z}$ under the convolution product

$$\mathbb{C}W \simeq H_d^{BM}(\mathcal{Z}; \mathbb{C}),$$

where $d = \dim(\mathcal{N}) = \dim(\widetilde{\mathcal{N}}) = \dim(\mathcal{Z})$. This realization of W can be converted into a sheaf-theoretic statement. The Springer sheaf

$$\mathbf{S} = \mu_* \mathbb{C}_{\widetilde{\mathcal{N}}}[d] \in \mathrm{Perv}(\mathcal{N}/G)$$

is the equivariant perverse sheaf on the nilpotent cone given by the pushforward of the (shifted) constant sheaf on the Springer resolution. Thanks to the definition of $\mathcal{Z}$ as the self-fiber-product $\mathcal{Z} = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$, a simple base-change calculation provides an isomorphism

$$H_d^{BM}(\mathcal{Z}; \mathbb{C}) \simeq \mathrm{End}_{\mathcal{N}/G}(\mathbf{S})$$

between the endomorphisms of $\mathbf{S}$ and the top homology of $\mathcal{Z}$, i.e., the group algebra $\mathbb{C}W$. Since the abelian category $\mathrm{Perv}(\mathcal{N}/G)$ is semisimple, all objects are projective and we may interpret this isomorphism as a full embedding of the abelian category of representations of W into equivariant perverse sheaves on the nilpotent cone,

$$\mathrm{Rep}(W) = \mathbb{C}W\text{-mod} \simeq \langle \mathbf{S} \rangle \subset \mathrm{Perv}(\mathcal{N}/G).$$

One important role for this embedding is provided by the representation theory of Chevalley groups. The universal unipotent principal series representation[1]

$$\mathbb{C}G(\mathbb{F}_q) \circlearrowright \mathbb{C}[\mathcal{B}(\mathbb{F}_q)]$$

has as endomorphism algebra the finite Hecke algebra

$$\mathcal{H}^f = \mathbb{C}[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)] = \mathrm{End}_{G(\mathbb{F}_q)}(\mathbb{C}[G(\mathbb{F}_q)/B(\mathbb{F}_q)]),$$

[1]Note that the finite Hecke algebra and hence the unipotent principal series is insensitive to Langlands duality. From our perspective it is in fact more natural to consider here representations of the Langlands dual Chevalley group $G^\vee(\mathbb{F}_q)$.

which (after choosing a square root of q) may be identified with $\mathbb{C}W$. Thus Springer theory provides a full embedding

$$\{\text{unipotent principal series of } G(\mathbb{F}_q)\} \simeq \mathcal{H}^f\text{-mod} \xrightarrow{\sim} \langle \mathbf{S} \rangle \subset \mathrm{Perv}(\mathcal{N}/G)$$

where we say a representation of $G(\mathbb{F}_q)$ is in the unipotent principal series if it is generated by its $B(\mathbb{F}_q)$-invariants.

1.2. **Affine Hecke algebras.** We now let G be a reductive group, Langlands dual to a split group $G^\vee(F)$ over a nonarchimedean local field F with ring of integers O and residue field $\mathbb{F}_q$. We write $\widetilde{G} = G \times \mathbb{G}_m$ as shorthand.

Definition 1.1. Let G be a reductive group with maximal torus T. The (extended) *affine Weyl group* of the dual group $G^\vee$ is the semidirect product $W_a = W \ltimes X_\bullet(T^\vee) = W \ltimes X^\bullet(T)$ of the finite Weyl group with the cocharacter lattice of $T^\vee$. The *affine Hecke algebra* $\mathcal{H}$ is a certain q-deformation of the group ring $\mathbb{C}W_a$ such that specializing q at a prime power gives the Iwahori-Hecke algebra:

$$\mathcal{H}_q = \mathbb{C}_c[I\backslash G^\vee(F)/I] = \mathrm{End}_{\mathrm{Rep}(G^\vee(F))}(\mathbb{C}_c[G^\vee(F)/I])$$

where $I \subset G^\vee(F)$ is an Iwahori subgroup. Explicit presentations of the affine Hecke algebra can be found, for example, in Section 7.1 of [CG97]. Unlike the finite Hecke algebra, $\mathcal{H}_q \not\simeq \mathbb{C}W_a$.

Our starting point is the celebrated theorem of Kazhdan-Lusztig [KL87] (as later extended and modified by Ginzburg (see [CG97] and Lusztig [Lu98]), providing a geometric realization of the affine Hecke algebra in terms of the Steinberg variety.

Theorem 1.2. [KL87, CG97, Lu98] *Suppose that G has simply connected derived subgroup. There is an isomorphism of algebras $\mathcal{H} \simeq K_0(\mathcal{Z}/\widetilde{G})$, compatible with the Bernstein isomorphism $Z(\mathcal{H}) \simeq \mathbb{C}[\widetilde{G}]^{\widetilde{G}} \simeq K_0^{\widetilde{G}}(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{C}$ between the center of $\mathcal{H}$ and the ring of equivariant parameters.*

Kazhdan and Lusztig famously applied Theorem 1.2 to prove the Deligne-Langlands conjecture, as refined by Lusztig. The category of representations of $\mathcal{H}_q$ is identified with the "Iwahori block", the (smooth) representations of $G^\vee(F)$ that are generated by their I-invariants (i.e., "appear in the decomposition of $\mathbb{C}_c[G^\vee(F)/I]$"). Equivalently this is the unramified principal series, the representations of $G^\vee(F)$ appearing in the parabolic induction of unramified characters of a split torus (i.e., "appear in the decomposition of $\mathbb{C}[G^\vee(F)/N^\vee(F)T^\vee(O)]$"). The Deligne-Langlands conjecture provides a classification of irreducible representations in the Iwahori block (i.e. with an Iwahori fixed vector), or equivalently irreducible $\mathcal{H}_q$ modules, in terms of Langlands parameters:

Theorem 1.3. [KL87, Re02] *The irreducible representations of $\mathcal{H}_q$ are in bijection with G-conjugacy classes of q-commuting pairs of semisimple and nilpotent elements in G*

$$\{s \in G^{ss}, n \in \mathcal{N} : gng^{-1} = qn\}/G,$$

together with a G-equivariant local system on the orbit of (s, n) which appears in the decomposition of a corresponding Springer sheaf.

For fixed (s, q) the variety $\mathcal{N}^{(s,q)}$ of (s, q)-fixed points on the nilpotent cone can be interpreted as a variety of *Langlands parameters*, representations of the Weil-Deligne group of F into G with fixed image of Frobenius. Representations with a fixed Langlands parameter (s, n) form an *L-packet*, and are described in terms of irreducible representations of the component group of the stabilizer. These representations can then be interpreted as equivariant local systems on the orbit of the Langlands parameter. Indeed general conjectures going back to work of Lusztig [Lu83], Zelevinsky [Ze81] and Vogan [Vo93] describe the representation theory of $G^\vee(F)$ at a fixed central character with the geometry of equivariant perverse sheaves on suitable spaces of Langlands parameters, generalizing the appearance of $\mathcal{N}^{(s,q)}$ above.

However, unlike the classical Springer theory story for $\mathcal{H}^f_q \simeq \mathbb{C}W$, the realization of $\mathcal{H}$ by equivariant K-theory in Theorem 1.2 does not immediately lead to a realization of $\mathcal{H}$ as endomorphisms of a sheaf, and therefore to a sheaf-theoretic description of the entire category of $\mathcal{H}$-modules. Rather, in applications equivariant K-theory is used as an intermediate step on the way to equivariant Borel-Moore homology, which leads back to variants of the Springer correspondence. Namely, by fixing a central character for $\mathcal{H}$, i.e. a Weyl group orbit of $(s, q) \in T \times \mathbb{G}_m$, the central completions of equivariant K-theory are identified by Lusztig [Lu88, Lu89] with graded Hecke algebras, which have a geometric description where we replace the nilpotent cone $\mathcal{N}$, Springer resolution $\widetilde{\mathcal{N}}$ and Steinberg variety $\mathcal{Z}$ by their (s, q)-fixed points. For example, the Chern character identifies the completion of $\mathcal{H}$ at the trivial central character with the $\widetilde{G} = G \times \mathbb{G}_m$-equivariant homology of the Steinberg variety $\mathcal{Z}$. This algebra is identified via Theorem 8.11 of [Lu95a] with the full Ext-algebra of the Springer sheaf in the equivariant derived category

$$\mathcal{H}^{gr} \simeq H_\bullet^{BM}(\mathcal{Z}/\widetilde{G}; \mathbb{C}) = R\Gamma(\mathcal{Z}/\widetilde{G}, \omega_{\mathcal{Z}/\widetilde{G}}) \simeq \mathrm{Ext}^\bullet_{\mathcal{N}/\widetilde{G}}(\mathbf{S}).$$

Moreover, by a theorem of Rider [Ri13] this Ext algebra is formal, hence we obtain a full embedding

$$\mathcal{H}^{gr}\text{-mod} \simeq \langle \mathbf{S} \rangle \subset \mathrm{Sh}(\mathcal{N}/\widetilde{G}) \tag{1.1}$$

of representations of $\mathcal{H}^{gr}$ into the equivariant derived category of the nilpotent cone. More generally, for $(s, q) \in T \times \mathbb{G}_m$, we have an identification

$$\mathcal{H}^{gr}_{(s,q)} \simeq H_\bullet^{BM}(\mathcal{Z}^{(s,q)}/\widetilde{G}^{(s,q)}; \mathbb{C}) \simeq \mathrm{Ext}^\bullet_{\mathcal{N}^{(s,q)}/\widetilde{G}^{(s,q)}}(\mathbf{S}^{(s,q)})$$

of the corresponding graded Hecke algebra in terms of an (s, q)-variant of the Springer sheaf. This provides a geometric approach to constructing and studying modules[2] of $\mathcal{H}$, see [CG97].

These developments give satisfying descriptions of the representation theory of $\mathcal{H}$ at a fixed central character. However there are numerous motivations to seek a description of *families* of representations of varying central character, including classical harmonic analysis (for example in the setting of spherical varieties [SV17]), K-theory and the Baum-Connes conjecture [ABPS17], and modular and integral representation theory [EH14, H20, HM18].

1.3. **Coherent Springer Theory.** In this paper we apply ideas from derived algebraic geometry to deduce from Theorem 1.2 a different, and in some sense simpler, geometric realization of the affine Hecke algebra, in which we first replace K-theory by Hochschild or cyclic homology, and then derive a description of its entire category of representations as a category of coherent sheaves (without the need for specifying central characters). For technical reasons, we will need to replace the nilpotent cone $\mathcal{N}$ with its formal completion $\widehat{\mathcal{N}} \subset \mathfrak{g}$, and likewise the Steinberg variety $\mathcal{Z} = \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ will be defined via a derived fiber product. For precise definitions of objects in this context, see Section 1.7.3.

Theorem 1.4 (Theorem 2.29, Corollary 2.38)**.** *Let k be an algebraically closed field of characteristic 0, and G a reductive algebraic group over k.*

(1) The trace map from connective K-theory to Hochschild homology on $\mathrm{Coh}(\mathcal{Z}/\widetilde{G})$ factors through an isomorphism of K_0 and $HH_\bullet$ (which is concentrated in cohomological degree zero):

$$\begin{array}{ccc} K(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})) \otimes_{\mathbb{Z}} k & \longrightarrow & HH_\bullet(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})) \\ \downarrow & & \downarrow \simeq \\ K_0(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})) \otimes_{\mathbb{Z}} k & \xrightarrow{\simeq} & HH_0(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})). \end{array}$$

[2]Further if one had an (s, q)-version of Rider's formality theorem, one could deduce a full embedding of the corresponding module categories into equivariant derived categories of constructible sheaves on $\mathcal{N}^{(s,q)}$. See Theorem 3.1 of [Kat15] for an accounting.

(2) *The Steinberg stack satisfies Hochschild-to-cyclic degeneration: fixing an isomorphism $H^\bullet(BS^1) \simeq k[[u]]$, there is an isomorphism*

$$\mathcal{H}[[u]] \simeq HN(\mathrm{Coh}(\mathcal{Z}/\widetilde{G}))$$

between the affine Hecke algebra and the negative cyclic homology of the Steinberg stack.

Remark 1.5. Our results also allow for an identification of monodromic variants of the affine Hecke category. See Remark 2.34 for details.

The Hochschild homology of categories of coherent sheaves admits a description in the derived algebraic geometry of loop spaces. In particular, we deduce an isomorphism of the affine Hecke algebra with volume forms on the derived loop space to the Steinberg stack,

$$\mathcal{H} \simeq R\Gamma(\mathcal{L}(\mathcal{Z}/\widetilde{G}), \omega_{\mathcal{L}(\mathcal{Z}/\widetilde{G})}).$$

More significantly, the geometry of derived loop spaces provides a natural home for the entire category of $\mathcal{H}$-modules, without fixing central characters.

Definition 1.6. Let $\widehat{\mathcal{N}} \subset \mathfrak{g}$ be the formal completion of the nilpotent cone, $\widetilde{\mathcal{N}}$ the usual (reduced) Springer resolution and $\mu : \widetilde{\mathcal{N}} \to \mathcal{N} \hookrightarrow \widehat{\mathcal{N}}$ the composition of the Springer resolution with the inclusion. The *coherent Springer sheaf* $\mathcal{S}_G \in \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ (or simply $\mathcal{S}$) is the pushforward of the structure sheaf under the loop map $\mathcal{L}\mu : \mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) \to \mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$:

$$\mathcal{S}_G = \mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})} \in \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

Equivalently, $\mathcal{S}_G$ is given by applying the parabolic induction correspondence

$$\mathcal{L}(\widehat{\{0\}}/T) \longleftarrow \mathcal{L}(\widehat{\mathfrak{n}}/B) = \mathcal{L}(\widehat{\widetilde{\mathcal{N}}}/G) \longrightarrow \mathcal{L}(\widehat{\mathcal{N}}/G)$$

to the (reduced) structure sheaf of $\mathcal{L}(\{0\}/T)$.

A priori the coherent Springer sheaf is only a complex of sheaves. However we show, using the theory of traces for monoidal categories in higher algebra, that its Ext algebra is concentrated in degree zero, and is identified with the affine Hecke algebra. This provides the following "coherent Springer correspondence", realizing the representations of the affine Hecke algebra as coherent sheaves.

Theorem 1.7 (Theorem 4.12)**.** *Let G be a reductive algebraic group over an algebraically closed field of characteristic 0.*

(1) *There is an isomorphism of algebras $\mathcal{H}_G \simeq \mathrm{End}_{\mathcal{L}(\mathcal{N}/\widetilde{G})}(\mathcal{S}_G)$ and all other* Ext *groups of $\mathcal{S}_G$ vanish.*

(2) *There is a full, compact-object preserving embedding*

$$D(\mathcal{H}_G) \simeq \langle \mathcal{S}_G \rangle \subset \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

(3) *The embedding takes the anti-spherical module to the projection of the dualizing sheaf to the Springer subcategory*

$$D(\mathcal{H}_G) \ni \mathrm{Ind}_{\mathcal{H}^f}^{\mathcal{H}}(\mathrm{sgn}) \longmapsto \mathrm{pr}_{\mathcal{S}_G}(\omega_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}) \in \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

(4) *The embedding is compatible with parabolic induction of affine Hecke algebras, i.e. if P is a parabolic subgroup of G with Levi quotient M, then there is a commuting diagram*

$$\begin{array}{ccc} D(\mathcal{H}_M) & \hookrightarrow & \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}_M/\widetilde{M})) \\ \downarrow{\scriptstyle \mathcal{H}_G \otimes_{\mathcal{H}_M} -} & & \downarrow{\scriptstyle \mathcal{L}\mu_* \circ \mathcal{L}\nu^*} \\ D(\mathcal{H}_G) & \hookrightarrow & \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}_G/\widetilde{G})), \end{array}$$

where $\mathcal{L}\mu_* \circ \mathcal{L}\nu^*$ is the pull-push along the correspondence obtained by applying $\mathcal{L}$ to the usual parabolic induction correspondence

$$\mathcal{L}(\widehat{\mathcal{N}}_M/\widetilde{M}) \xleftarrow{\mathcal{L}\mu} \mathcal{L}(\widehat{\mathcal{N}}_P/\widetilde{P}) \xrightarrow{\mathcal{L}\nu} \mathcal{L}(\widehat{\mathcal{N}}_G/\widetilde{G}).$$

In particular, $\mathcal{L}\mu_*\mathcal{L}\nu^*\mathcal{S}_M \simeq \mathcal{S}_G$.

(5) *Fixing an isomorphism* $H^\bullet(BS^1) \simeq k[[u]]$, *there is a full embedding of the derived category of perfect modules for the trivial* u*-deformation* $\mathcal{H}[[u]]$ *into the* u*-deformation* $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))^{S^1}$.

One consequence of the theorem is an interpretation of the coherent Springer sheaf as a universal family of $\mathcal{H}$-modules.

We also conjecture (Conjecture 4.15) – and check for SL_2 – that $\mathcal{S}$ is actually a coherent sheaf (i.e., lives in the heart of the standard t-structure on coherent sheaves). The vanishing of all nonzero Ext groups of $\mathcal{S}$ suggests the existence of a natural "exotic" t-structure for which $\mathcal{S}$ is a compact projective object in the heart. For such a t-structure we would then automatically obtain a full embedding of the abelian category $\mathcal{H}$-mod into "exotic" coherent sheaves, where one could expect a geometric description of simple objects. See Section 5.3 for a discussion.

We will explain in Section 1.5 how equivariant localization and Koszul duality patterns in derived algebraic geometry (as developed in [BN13, Ch20a, Ch21]) provides the precise compatibility between this coherent Springer theory and the usual perverse Springer theory, one parameter at a time.

1.4. **Applications to the local Langlands correspondence.** We will consider a derived stack $\mathbb{L}^u_{q,G}$ of *unipotent Langlands parameters*, which parametrizes the unipotent Weil-Deligne representations for a local field F with residue field $\mathbb{F}_q$, and whose set of k-points is a variant of the set of Deligne-Langlands parameters in Theorem 1.3 (with semisimplicity of s dropped). Note that the following notions make sense for any $q \in \mathbb{C}$, with applications to local Langlands when q is a prime power, and that, in line with expectations, the stack of *unipotent* Langlands parameters depends only on order of the residue field of F.

Definition 1.8. Let $q = p^r$ be a prime power.

(1) The *stack of unipotent Langlands parameters* $\mathbb{L}^u_{q,G} = (\widehat{\mathcal{N}}/G)^q$ (or simply $\mathbb{L}^u_q$) is the derived fixed point stack of multiplication by $q \in \mathbb{G}_m$ on $\widehat{\mathcal{N}}/G$. Equivalently, it is the fiber of the loop (or derived inertia) stack of the nilpotent cone over $q \in \mathbb{G}_m$,

$$\begin{array}{ccc} \mathbb{L}^u_{q,G} & \longrightarrow & \mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}) \\ \downarrow & & \downarrow \\ \{q\} & \longrightarrow & \mathcal{L}(\mathrm{pt}/\mathbb{G}_m) = \mathbb{G}_m/\mathbb{G}_m. \end{array}$$

Thus informally

$$\mathbb{L}^u_{q,G} \simeq \{g \in G, n \in \mathcal{N} : gng^{-1} = qn\}/G.$$

By Proposition 4.3 this a priori derived stack has no derived nor infinitesimal structure, i.e. $(\widehat{\mathcal{N}}/G)^q = (\mathfrak{g}/G)^q$ and we may equivalently define $\mathbb{L}^u_{q,G}$ using the classical fiber product of the reduced nilpotent cone $\mathcal{N}$.

(2) The *q-coherent Springer sheaf* $\mathcal{S}_{q,G} \in \mathrm{Coh}(\mathbb{L}^u_q)$ (or simply $\mathcal{S}_q$) is the $*$-specialization of $\mathcal{S}_G$ to the fiber $\mathbb{L}^u_q$ over q. Equivalently, $\mathcal{S}_{q,G}$ is given by applying the parabolic induction correspondence

$$\mathbb{L}^u_{q,T} \longleftarrow \mathbb{L}^u_{q,B} \longrightarrow \mathbb{L}^u_{q,G}$$

to the structure sheaf of $\mathbb{L}^u_{q,T} \simeq T \times BT$.

Specializing Theorem 1.7 to $q \in \mathbb{G}_m$ we obtain the following. Note that we implicitly identify, in this case where $q \in \mathbb{R}^+$, the category of perfect complexes for $\mathcal{H}_q$ with the category of complexes with coherent cohomology via Theorem 2.2 of [OS09].

Theorem 1.9 (Theorem 4.12)**.** *Suppose that* $q = p^r$ *is a prime power, and let* G *be a reductive algebraic group an algebraically closed field of characteristic 0.*

(1) *There an isomorphism of algebras* $\mathcal{H}_{q,G} \simeq \mathrm{End}_{\mathbb{L}_{q,G}}(\mathcal{S}_{q,G})$ *and a full embedding*
$$D_{f.g.}(\mathcal{H}_{q,G}) \simeq \langle \mathcal{S}_{q,G} \rangle \subset \mathrm{Coh}(\mathbb{L}^u_{q,G}).$$
In particular, this gives a full embedding of the principal block of $G^\vee(F)$ *into coherent sheaves on the stack of unipotent Langlands parameters.*
(2) *The embedding takes the anti-spherical module to the structure sheaf* $\mathcal{O}_{\mathbb{L}^u_{q,G}} \in \mathrm{Coh}(\mathbb{L}^u_{q,G})$.
(3) *The embedding is compatible with parabolic induction, i.e. if* $P^\vee \subset G^\vee$ *is a parabolic with quotient Levi* $M^\vee$*, then we have a commutative diagram*
$$\begin{array}{ccc} \{\textit{unramified principal series of } M^\vee(F)\} \simeq D_{f.g.}(\mathcal{H}_{q,M}) & \longrightarrow & \mathrm{Coh}(\mathbb{L}^u_{q,M}) \\ \downarrow{\scriptstyle i^{G^\vee}_{P^\vee}} & & \downarrow{\scriptstyle (\mu^q)_* \circ (\nu^q)^*} \\ \{\textit{unramified principal series of } G^\vee(F)\} \simeq D_{f.g.}(\mathcal{H}_{q,G}) & \longrightarrow & \mathrm{Coh}(\mathbb{L}^u_{q,G}), \end{array}$$
where $i^{G^\vee}_{P^\vee} : \mathrm{Rep}^{sm}_{f.g.}(M^\vee(F)) \to \mathrm{Rep}^{sm}_{f.g.}(G^\vee(F))$ *is the parabolic induction functor from smooth finitely-generated*[3] *reprentations of* $M^\vee(F)$ *to* $G^\vee(F)$ *restricted to the unramified principal series, and the map* $(\mu^q)_* \circ (\nu^q)^*$ *is the pull-push along the correspondence obtained by applying taking derived* q*-invariants of the usual parabolic induction correspondence*
$$\mathbb{L}^u_{q,M} \xleftarrow{\mu^q} \mathbb{L}^u_{q,P} \xrightarrow{\nu^q} \mathbb{L}^u_{q,G}.$$
In particular, $(\mu^q)_*(\nu^q)^*\mathcal{S}_{q,M} \simeq \mathcal{S}_{q,G}$.

Note that due to Proposition 4.3, in the q-specialized setting of the above theorem the stack of parameters has no infinitesimal structure, i.e. $(\mathfrak{g}/G)^q = (\widehat{\mathcal{N}}/G)^q$. This has two consequences: first, due to Proposition 3.12, which does not apply in the context of Theorem 1.7, we may identify the anti-shperical sheaf at specialized q with the structure sheaf, which is equivalent to the dualizing sheaf. Second, the anti-spherical sheaf at specialized q is a compact object in the category, i.e. a coherent sheaf, whereas the sheaf appearing in Theorem 1.7 is not.

The existence of such an equivalence was conjectured independently by Hellmann in [He20], whose work we learned of at a late stage in the preparation of his paper. Indeed, the above result resolves Conjecture 3.2 of [He20]. Hellmann's work also gives an alternative characterization of the (q-specialized) coherent Springer sheaf as the Iwahori invariants of a certain family of admissible representations on $\mathbb{L}^u_{q,G}$ constructed by Emerton and the third author in [EH14].

A much more general categorical form of the local Langlands correspondence is formulated by Fargues-Scholze [FS21] and Zhu [Zh20], as well as compatibility with a categorical global Langlands correspondence. In *loc. cit.* a forthcoming proof by Hemo and Zhu [HZ] of a result closely parallel to ours is also announced.

Remark 1.10. The local Langlands correspondence depends on a choice of Whittaker normalization; that is, a choice of a pair (U, ψ), where U is the unipotent radical of a Borel subgroup of $G^\vee$ and ψ is a generic character of $U(F)$, up to $G^\vee(F)$-conjugacy, and indeed, the conjecture in [He20] and the announced result in [HZ] depend on such a choice. In the formulation of Theorem 1.9 no such choice appears explicitly, but instead comes from the integral structure on $G^\vee$, which in particular gives us a distinguished hyperspecial subgroup $G^\vee(O)$ of $G^\vee(F)$.

Indeed, for any unramified group $G^\vee$ over F there is a natural bijection between $G^\vee(F)$-conjugacy classes of Whittaker data (U, ψ) for $G^\vee$ and $G^\vee(F)$-conjugacy classes of triples (K_x, U_x, ψ_x), where K_x is a hyperspecial subgroup of $G^\vee(F)$, U_x is the unipotent radical of a Borel subgroup of the reductive quotient $G^\vee_x$ of K_x, and ψ_x is a generic character of U_x. This bijection has the property that if (U, ψ) corresponds to (K_x, U_x, ψ_x), then the summand of the compact induction $\mathrm{cInd}_{U(F)}^{G^\vee(F)} \psi$ corresponding to the unipotent principal series block is isomorphic to $\mathrm{cInd}_{K_x}^{G^\vee(F)} \mathrm{St}_x$, where St_x denotes the inflation to K_x of the Steinberg representation of the reductive quotient $G^\vee_x$. In particular the "unipotent principal series part" of $\mathrm{cInd}_{U(F)}^{G^\vee(F)} \psi$

[3] I.e. the corresponding modules for Hecke algebras are finitely generated.

depends only on the conjugacy class of hyperspecial subgroup associated to (U, ψ), and not the whole tuple (K_x, U_x, ψ_x). This means that the restriction of the local Langlands correspondence to the unramified principal series depends only on a choice of hyperspecial subgroup (which we have fixed).

Note in particular that for any choice of Whittaker datum (U, ψ) compatible with our hyperspecial subgroup $G^\vee(O)$, the $\mathcal{H}_{q,G}$-module associated to the compact induction $\mathrm{cInd}_{U(F)}^{G^\vee(F)} \psi$ is precisely the antispherical module, so property (2) of Theorem 1.9 is consistent with (and indeed, equivalent to) the Whittaker normalization appearing in [He20].

In the case of the general linear group and its Levi subgroups, one can go much further. Namely, in Section 6 we combine the local Langlands classification of irreducible representations due to Harris-Taylor and Henniart with the Bushnell-Kutzko theory of types and the ensuing inductive reduction of all representations to the principal block. The result is a spectral description of the entire category of smooth $\mathrm{GL}_n(F)$ representations. To do so it is imperative to first have a suitable stack of Langlands parameters. These have been studied extensively in mixed characteristic, for instance in [H20] in the case of GL_n, or more recently in [BG19, BP19], and [DHKM20] for more general groups. Since in our present context we work over $\mathbb{C}$, the results we need are in general simpler than the results of the above papers, and have not appeared explicitly in the literature in the form we need.

Theorem 1.11 ([H20])**.** *Let F be a local field with residue field $\mathbb{F}_q$. There is a classical Artin stack locally of finite type $\mathbb{L}_{F,\mathrm{GL}_n}$, with the following properties:*

1. *The k-points of $\mathbb{L}_{F,\mathrm{GL}_n}$ are identified with the groupoid of continuous n-dimensional representations of the Weil-Deligne group of F.*
2. *The formal deformation spaces of Weil-Deligne representations are identified with the formal completions of $\mathbb{L}_{F,\mathrm{GL}_n}$.*
3. *The stack $\mathbb{L}^u_{q,\mathrm{GL}_n}$ of unipotent Langlands parameters is a connected component of $\mathbb{L}_{F,\mathrm{GL}_n}$.*

We then deduce a categorical local Langlands correspondence for GL_n and its Levi subgroups as follows:

Theorem 1.12 (Theorems 6.13, 6.15, and 6.17)**.** *For each Levi subgroup M of $\mathrm{GL}_n(F)$, there is a full embedding*

$$D(M) \hookrightarrow \mathrm{QC}^!(\mathbb{L}_{F,M})$$

of the derived category of smooth M-representations into ind-coherent sheaves on the stack of Langlands parameters, uniquely characterized by the following properties.

1. *If π is an irreducible cuspidal representation of M, then the image of π under this embedding is the skyscraper sheaf supported at the Langlands parameter associated to π.*
2. *Let M' be a Levi subgroup of G, and let P be a parabolic subgroup of M' with Levi subgroup M. There is a commutative diagram of functors:*

$$\begin{array}{ccc} D(M) & \hookrightarrow & \mathrm{QC}^!(\mathbb{L}_{F,M}) \\ {\scriptstyle i_M^{M'}}\downarrow & & \downarrow{\scriptstyle \mu_*\nu^*} \\ D(M') & \hookrightarrow & \mathrm{QC}^!(\mathbb{L}_{F,M'}) \end{array}$$

in which $i_M^{M'}$ is the parabolic induction functor and the right-hand map is obtained by applying the correspondence

$$\mathbb{L}_{F,M} \xleftarrow{\mu} \mathbb{L}_{F,P} \xrightarrow{\nu} \mathbb{L}_{F,M'}.$$

Note that the local Langlands correspondence for cuspidal representations of GL_n and its Levis, is an *input* to the above result. We do not expect the functor to be an equivalence, see Remark 4.13.

As with Theorem 1.9 our results here were independently conjectured by Hellmann (see in particular Conjecture 3.2 of [He20]) for more general groups G; these results also fit the general categorical form of the local Langlands correspondence formulated by Fargues-Scholze [FS21] and Zhu [Zh20].

1.4.1. *Discussion: Categorical Langlands Correspondence.* Theorems 1.9 and 1.12 match the expectation in the Langlands program that has emerged in the last couple of years for a strong form of the local Langlands correspondence, in which categories of representations of groups over local fields are identified with categories of coherent sheaves on stacks of Langlands parameters. Such a coherent formulation of the real local Langlands correspondence was discovered in [BN13], while the current paper finds a closely analogous picture in the Deligne-Langlands setting. As this paper was being completed Xinwen Zhu shared the excellent overview [Zh20] on this topic and Laurent Fargues and Peter Scholze completed the manuscript [FS21], to which we refer the reader for more details. We only briefly mention three deep recent developments in this general spirit.

The first derives from the work of V. Lafforgue on the global Langlands correspondence over function fields [La18a, La18b]. Lafforgues' construction in Drinfeld's interpretation (cf. [LaZ19, Section 6], [La18b, Remark 8.5] and [Ga16]) predicts the existence of a universal quasicoherent sheaf $\mathfrak{A}_X$ on the stack of representations of $\pi_1(X)$ into G corresponding to the cohomology of moduli spaces of shtukas. The theorem of Genestier-Lafforgue [GL18] implies that the category of smooth $G^\vee(F)$ representations sheafifies over a stack of local Langlands parameters, and the local version $\mathfrak{A}$ of the Drinfeld-Lafforgue sheaf is expected [Zh20] to be a universal $G^\vee(F)$-module over the stack of local Langlands parameters. In other words, the fibers $\mathfrak{A}_\sigma$ are built out of the $G^\vee(F)$-representations in the L-packet labelled by σ. The expectation is that the coherent Springer sheaf, which by our results is naturally enriched in $\mathcal{H}_q$-modules, is identified with the Iwahori invariants of the local Lafforgue sheaf $\mathcal{S}_q \simeq \mathfrak{A}^I$.

The second is the theory of categorical traces of Frobenius as developed in [Ga16, Zh18, GKRV20]. When applied to a suitably formulated local geometric Langlands correspondence, we obtain an expected equivalence between an automorphic and spectral category. The automorphic category is $\mathrm{Sh}(G^\vee(F)/^{\mathrm{Fr}}G^\vee(F))$, the category of Frobenius-twisted adjoint equivariant sheaves on $G^\vee(F)$, with orbits given by the Kottwitz set $B(G^\vee)$ of isomorphism classes of $G^\vee$-isocrystals. The spectral category is expected to be a variant of a category $\mathrm{QC}^!(\mathbb{L}_{F,G})$ of ind-coherent sheaves over the stack $\mathbb{L}_{F,G}$ of Langlands parameters into G. The former category contains the categories of representations of $G^\vee(F)$ and its inner forms as full subcategories, hence we expect a spectral realization in the spirit of Theorems 1.9 and 1.12.

The last of these developments is the program of Fargues-Scholze [Fa16], [FS21] in the context of p-adic groups, which interprets the local Langlands correspondence as a geometric Langlands correspondence. On the automorphic side one considers sheaves on the stack $\mathrm{Bun}_{G^\vee}$ of bundles on the Fargues-Fontaine curve, whose isomorphism classes $|\mathrm{Bun}_{G^\vee}| = B(G^\vee)$ are given as before by the Kottwitz set of $G^\vee$-isocrystals. This category of sheaves admits a semiorthogonal decomposition indexed by $B(G^\vee)$, in which the factor corresponding to $b \in B(G^\vee)$ is naturally equivalent to the category of smooth representations of the inner form $G_b^\vee(F)$ arising from b. On the spectral side of the picture is the same category of ind-coherent sheaves on the moduli stack of Langlands parameter that we study. Fargues-Scholze construct a spectral action of the category of perfect complexes on this moduli stack on the category of ℓ-adic sheaves on $\mathrm{Bun}_{G^\vee}$, and conjecture that there is an equivalence of this category with the category of ind-coherent sheaves on the moduli stack of Langlands parameters compatible with this spectral action. Such an equivalence necessarily has the properties given in Theorem 1.12, although we do not attempt to verify that our construction is compatible with that of Fargues-Scholze.

1.5. **Compatibility of coherent and perverse Springer theory.** In this section we explain how equivariant localization and Koszul duality patterns in derived algebraic geometry (as developed in [BN13, Ch20a, Ch21]) provide the precise compatibility between this coherent version of the local Langlands correspondence and the more familiar model [Ze81, Lu83, Vo93] for local Langlands categories with fixed central character via categories of perverse sheaves. This pattern was developed in the context of the real local Langlands correspondence: the work of Adams, Barbasch and Vogan [ABV92, Vo93] and Soergel's conjecture [So01] describe representations of real groups with fixed infinitesimal character by equivariant perverse sheaves on spaces of Langlands parameters, while [BN13] gives a conjectural description of the full categories of representations in terms of coherent sheaves. Likewise, the solution to the Deligne-Langlands conjecture in [KL87] realizes the irreducible representations of affine Hecke algebras, one central parameter (s,q) at a time, in terms of simple equivariant perverse sheaves (or equivalently $\mathcal{D}$-modules) on a collection of spaces $\mathcal{N}^{(s,q)}$. On the other hand, Theorem 1.7 provides a uniform description of all representations of $\mathcal{H}$ in terms of *coherent* sheaves on single parameter space.

The underlying mechanism in passing between the coherent sheaves on our algebro-geometric parameter space $\mathbb{L}_G^u$ and perverse sheaves or $\mathcal{D}$-modules on variants of the nilpotent cone $\mathcal{N}_G^{(s,q)}$ is the interpretation of $\mathcal{D}$-modules in the derived algebraic geometry of loop spaces [BN12, BN13, TV11, TV15, Pr15, Ch20a], a unification of Connes' description of de Rham cohomology as periodic cyclic homology and of the Koszul duality between $\mathcal{D}$-modules and modules for the de Rham complex [BeDr91, Ka91]. Recall that the *loop space*, or derived inertia, of a stack X is defined by the mapping space from the circle, or equivalently the (derived) self-intersection of the diagonal

$$\mathcal{L}X = \mathrm{Map}(S^1, X) = X \times_{X\times X} X = \Delta \cap \Delta.$$

For $X = \mathrm{Spec}(R)$ affine, the loop space is the spectrum of the (derived) algebra of Hochschild chains $HH_\bullet(R) = R \otimes_{R\otimes R} R$. More generally for any scheme X, we have the (Hochschild-Kostant-Rosenberg) identification

$$\mathcal{L}X \simeq \mathbb{T}_X[-1] = \mathrm{Spec}_X \mathrm{Sym}_X^\bullet(\Omega_X^1[1])$$

of the loop space with the relative spectrum of (derived) differential forms. Under this identification the loop rotation action of S^1 on $\mathcal{L}X$ (Connes' B-differential on the level of Hochschild homology) becomes encoded by the de Rham differential.

Theorem 1.13 (Koszul duality [BN12, TV11, Pr15])**.** *For X an algebraic space almost of finite type over k a field of characteristic zero, there is a natural equivalence of $k((u))$-linear categories*

$$\mathrm{Coh}(\mathcal{L}X)^{S^1} \otimes_{k[[u]]\text{-mod}} k((u))\text{-mod} \simeq \mathcal{D}_X\text{-perf} \otimes_{k\text{-mod}} k((u))\text{-mod}$$

where $u \in H^\bullet(BS^1;k)$ is the degree 2 Chern class.

When X is a stack, we only have an equivalence between $\mathcal{D}$-modules and S^1-equivariant sheaves on the formal loop space $\widehat{\mathcal{L}}X$, i.e. the formal completion of the loop space $\mathcal{L}X$ at constant loops. The loop space of a smooth global quotient stack $\mathcal{L}(X/G)$ lies over a parameter space $\mathcal{L}(BG) = G/G$, and the equivariant localization patterns in [Ch20a] realize the formal completion (resp. specialization) of $\mathcal{L}(X/G)$ over a semisimple parameter $z \in G/G$ as the formal loop space of the G^z-equivariant classical z-fixed points $\widehat{\mathcal{L}}(X^z/G^z)$ (resp. the non-equivariant loop space $\mathcal{L}(X^z)$). In particular, in the setting of Deligne-Langlands, specializing at a parameter z recovers the loop space of the fixed point schemes $\mathcal{L}(\widehat{\mathcal{N}}^{s,q})$, and we can pass to $\mathcal{D}$-modules on the correpsonding analytic space via Koszul duality.

In order to formulate the equivalence at completed parameters, we need to renormalize the category of coherent sheaves to include objects such as the structure sheaf or sheaf of distributions on formal completions. This form of Koszul duality is developed by one of the authors in [Ch21] (see Section 5.2 for the details). We call objects in this category *Koszul-perfect sheaves* $\mathrm{KPerf}(\widehat{\mathbb{T}}_X[-1])$ on the formal odd tangent bundle, and they have the following favorable properties: (1) they are preserved by smooth pullback and proper pushforward in X, (2) for a smooth Artin stack X, Koszul-perfect objects are those which pull back to Koszul-perfect objects along a smooth atlas and (3) for smooth schemes X they are just the coherent complexes. These properties mirror the properties enjoyed by the subcategory of *coherent* $\mathcal{D}$-modules on QCA stacks (which do not coincide with compact objects in general, see [DG13]).

Theorem 1.14 (Theorem 5.23, [Ch21])**.** *Let X/G be a global quotient stack and let $F\breve{\mathcal{D}}(X/G)$ denote the category of filtered renormalized (i.e. ind-coherent) $\mathcal{D}$-modules on the global quotient stack X/G. There is an equivalence of categories*

$$\mathrm{KPerf}(\widehat{\mathcal{L}}(X/G))^{B\mathbb{G}_a \rtimes \mathbb{G}_m} \simeq F\breve{\mathcal{D}}(X/G).$$

Applying this theorem requires choosing, at each parameter, a graded lift of the z-completed (or specialized) coherent Springer sheaf. There is a natural *geometric* or *Hodge graded lift*, and using this lift, we establish in Corollary 5.3 that the coherent Springer sheaf is Koszul dual at each parameter to the corresponding perverse Springer sheaves:

Corollary 1.15. *Fix a semisimple parameter $(s,q) \in \widetilde{G}$, and let $d(s,q) = \dim(\mathcal{N}^{(s,q)})$. Then the (s,q)-specialization of the coherent Springer sheaf $\mathcal{S}$ is Koszul dual to the (s,q)-Springer sheaf $\mu^z_* \mathbb{C}_{\widetilde{\mathcal{N}}^{(s,q)}}[d(s,q)]$, i.e. the pushforward of the (shifted) constant sheaf along (s,q)-fixed points of the Springer resolution.*

More precisely, the (s,q)-specialization $\mathcal{S}(s,q)$ of $\mathcal{S}$ has a Hodge graded lift, which is Koszul dual to the (s,q)-Springer sheaf $\mathbf{S}(s,q)$ equipped with its Hodge filtration. Likewise, the Hodge graded lift of the (s,q)-completion $\mathcal{S}(\widehat{s,q})$ is naturally isomorphic to the $\widetilde{G}^{(s,q)}$-equivariant (s,q)-Springer sheaf $\mathbf{S}(\widehat{s,q})$ equipped with the Hodge filtration.

1.6. **Methods.** We now discuss the techniques underlying the proofs of Theorems 1.4 and 1.7 – namely, Bezrukavnikov's Langlands duality for the affine Hecke category and the theory of traces of monoidal dg categories.

1.6.1. *Bezrukavnikov's theorem.* The Kazhdan-Lusztig theorem (Theorem 1.2) has been famously categorified in the work of Bezrukavnikov [Bez06, Bez16], with numerous applications in representation theory and the local geometric Langlands correspondence.

Theorem 1.16. [Bez16] *Let $F = \overline{\mathbb{F}_q}((t))$. Let $I \subset G(F)$ be an Iwahori subgroup, and define the Steinberg stack $\mathcal{Z}/G$ over $\overline{\mathbb{Q}}_\ell$. There is an equivalence of monoidal dg categories*

$$D(I\backslash G^\vee(F)/I; \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Coh}(\mathcal{Z}/G)$$

intertwining the automorphisms pullback by geometric Frobenius and pullback by multiplication by q.

Remark 1.17. In view of Theorem 1.16, we define the *affine Hecke category* to be $\mathbf{H} := \mathrm{Coh}(\mathcal{Z}/G)$. It is natural to expect a mixed version, identifying the *mixed affine Hecke category* $\mathbf{H}^{\mathrm{m}} := \mathrm{Coh}(\mathcal{Z}/\widetilde{G})$ with the mixed Iwahori-equivariant sheaves on the affine flag variety (as studied in [BY13]). Indeed such a version is needed to directly imply the Kazhdan-Lusztig Theorem 1.2 by passing to Grothendieck groups, rather than its specialization at $q = 1$.

Theorem 1.16 establishes the "principal block" part of the local geometric Langlands correspondence. Namely, it implies a spectral description of module categories for the affine Hecke category (the geometric counterpart of unramified principal series representations) as suitable sheaves of categories on stacks of Langlands parameters.

We apply Theorem 1.16 in Section 2 to construct a semiorthogonal decomposition of the affine Hecke category. This allows us to calculate its Hochschild and cyclic homology and to establish the comparison with algebraic K-theory.

1.6.2. *Trace Decategorifications.* To prove Theorem 1.7 we use the relation between the "horizontal" and "vertical" trace decategorifications of a monoidal category, and the calculation of the subtler horizontal trace of the affine Hecke category in [BNP17b].

Let $(\mathbf{C}, *)$ denote a monoidal dg category. Then we can take the trace (or Hochschild homology) $\mathrm{tr}(\mathbf{C}) = HH(\mathbf{C})$ of the underlying (i.e. ignoring the monoidal structure) dg category $\mathbf{C}$, which forms an associative (or A_∞-)algebra $(\mathrm{tr}(\mathbf{C}), *)$ thanks to the functoriality (specifically the symmetric monoidal structure) of Hochschild homology, as developed in [TV15, HSS17, CP19, GKRV20]. This is the naive or "vertical" trace of $\mathbf{C}$. On the other hand, a monoidal dg category has another trace or Hochschild homology $\mathbf{Tr}(\mathbf{C}, *)$ using the monoidal structure which is itself a dg category – the categorical or "horizontal" trace of $(\mathbf{C}, *)$. This is the dg category which is the universal receptacle of a trace functor out of the monoidal category $\mathbf{C}$. In particular, the trace of the monoidal unit of $\mathbf{C}$ defines an object $[1_{\mathbf{C}}] \in \mathbf{Tr}(\mathbf{C}, *)$ – i.e., $\mathbf{Tr}(\mathbf{C}, *)$ is a pointed (or E_0-)category[4]. Moreover, as developed in [CP19, GKRV20] the categorical trace provides a "delooping" of the naive trace: we have an isomorphism of associative algebras

$$(\mathrm{tr}(\mathbf{C}), *) \simeq \mathrm{End}_{\mathbf{Tr}(\mathbf{C},*)}([1_{\mathbf{C}}]).$$

In particular taking Hom from $[1_{\mathbf{C}}]$ defines a functor

$$\mathrm{Hom}([1_{\mathbf{C}}], -) : \mathbf{Tr}(\mathbf{C}, *) \longrightarrow (HH(\mathbf{C}), *)\text{-mod}.$$

Under suitable compactness assumptions the left adjoint to this functor embeds the "naive" decategorification (the right hand side) as a full subcategory of the "smart" decategorification (the left hand side).

More generally, given a monoidal endofunctor F of $(\mathbf{C}, *)$, we can replace Hochschild homology (trace of the identity) by trace of the functor F, obtaining two decategorifications (vertical and horizontal) with a similar relation

$$\mathrm{Hom}([1_{\mathbf{C}}], -) : \mathbf{Tr}((\mathbf{C}, *), F) \longrightarrow (\mathrm{tr}(\mathbf{C}, F), *)\text{-mod}. \tag{1.2}$$

Remark 1.18 (Trace of Frobenius). When $\mathbf{C}$ is a category of ℓ-adic sheaves on a stack over $\overline{\mathbb{F}_q}$ and Fr is the Frobenius morphism, Gaitsgory has explained [Ga16] that one expects a formalism of categorical traces to hold realizing the function-sheaf correspondence – i.e. $\mathrm{tr}(\mathrm{Sh}(X), \mathrm{Fr}^*)$ should be the space of functions on $X(\mathbb{F}_q)$. Likewise the monoidal version of trace decategorification would then allow us to pass from Hecke categories to categories of representations directly. Zhu [Zh18] explains some of the rich consequences of this formalism that can already be proved directly.

Example 1.19 (Finite Hecke Categories and unipotent representations)**.** For the finite Hecke category $\mathbf{C} = \mathrm{Sh}(B\backslash G/B)$, the main theorem of [BN15] identifies $\mathbf{Tr}(\mathbf{C}, *)$ with the full category of Lusztig unipotent character sheaves on G. The object $[1_{\mathbf{C}}]$ is the Springer sheaf itself, and modules for the naive decategorification $(\mathrm{tr}((\mathbf{C}), *), \mathrm{id}_{\mathbf{C}})$ gives the Springer block, or unipotent principal series character sheaves, as modules for the graded Hecke algebra. Likewise the trace of Frobenius on $(\mathbf{C}, *)$ is studied in [Zh18, Section 3.2] (see also [Ga16, Section 3.2]) – here the categorical trace is the category of all unipotent representations of $G(\mathbb{F}_q)$, not only those in the principal series.

1.6.3. *Trace of the affine Hecke category.* We now consider the two kinds of trace decategorification for the affine Hecke category $\mathbf{H}$. First our description of the Hochschild homology of the Steinberg stack provides a precise sense in which the affine Hecke category categorifies the affine Hecke algebra. The following Corollary is a result of Theorems 1.16 and 1.4.

Corollary 1.20. *The (vertical/naive) trace of Frobenius on the affine Hecke category is identified with the affine Hecke algebra* $\mathcal{H} \simeq \mathrm{tr}(\mathbf{H}, \mathrm{Fr}^*)$. *Hence the naive decategorification of* $\mathbf{H}$-mod *is the category of unramified principal series representations of* $G^\vee(F)$.

[4]The horizontal trace is also the natural receptacle for characters of $\mathbf{C}$-module categories, and $[\mathbf{C}]$ appears as the character of the regular left $\mathbf{C}$-module, see Definition 3.2.

Remark 1.21. Note that this corollary would follow directly from Theorem 1.16 if we had available the hoped-for function-sheaf dictionary for traces of Frobenius on categories of ℓ-adic sheaves (Remark 1.18). After this paper was complete Xinwen Zhu informed us that Hemo and he have a direct argument for this corollary, see the forthcoming [HZ]. Combined with Bezrukavnikov's theorem and Theorem 1.22 this gives an alternative argument for the identification of $\mathcal{H}_q$ with the Ext algebra of the coherent Springer sheaf.

The results of [BNP17b] (based on the technical results of [BNP17a]) provide an affine analog of the results of [BN15, BFO12] for finite Hecke categories and (thanks to Theorem 1.16) a spectral description of the full decategorification of $\mathbf{H}$. Statement (1) is directly taken from Theorem 4.4.1 in [BNP17b], statements (2)-(3) follow immediately from the same techniques and Theorem 3.8.5 of [GKRV20] (see Theorems 3.4 and 3.23 and Lemma 3.24), and the absence of a singular support condition is discussed in Remark 4.14.

Theorem 1.22 ([BNP17b]). *Let G be a reductive group over a field of characteristic 0.*

(1) *The (horizontal/categorical) trace of the monoidal category* $(\mathrm{Coh}(\mathcal{Z}/G), *)$ *is identified as*
$$\mathbf{Tr}(\mathrm{Coh}(\mathcal{Z}/G), *) = \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/G)).$$
The same assertion holds with G replaced by $\widetilde{G} = G \times \mathbb{G}_m$.

(2) *The trace of multiplication by $q \in \mathbb{G}_m$ acting on the monoidal category* $(\mathrm{Coh}(\mathcal{Z}/G), *)$ *is identified as*
$$\mathbf{Tr}((\mathrm{Coh}(\mathcal{Z}/G), *), q^*) = \mathrm{Coh}(\mathbb{L}_q^u).$$

(3) *The distinguished object $[1_{\mathbf{C}}]$ in each of these trace decategorifications is given by the coherent Springer sheaf $\mathcal{S}$ (or its q-specialized version $\mathcal{S}_q$). Hence the endomorphisms of the coherent Springer sheaf recover the affine Hecke algebra (the vertical trace, as in Theorem 1.7), and the natural functor in Theorem 3.4 is identified with*
$$\mathrm{Hom}(\mathcal{S}_q, -) : \mathrm{Coh}(\mathbb{L}_q^u) \longrightarrow \mathcal{H}_q\text{-mod}.$$

In other words, we identify the entire category of coherent sheaves on the stack of unipotent Langlands parameters as the categorical trace of the affine Hecke category. Inside we find the unramified principal series as modules for the naive trace (the Springer block). Just as the decategorification of the finite Hecke category (Example 1.19) knows all unipotent representations of Chevalley groups, the horizontal trace $\mathrm{Coh}(\mathbb{L}_q^u)$ of the affine Hecke category contains in particular all unipotent representations of $G^\vee(F)$ – i.e., the complete L-packets of unramified principal series representations – thanks to Lusztig's remarkable Langlands duality for unipotent representations:

Theorem 1.23 ([Lu95b]). *The irreducible unipotent representations of $G^\vee(F)$ are in bijection with G-conjugacy classes of triples (s, n, χ) with s, n q-commuting as in Theorem 1.3 and χ an* arbitrary *G-equivariant local system on the orbit of (s, n).*

It would be extremely interesting to understand Theorem 1.23 using trace decategorification of Bezrukavnikov's Theorem 1.16. In particular we expect the full category of unipotent representations to be embedded in $\mathrm{QC}^!(\mathbb{L}_q^u)$ as well as its cyclic deformation $\mathrm{QC}^!(\mathbb{L}_q^u)^{S^1}$.

1.7. **Assumptions and notation.** We work throughout over a field k of characteristic zero. We will sometimes work in the specific case of $k = \overline{\mathbb{Q}}_\ell$ (e.g. in Section 2.2), and our main results require in addition that the field is algebraically closed. This requirement that k is algebraically closed is also used in Section 5 in order to apply equivariant localization. All functors and categories are dg derived unless noted otherwise. All (co)chain complexes are cohomologically indexed, even if referred to as a chain complex. We abusively use HH to denote the Hochschild chain complex rather than its homology groups, and use $H^\bullet(HH)$ to denote the latter (and similarly for its cyclic variants HC, HP).

1.7.1. *Categories.* Let A be a Noetherian dg algebra. We let A-mod denote the dg derived category of A-modules, A-perf denote the full subcategory of perfect complexes, and A-coh denote the subcategory of coherent objects, i.e. cohomologically bounded complexes with coherent cohomology over $\pi_0(A) = H^0(A)$. Let $\mathbf{C}$ denote a symmetric monoidal dg category, and $A \in \mathrm{Alg}(\mathbf{C})$ an algebra object. We denote by $A\text{-mod}_{\mathbf{C}}$ the category of A-module objects in $\mathbf{C}$. We denote the compact objects in a stable ∞-category $\mathbf{C}$ by $\mathbf{C}^\omega$, i.e. the objects $X \in \mathbf{C}$ for which $\mathrm{Hom}_{\mathbf{C}}(X, -)$ commutes with all infinite direct sums (i.e. at least the countable cardinal ω).

Let $\mathbf{C}$ be a stable k-linear ∞-category (or a k-linear triangulated category or a pretriangulated dg category). These come in two primary flavors, "big" and "small": $\mathbf{dgCat}_k$ is the ∞-category of presentable stable k-linear ∞-categories (with colimit-preserving functors), and $\mathbf{dgcat}_k$ is the ∞-category of small idempotent-complete stable k-linear ∞-categories (with exact functors). Both $\mathbf{dgCat}_k$ and $\mathbf{dgcat}_k$ are symmetric monoidal ∞-categories under the Lurie tensor product, with units $\mathbf{Vect}_k = k\text{-mod}$ and $\mathbf{Perf}_k = k\text{-perf} = k\text{-coh}$ the dg categories of chain complexes of k-vector spaces and perfect chain complexes, respectively. We have a symmetric monoidal ind-completion functor:

$$\mathrm{Ind} : \mathbf{dgcat}_k \to \mathbf{dgCat}_k.$$

It defines an equivalence between $\mathbf{dgcat}_k$ and the subcategory of $\mathbf{dgCat}_k$ defined by *compactly generated* categories and compact functors (functors preserving compact objects, or equivalently, possessing colimit preserving right adjoints).

Assume that $\mathbf{C}$ is either small or that it is compactly generated, and let $X \in \mathbf{C}$ be an object, which we require to be compact in the latter case. We denote by $\langle X \rangle$ the subcategory (classicaly or weakly) generated by X.

1.7.2. *Algebraic geometry.* We work in the setting of derived algebraic geometry over a field k of characteristic zero, in the setting presented in [GR17]. Namely, this is a version of algebraic geometry in which functors of (discrete) categories from rings to sets are replaced by *prestacks*, functors of (∞-)categories from connective commutative dg k-algebras to simplicial sets. Examples of prestacks are given by both classical schemes and stacks and topological spaces (or rather the corresponding simplicial sets of singuar chains) such as S^1, considered as constant functors.

We will only be concerned with *QCA (derived) stacks* as in [DG13], i.e., quasi-compact stacks of finite presentation with affine[5] finitely-presented diagonal (in fact only with quotients of schemes by affine group-schemes), and use the term *stack* to refer to such an object. A stack X carries a symmetric monoidal ∞-category (i.e., a commutative algebra object in $\mathbf{dgCat}_k$) $\mathrm{QC}(X)$ of quasicoherent sheaves, defined by right Kan extension from the case of representable functors $X = \mathrm{Spec}(R)$ which are assigned $\mathrm{QC}(\mathrm{Spec}\, R) = R\text{-mod}$. For all stacks we will encounter (and more generally for *perfect* stacks in the sense of [BFN10]), we have $\mathrm{QC}(X) \simeq \mathrm{Ind}(\mathrm{Perf}(X))$, i.e., quasicoherent sheaves are compactly generated and the compact objects are perfect complexes ($\mathrm{Perf}(X) \in \mathbf{dgCat}_k$ forms a small symmetric monoidal dg category).

We can also consider the category $\mathrm{QC}^!(X) = \mathrm{Ind}(\mathrm{Coh}(X)) \in \mathbf{dgCat}_k$ of *ind-coherent sheaves*, whose theory is developed in detail in the book [GR17] (see also the earlier [Ga13]). The category $\mathrm{QC}^!(X)$ (under our assumption that X is QCA) is compactly generated by $\mathrm{Coh}(X)$, the objects which are coherent after smooth pullback to a scheme (see Theorem 3.3.5 of [DG13]). For smooth X, the notions of coherent and perfect, hence ind-coherent and quasicoherent, sheaves are equivalent.

A crucial formalism developed in detail in [GR17] is the functoriality of $\mathrm{QC}^!$. Namely for a map $p : X \to Y$ of stacks, we have colimit-preserving functors of pushforward $p_* : \mathrm{QC}^!(X) \to \mathrm{QC}^!(Y)$ and exceptional pullback $p^! : \mathrm{QC}^!(Y) \to \mathrm{QC}^!(X)$, which form an adjoint pair $(p_*, p^!)$ for p proper. These functors satisfy a strong form of base change, which makes $\mathrm{QC}^!$ a functor

[5]The notion of a QCA stack in [DG13] is slightly more general; only automorphism groups at geometric points are required to be affine, and they are not required to be of finite presentation.

– in fact a symmetric monoidal functor[6] – out of the category of correspondences of stacks (the strongest form of this result is [GR17, Theorem III.3.5.4.3, III.3.6.3]).

See Definition 2.3.1 of [Ch20a] for a definition of the derived loop space $\mathcal{L}(-)$. For a stack X with a self-map f, we define X^f to be the derived fixed points of f, i.e. the derived fiber product

$$\begin{array}{ccc} X^f & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (f,\mathrm{id}_X)} \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

When $f = \mathrm{id}_X$, we have $X^f = \mathcal{L}X$. Given a group action G on a scheme X, and $f : X \to X$ commuting with the G-action, we have via Proposition 2.1.8 of [Ch20a]:

$$\begin{array}{ccc} (X/G)^f & \longrightarrow & (X \times G)/G \\ \downarrow & & \downarrow{\scriptstyle (f\circ\alpha,\mathrm{id}_X)} \\ X/G & \xrightarrow{\Delta} & (X \times X)/G \end{array}$$

where α is the action map.

When f is multiplication by $g \in G$, we sometimes write $X^g = \mathcal{L}_g(X) \simeq \mathcal{L}(X/G) \times_{\mathcal{L}(BG)} \{g\}$, as in Definition 4.1. Depending on whether we view the group G as an input to the construction or not, we adopt slightly different notation, e.g. the specialized loop space in Definition 5.1 is denoted $\mathcal{L}'_g(X/G)$, and we have $\mathcal{L}_g(X) = \mathcal{L}'_g(X/G)$.

1.7.3. *Representation theory.* In Sections 1-4 of the paper, unless otherwise noted, G denotes a reductive group over a field k of characteristic 0 with Borel B and torus $T \subset B$ with universal Cartan H and (finite) universal Weyl group W_f. The extended affine Weyl group is denoted $W_a := X^\bullet(T) \rtimes W_f$. We denote by $\mathrm{Rep}(G) = \mathrm{QC}(BG)$ the derived category of rational representations of G.

Morally, we view G as a group on the spectral side of Langlands duality. On the automorphic side, one is interested in representations of $G^\vee(F)$, where we let F denote a non-archimedian local field with ring of integers O. We will sometimes denote $G^\vee(F)$ by $\mathbf{G}$, with corresponding Iwahori $\mathbf{I}$ (and pro-unipotent radical $\mathbf{I}^0$), defined by the fixed Borel subgroup $B^\vee \subset G^\vee$ and maximal hyperspecial $G^\vee(O) \subset G^\vee(F)$. *In Section 6.1, we will reverse this convention for ease of reading*, and G will denote a split reductive group over a the non-archimedian local field F.

We will often be interested in equivariance with respect to the trivial extension of G, which we denote[7] $\widetilde{G} = G \times \mathbb{G}_m$. Likewise, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$, et cetera.

Let $\mathcal{B} = G/B$ denote the flag variety, $\mathcal{N}_G$ denote the nilpotent cone, and $\widehat{\mathcal{N}}_G$ its formal neighborhood inside $\mathfrak{g}$ *formal neighborhood* of the nilpotent cone of $\mathfrak{g}$. We let $\widetilde{\mathcal{N}}_G$ denote the (reduced) Springer resolution, and denote by $\mu : \widetilde{\mathcal{N}}_G = T^*(G/B) \to \mathcal{N}_G \hookrightarrow \widehat{\mathcal{N}}_G$ the composition of the Springer resolution with the inclusion, and $\widetilde{\mathfrak{g}}$ the Grothendieck-Springer resolution, which is $\widetilde{G}$-equivariant. Sometimes, we take the codomain of μ to be all of $\mathfrak{g}$. Let $\mathcal{Z}_G = \widetilde{\mathcal{N}}_G \times_{\mathfrak{g}} \widetilde{\mathcal{N}}_G$ denote the *derived Steinberg scheme*, $\mathcal{Z}'_G = \widetilde{\mathcal{N}}_G \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ denote the *non-reduced Steinberg scheme*, and $\mathcal{S}^\wedge_G = (\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}})^\wedge$ denote the *formal Steinberg scheme* via completing along the nilpotent

[6] In general $\mathrm{QC}^!$ is only right-lax symmetric monoidal but thanks to [DG13] it is strict on QCA stacks. Also the full correspondence formalism in [GR17] only includes pushforward for [inf,ind-]schematic maps.

[7] We explain this choice of notation. In the usual convention (opposite to ours), G denotes a group on the "automorphic" side of Langlands and ${}^LG$ is used to denote its Langlands dual on the "spectral" side. It was proposed in [BG14] [Ber20] to replace G with a (possibly nontrivial) central extension of G by $\mathbb{G}_m$, denoted $\widetilde{G}$, whose Langlands dual would be denoted ${}^CG$. When G is adjoint (therefore ${}^LG$ simply connected), the center is trivial and therefore $\widetilde{G} = G \times \mathbb{G}_m$ is a trivial extension, and ${}^CG = {}^LG \times \mathbb{G}_m$. Note that in our work is mostly on the spectral side so we depart from this convention in using G to denote a group on the spectral side rather than ${}^LG$ for convenience. We note there is an inherent asymmetry since taking Langlands duals flips the ordering in the short exact sequence $1 \to \mathbb{G}_m \to G \to \widetilde{G} \to 1$.

elements. We denote by $\pi_0(\mathcal{Z}_G)$ the *classical Steinberg variety*, which coincides with $(\mathcal{Z}'_G)^{red} = (\mathcal{Z}^\wedge_G)^{red}$. We will drop the subscript if there is no ambiguity regarding the group G in discussion.

We denote the *affine Hecke algebra* by $\mathcal{H}_G$; we use a Coxeter presentation, i.e. a definition on the spectral side, which can be found e.g. in Definition 7.1.9 of [CG97]. It is a $k[q,q^{-1}]$-algebra whose specializations at prime powers $q = p^r$ are isomorphic to the *Iwahori-Hecke algebras* $\mathcal{H}_{q,G} \simeq \mathcal{H}(G^\vee(F), I) := \mathrm{Fun}_c(I\backslash G^\vee(F)/I)$ of compactly supported Iwahori-biequivariant functions on a loop group (or p-adic group). More generally, for a locally compact totally disconnected group $\mathbf{G}$, a compact open subgroup $\mathbf{K} \subset \mathbf{G}$ and a representation τ of $\mathbf{K}$, we denote its Hecke algebra by $\mathcal{H}(\mathbf{G}, \mathbf{K}, \tau) := \mathrm{End}_{\mathbf{G}}(\mathrm{cInd}_{\mathbf{K}}^{\mathbf{G}} \tau)$.

The *mixed affine Hecke category* is defined by $\mathbf{H}_G^{\mathrm{m}} := \mathrm{Coh}(\mathcal{Z}/\widetilde{G})$, while the *affine Hecke category* is defined to be $\mathbf{H}_G := \mathrm{Coh}(\mathcal{Z}/G)$. Note that we define these categories directly on the spectral side of Langlands duality, while they are usually defined on the automorphic side. That is, we implicitly pass through (proven and conjectural versions of) Bezrukavnikov's theorem (Theorem 1.16).

We define the *coherent Springer sheaf* and the *coherent q-Springer sheaf* by:

$$\mathcal{S}_G := \mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})} \simeq \mathcal{L}\mu_* \omega_{\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})} \in \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})),$$

$$\mathcal{S}_{q,G} := (\mu^q)_* \mathcal{O}_{(\widetilde{\mathcal{N}}/G)^q} \simeq (\mu^q)_* \omega_{(\widetilde{\mathcal{N}}/G)^q} \in \mathrm{Coh}((\widehat{\mathcal{N}}/G)^q).$$

The coherent q-Springer sheaf is a coherent sheaf on the *stack of unipotent Langlands parameters*:

$$\mathbb{L}^u_{q,G} := (\widehat{\mathcal{N}}_G/G)^q = \mathcal{L}_q(\widehat{\mathcal{N}}_G/G).$$

Note that this definition is functorial and makes sense for *any* affine algebraic group G (still completing along nilpotents), and thus the coherent q-Springer sheaf may be realize by applying parabolic induction

$$\mathbb{L}^u_{q,H} \xleftarrow{\ \nu\ } \mathbb{L}^u_{q,B} \xrightarrow{\ \mu\ } \mathbb{L}^u_{q,G}$$

to the structure sheaf of $\mathbb{L}^u_{q,H}$, i.e. $\mathcal{S}_{q,G} = \mu_* \nu^* \mathcal{O}_{\mathbb{L}^u_{q,H}}$.

By Proposition 4.3, if G is reductive then $\mathbb{L}^u_{q,G}$ is a classical stack (i.e. no derived and no infinitesimal structure) when q is not a root of unity. Note that other authors [BG19, BP19, H20, DHKM20, Zh20] have defined a moduli stack of Langlands parameters $X_{F,G}$ for a given local field F and a reductive group $G^\vee$ with coefficients in F. Our stack embeds as a connected component of tame Langlands parameters.

We fix once and for all a coordinate $z \in \mathbb{G}_m$. For any geometric vector space or bundle V (e.g. the Springer resolution), by convention the coordinate will act on geometric fibers by weight -1, i.e. $z \cdot x = z^{-1}x$ for $x \in V$, and therefore on functions by weight 1 (i.e. $z \cdot f(-) = zf(-)$ for $f \in V^*$). This negative sign convention is forced by the requirement that the $z = q$ fixed points of $\mathcal{N}/\widetilde{G}$ correspond to unipotent Langlands parameters (s, N) for a local field with residue $\mathbb{F}_q$, i.e. $(s,N,q) \cdot N = sNs^{-1}q^{-1} = N$. We note that, given an identification $\mathcal{H} \simeq \mathrm{tr}(\mathbf{H}^{\mathrm{m}}, \mathrm{id}_{\mathbf{H}^{\mathrm{m}}})$ as in Theorem 2.29, this implies an identification $\mathcal{H}_q \simeq \mathrm{tr}(\mathbf{H}, q_*) = \mathrm{tr}(\mathbf{H}, \mathbf{q}^*) \simeq \mathrm{tr}(\mathbf{H}, \mathrm{Fr}^*)$, where q denotes the action of $q \in \mathbb{G}_m$, while $\mathbf{q}$ denotes the multiplication by q map corresponding to the geometric Frobenius Fr under Bezrukavnikov's equivalence in [Bez16]. This convention is compatible with [KL87, CG97, AB09, Bez16].

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2. Hochschild homology of the affine Hecke category

In this section we calculate the Hochschild and cyclic homology of the affine Hecke category. In particular in Corollary 2.26 we prove that the Chern character from K-theory factors through an isomorphism between K_0 and Hochschild homology. For this we use Bezrukavnikov's Langlands duality for the affine Hecke category to construct a semiorthogonal decomposition on the equivariant derived category of the Steinberg stack with simple components, from which the calculation of localizing invariants is immediate.

The results of Subsection 2.1.1 apply for any field k of characteristic zero. The results of Subsections 2.1.4 and 2.2 specifically apply to the case $k = \overline{\mathbb{Q}}_\ell$. In Corollary 2.26 we will pass to Hochschild homology, where statements will hold for any field of characteristic zero. Finally, in Subsection 2.4 we will use a theorem of Ginzburg-Kazhdan-Lusztig which further requires k to be algebraically closed.

2.1. **Background.** We first review some standard notions regarding Hochschild homology and equivariant ℓ-adic sheaves that we need for our arguments.

2.1.1. *Trace decategorifications and Hochschild homology.* An extended discussion of the notions of this subsection can be found in [GKRV20], [BN19] and [Ch20a]. We recall the notion of a *dualizable object* X of a symmetric monoidal ∞-category $\mathbf{C}_\otimes$ with monoidal unit $1_\otimes$.

Definition 2.1. The object X is *dualizable* if there exists an object $X^\vee$ and coevaluation and evaluation morphisms

$$\eta_X : 1_\otimes \to X \otimes X^\vee, \qquad \epsilon_X : X^\vee \otimes X \to 1_\otimes$$

satisfying a standard identity. Dualizability is a *property* rather than an additional structure on X (see Remark 2.7). The *trace* of an endomorphism $f \in \mathrm{End}_{\mathbf{C}}(X)$ of a dualizable object is defined by

$$\mathrm{tr}(X, f) = \epsilon_X \circ (f \otimes 1) \circ \eta_X \in \mathrm{End}_{\mathbf{C}_\otimes}(1_\otimes).$$

We are interested in the case when X is an algebra object in the symmetric monoidal ∞-category $\mathbf{C}_\otimes$, and the resulting algebra structure on traces. To formulate this, we note that traces are canonically *symmetric* monoidal with respect to the monoidal structure in $\mathbf{C}_\otimes$ and composition in $\mathrm{End}_{\mathbf{C}_\otimes}(1_\otimes)$. In addition, we require a natural functoriality enjoyed by the abstract construction of traces in the higher-categorical setting, see [TV15, HSS17, GKRV20] (see also [BN19] for an informal discussion). Namely the trace of an object is covariantly functorial under right-dualizable morphisms.

Definition 2.2. A *morphism of pairs* $(F, \psi) : (X, f) \to (Y, g)$ is a right-dualizable morphism $F : X \to Y$ (i.e. has a right adjoint G) along with a commuting structure[8] $\psi : F \circ f \to g \circ F$. Given a morphism of pairs (F, ψ), it defines a map $\mathrm{tr}(F, \psi)$ on traces via the composition

$$\mathrm{tr}(X, f) \xrightarrow{\mathrm{tr}(X, \eta_F \mathrm{id}_f)} \mathrm{tr}(X, GFf) \xrightarrow{\mathrm{tr}(X, \mathrm{id}_G \psi)} \mathrm{tr}(X, GgF) \xrightarrow{\simeq} \mathrm{tr}(Y, gFG) \xrightarrow{\mathrm{tr}(Y, \mathrm{id}_g \epsilon_F)} \mathrm{tr}(Y, g)$$

where η_F and ϵ_F are the unit and counit of the adjunction (F, G), and the equivalence in the middle is via cyclic symmetry of traces (see also Definition 3.24 of [BN19]).

Thus, the trace construction enhances to a symmetric monoidal functor from the ∞-category of endomorphisms of dualizable objects in $\mathbf{C}_\otimes$ to endomorphisms of the unit $\mathbf{End}_{\mathbf{C}_\otimes}(1_\otimes)$, see [HSS17, 2], [TV15, 2.5] and [GKRV20, 3] for details. In particular, if X is an algebra object in $\mathbf{C}_\otimes$ and f is an algebra endofunctor, then $\mathrm{tr}(X, f)$ is an algebra object in $\mathbf{End}_{\mathbf{C}_\otimes}(1_\otimes)$.

In this paper, we consider the case $\mathbf{C}_\otimes = \mathbf{dgCat}_k$, the ∞-category of cocomplete k-linear dg categories, with morphisms given by left adjoint (i.e. cocomplete) functors, with monoidal product the Lurie tensor product. We now specialize to this case.

[8]Note we do not require this to be an equivalence, though it always will be in this paper.

Example 2.3. Any compactly generated dg category $\mathbf{C} = \mathrm{Ind}(\mathbf{C}^\omega) \in \mathbf{dgCat}_k$ is dualizable, with dual given by taking the ind-completion of the opposite of compact objects $\mathbf{C}^\vee = \mathrm{Ind}(\mathbf{C}^{\omega,op})$. Thus we may speak of traces of its endofunctors, which are endomorphisms of the unit, i.e. *chain complexes*

$$\mathrm{End}_{\mathbf{dgCat}_k}(\mathbf{Vect}_k) \simeq \mathbf{Vect}_k.$$

Furthermore, note that a morphism of pairs of compactly generated dg categories is a functor that has a *continuous* right adjoint, or equivalently for compactly generated categories, a functor which preserves compact objects.

Definition 2.4. The *Hochschild homology* of a dualizable (for instance, compactly generated) k-linear dg category $\mathbf{C} \in \mathbf{dgCat}_k$ is the trace of the identity functor

$$HH(\mathbf{C}/k) := \mathrm{tr}(\mathbf{C}, \mathrm{id}_{\mathbf{C}}) \in \mathbf{Vect}_k.$$

We often omit k from the notation above. More generally, the Hochschild homology of $\mathbf{C}$ with coefficients in a continuous endofunctor F is $HH(\mathbf{C}, F) = \mathrm{tr}(\mathbf{C}, F) \in \mathbf{Vect}_k$.

Remark 2.5 (Large vs. small categories). The above definition is formulated in terms of large categories, but can be defined for small categories by taking ind-completions. Since every compactly generated category is dualizable but not conversely, the notion of Hochschild homology for large categories is general. We will often not distinguish between the two.

We have a notion of characters of compact objects in categories, defined via functoriality of traces.

Definition 2.6. Let $\mathbf{C} \in \mathbf{dgCat}_k$ be dualizable, and $F : \mathbf{C} \to \mathbf{C}$ an endofunctor. Any object $c \in \mathrm{Ob}(\mathbf{C})$ defines a functor $\alpha_c : \mathbf{Vect}_k \to \mathbf{C}$ by action on the object c, and a map $\psi : c \to F(c)$ defines a commuting structure. If $\mathbf{C}$ is compactly generated and c is a compact object, then α_c is right dualizable. Thus, by functoriality of traces, we have a map

$$\mathrm{tr}(\alpha_c, \psi) : HH(\mathbf{Vect}_k) = k \longrightarrow HH(\mathbf{C}, F)$$

and we define the *character*[9] $[c] = \mathrm{tr}(\alpha_c, \psi)(1)$ of c to be the image of $1 \in k$ under this map.

Remark 2.7. We highlight a few properties of Hochschild homology which we use in our arguments:

(1) Hochschild homology is computed via a choice of dualizing structure, and the space of such choices is contractible by Proposition 4.6.1.10 in [Lur18]. In particular, for any two choices there is a canonical quasi-isomorphism of Hochschild chain complexes. Our arguments will play off two dualizing structures on the category $\mathrm{Coh}(Z)$ where Z is a reasonable stack. One is categorical or algebraic (and makes sense for any dg category), while the other uses the geometry of Z.
(2) Hochschild homology is additive and exact in the Morita model structure (in the language of [BGT13], it is a localizing invariant) by Theorem 5.2 of [Ke06], and in particular in the explicit algebraic model of Definition 2.11 one can replace $\mathrm{Ob}(\mathbf{C})$ with any set of generating objects.
(3) Hochschild homology takes (possibly bi-infinite) F-stable semiorthogonal decompositions (see Section 2.3) of $\mathbf{C}$ to direct sums. This is a consequence of (2) since semiorthogonal decompositions give rise to split exact sequences of categories.
(4) Let A be a dg algebra, M an dg A-bimodule, and define $F_M(-) = M \otimes_A -$. Then, $HH(A\text{-perf}, F_M) = A \otimes^L_{A \otimes_k A^{op}} M$. This derived tensor product can be computed via a bar resolution or otherwise.
(5) The Hochschild homology receives an S^1-equivariant Chern character map from the connective K-theory spectrum (see Definition 2.14).

Example 2.8. We give a toy example to illustrate a canonical identification of two calculations of Hochschild homology. Let $\mathbf{C} = \mathrm{Coh}(\mathbb{P}^1)$. It is well-known that $\mathcal{O}(-1) \oplus \mathcal{O}$ generates

[9] This may also sometimes be referred to as a trace, but we call it a character to avoid overloading the term.

the category, with endomorphism algebra represented by the Kronecker quiver. Since the Kronecker quiver has no cycles, we have an identification $HH(\mathrm{Coh}(\mathbb{P}^1)) \simeq k^2$. The character map is the (twisted) algebraic Euler characteristic: $[\mathcal{L}] = (\chi(\mathbb{P}^1, \mathcal{L}(1)), \chi(\mathbb{P}^1, \mathcal{L}))$. On the other hand, Hochschild-Kostant-Rosenberg produces an identification $HH(\mathrm{Coh}(\mathbb{P}^1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \oplus H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \simeq k^2$. The character map is the Chern character, i.e. $[\mathcal{O}(n)] = (1, n)$; compatibility of traces forces a particular identification $H^{0,0}(\mathbb{P}^1) \oplus H^{1,1}(\mathbb{P}^1) \simeq \mathrm{End}(\mathcal{O}(-1)) \oplus \mathrm{End}(\mathcal{O})$.

2.1.2. *De-equivariantization and the Block-Getzler complex.* Hochschild homology has an algebraic realization via the cyclic bar complex. For dg categories with a $\mathrm{Rep}(G)$-action, there is an explicit algebraic model for the Hochschild homology due to Block and Getzler [BG94] obtained by passing to the de-equivariantization. We fix the following set-up for the rest of the subsection.

Definition 2.9. Let G be a reductive group over a field k of characteristic zero, and $\mathbf{C}$ a compactly generated cocomplete dg category with a $\mathrm{Rep}(G)$-action. We define the *de-equivariantization* to be $\mathbf{C}^{\mathrm{deq}} := \mathbf{C} \otimes_{\mathrm{Rep}(G)} \mathbf{Vect}_k$, where $\mathrm{Rep}(G)$ acts on $\mathbf{Vect}_k$ trivially. There is a canonical functor $\mathbf{C} \to \mathbf{C}^{\mathrm{deq}}$ and we denote its image by $\mathbf{C}^{\mathrm{deq}}_0$; this category is naturally enriched in $\mathrm{Rep}(G)$, which we establish below.

Lemma 2.10. *The de-equivariantization functor $\mathbf{C} \to \mathbf{C}^{\mathrm{deq}}$ preserves compact objects, and $\mathbf{C}^{\mathrm{deq}}$ is compactly generated, and generated under colimits by $\mathbf{C}^{\mathrm{deq}}_0$. Furthermore, $\mathbf{C}^{\mathrm{deq}}_0$ is naturally enriched in* $\mathrm{Rep}(G)$*, and we have*

$$\mathrm{Hom}_{\mathbf{C}}(X, Y) = \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}_0}(X, Y)^G.$$

In particular, if $E \in \mathbf{C}$ is a compact generator for $\mathbf{C}^{\mathrm{deq}}$, then E is a compact $\mathrm{Rep}(G)$*-generator of $\mathbf{C}$, i.e. $\mathbf{C}$ is equivalent to modules in* $\mathrm{Rep}(G)$ *for the internal endomorphism algebra*

$$\mathcal{A} = \underline{\mathrm{End}}(E)^{op} \in \mathrm{Alg}(\mathrm{Rep}(G)).$$

Proof. The lemma is an application of the rigidity of $\mathrm{Rep}(G)$ and the Barr-Beck-Lurie monadicity theorem. Explicitly, recall (e.g. in Chapter 1, Definition 9.1.2 of [GR17]) that by Corollary 9.3.3 of *op. cit.*, rigidity implies that the deequivariantization functor $F : \mathbf{C} \to \mathbf{C}^{\mathrm{deq}}$ has a continuous right adjoint $G : \mathbf{C}^{\mathrm{deq}} \to \mathbf{C}$ given by tensoring with the regular representation $\mathcal{O}(G)$, and hence preserves compact objects. Furthermore, since $\mathbf{C}^{\mathrm{deq}} = \mathbf{C} \otimes_{\mathrm{Rep}(G)} \mathbf{Vect}_k$, it is the colimit of the usual cyclic bar complex, thus it is generated under colimits by the image of F. This implies compact generation as well, since F preserves compact objects.

The internal Hom may be defined in the following way. For any $X \in \mathbf{C}$, the functor $\mathrm{act}_X : \mathrm{Rep}(G) \to \mathbf{C}$ given by action on X has a $\mathrm{Rep}(G)$-linear continuous right adjoint $\Psi_X(-) = \underline{\mathrm{Hom}}_{\mathrm{Rep}(G)}(X, -)$. We define $\underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(F(X), F(Y)) = \Psi_X(Y)$. More explicitly, we have

$$\underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(F(X), F(Y)) = \mathrm{Hom}_{\mathbf{C}}(X, Y \otimes \mathcal{O}(G)) = \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Hom}_{\mathbf{C}}(X, Y \otimes V) \otimes V^*.$$

Note that the trivial isotypic component is given by the summand $\mathrm{Hom}_{\mathbf{C}}(X, Y)$.

For the second claim, note that Ψ_E takes E to the internal endomorphism algebra, which represents the corresponding monad $\Psi_E \circ \mathrm{act}_E$ on $\mathrm{Rep}(G)$. Since $F(E)$ is a compact generator for $\mathbf{C}^{\mathrm{deq}}$, the functor

$$\Psi_E^{\mathrm{deq}}(-) = \mathrm{Hom}_{\mathbf{C}^{\mathrm{deq}}}(F(E), -) : \mathbf{C}^{\mathrm{deq}} \to A\text{-mod}$$

is an equivalence, giving us the commuting square of left adjoint functors:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\Psi_E} & A\text{-mod}_{\mathrm{Rep}(G)} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F'} \\ \mathbf{C}^{\mathrm{deq}} & \underset{\simeq}{\xrightarrow{\Psi_E^{\mathrm{deq}}}} & A\text{-mod} \end{array}$$

where F' is the forgetful functor. Applying Barr-Beck to the functors F, F' and their right adjoints, the comonads in $\mathbf{C}^{\mathrm{deq}}$ and A-mod are identified under the equivalence Ψ_E^{deq} and therefore $\Psi_E : \mathbf{C} \to A\text{-mod}_{\mathrm{Rep}(G)}$ is an equivalence. □

Block and Getzler defined a chain complex in [BG94] associated to any dg category $\mathbf{C}'$ enriched in $\mathrm{Rep}(G)$ (morally, $\mathbf{C}' = \mathbf{C}_0^{\mathrm{deq},\omega}$ is the image of the compact objects of a $\mathrm{Rep}(G)$-category $\mathbf{C}$ in its de-equivariantization). We review this notion here.

Definition 2.11. Let G be a reductive group, and let $\mathbf{C}'$ be a small dg category enriched in $\mathrm{Rep}(G)$ equipped with an dg-endofunctor F. For any $V \in \mathrm{Rep}(G)$, we abusively denote by $\gamma : V \to V \otimes k[G]$ the coaction map. The *Block-Getzler complex* (over k) $BG^\bullet(\mathbf{C}', F; G)$ is defined[10] to be the sum totalization of the simplicial object in chain complexes with

$$BG^{-n}(\mathbf{C}', F; G) = \bigoplus_{X_0,\dots,X_n \in \mathrm{Ob}(\mathbf{C}')} \Big(\underline{\mathrm{Hom}}^\bullet(X_0, X_1) \otimes \cdots \otimes \underline{\mathrm{Hom}}^\bullet(X_n, F(X_0)) \otimes k[G]\Big)^G$$

where the face maps $d_i : BG^{-n} \to BG^{-(n-1)}$ (for $i = 0, \dots, n$) compose morphisms, i.e.

$$d_i(f_0 \otimes \cdots \otimes f_n \otimes g) = f_0 \otimes \cdots f_i f_{i+1} \otimes \cdots \otimes f_n \otimes g, \qquad i = 0, \dots, n-1$$
$$d_n(f_0 \otimes \cdots \otimes f_n \otimes g) = \gamma(f_n) F(f_0) \otimes F(f_1) \otimes \cdots \otimes F(f_{n-1}) \otimes g.$$

We define the *enhanced Block-Getzler complex* to $\underline{BG}^\bullet(\mathbf{C}', F; G)$ to be the complex above, but without taking G-invariants.[11] Finally, for a specified $g \in G(k)$ we define

$$BG_g^\bullet(\mathbf{C}', F; G) = \underline{BG}^\bullet(\mathbf{C}', F; G) \otimes_{k[G]} k_g$$

where k_g is the skyscraper module at $g \in G$. Note that there is a canonial map

$$BG^\bullet(\mathbf{C}', F; G) \hookrightarrow \underline{BG}^\bullet(\mathbf{C}', F; G) \to BG_g^\bullet(\mathbf{C}', F; G).$$

When it is understood, we often omit G from the notation.

We are interested in comparing the Hochschild homology of $\mathbf{C}$ with the Hochschild homology of $\mathbf{C}^{\mathrm{deq}}$ twisted by the action of a particular $g \in G$. If $\mathbf{C}$ has a $\mathrm{Rep}(G)$-action, then any fixed $g \in G$ determines an endofunctor $g_* : \mathbf{C}_0^{\mathrm{deq}} \to \mathbf{C}_0^{\mathrm{deq}}$ and an equivalence $\psi : g_* \simeq \mathrm{id}_{\mathbf{C}_0^{\mathrm{deq}}}$.[12] Let F be a $\mathrm{Rep}(G)$-linear endofunctor; this provides a canonical identification $F_g := F \circ g_* \simeq g_* \circ F$. We have a natural map of pairs $(\mathbf{C}, F) \to (\mathbf{C}^{\mathrm{deq}}, F_g)$, with commuting structure given by ψ above, and we have the following compatibility.

Proposition 2.12. *Let G be a reductive group (over k) and let $\mathbf{C}$ be a dg category with a $\mathrm{Rep}(G)$-action. Then, the map $BG^\bullet(\mathbf{C}_0^{\mathrm{deq},\omega}, F^{\mathrm{deq}}) \to BG_g^\bullet(\mathbf{C}_0^{\mathrm{deq},\omega}, F_g^{\mathrm{deq}})$ computes the map in Hochschild homology $HH(\mathbf{C}, F) \to HH(\mathbf{C}^{\mathrm{deq}}, F_g^{\mathrm{deq}})$.*

Proof. The first claim is similar to Proposition 2.3.6 of [Ch20a]. Let S be a set of compact objects of $\mathbf{C}$ that generate under $\mathrm{Rep}(G)$. By Lemma 2.10, their images under the de-equivariantization functor S^{deq} generate $\mathbf{C}^{\mathrm{deq}}$. We can use the cyclic bar complex on the generators S to compute Hochschild homology, whose nth term is

$$\bigoplus_{X_i \in S} \bigoplus_{V_i \in \mathrm{Irr}(G)} \mathrm{Hom}_{\mathbf{C}}(X_0 \otimes V_0, X_1 \otimes V_1) \otimes_k \cdots \otimes_k \mathrm{Hom}_{\mathbf{C}}(X_n \otimes V_n, F(X_0) \otimes V_0)$$
$$\simeq \bigoplus_{X_i \in S} \bigoplus_{V_i \in \mathrm{Irr}(G)} \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_0 \otimes V_0, X_1 \otimes V_1)^G \otimes_k \cdots \otimes_k \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_n \otimes V_n, F(X_0) \otimes V_0)^G$$
$$\simeq \bigoplus_{X_i \in S} \bigoplus_{V_i \in \mathrm{Irr}(G)} (V_0^* \otimes \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_0, X_1) \otimes V_1)^G \otimes_k \cdots \otimes_k (V_n^* \otimes \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_n, F(X_0)) \otimes V_0)^G.$$

By Proposition 2.3.2 of *op. cit.* we have

$$\simeq \bigoplus_{X_i \in S^{\mathrm{deq}}} \bigoplus_{V_0 \in \mathrm{Irr}(G)} (V_0^* \otimes \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_0, X_1) \otimes_k \cdots \otimes_k \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_n, F(X_0)) \otimes V_0)^G.$$

[10]Note that we use cohomological gradings; thus the index has a negative sign.

[11]Note that if F is the identity functor, then the the Block-Getzler simplicial chain complex is a cyclic object, and thus the associated chain complex has the natural structure of a mixed complex. However, the enhanced Block-Getzler complex is not cyclic, since the "rotation" twists by the coaction γ which can be nontrivial on nontrivial G-isotypic components. One can view this object as an S^1-equivariant object in $\mathrm{QC}(G/G)$.

[12]This arises via de-equivariantization: the category $\mathbf{C}^{\mathrm{deq}}$ is a $\mathbf{Vect}_k^{\mathrm{deq}} = \mathrm{QC}(G)$-module category, and the functor is given by action by the skyscraper sheaf at $g \in G$.

By Peter-Weyl, we have

$$\bigoplus_{X_i \in S^{\mathrm{deq}}} (\underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_0, X_1) \otimes_k \cdots \otimes_k \underline{\mathrm{Hom}}_{\mathbf{C}^{\mathrm{deq}}}(X_n, F(X_0)) \otimes k[G])^G.$$

We leave to the reader the verification that these identifications are compatible with the face maps. The second claim follows from the observation that $BG_g^\bullet(\mathbf{C}^{\mathrm{deq}}, F_g^{\mathrm{deq}})$ is just the cyclic bar complex, passing through the identification $g_* X_0 \simeq X_0$. Verification that the equivalence in *loc. cit.* is functorial for the above map is left to the reader. □

Example 2.13. Recall the standard examples:

$$\mathrm{Rep}(G)^{\mathrm{deq}} \simeq \mathbf{Vect}_k, \qquad \mathbf{Vect}_k^{\mathrm{deq}} \simeq \mathrm{Rep}(G).$$

The Block-Getzler complex for $\mathrm{Rep}(G)^{\mathrm{deq}} \simeq \mathbf{Vect}_k$ (where Homs are equipped with the trivial G-action) is simply $BG^\bullet(\mathbf{Vect}_k) = k[G]^G$, which by Peter-Weyl is equivalent to the cyclic bar complex for $\mathrm{Rep}(G)$, i.e. $\bigoplus_{V \in \mathrm{Irr}(G)} k$. On the other hand, one can check (e.g. via the argument in *loc. cit.*) that the Block-Getzler complex for $\mathbf{Vect}_k^{\mathrm{deq}} \simeq \mathrm{Rep}(G)$ is quasi-equivalent to k.

2.1.3. *Chern character from K-theory to Hochschild homology.* Finally, we will use the universal S^1-equivariant trace map from connective K-theory to Hochschild homology constructed in [BGT13].

Definition 2.14. For any small k-linear dg-category $\mathbf{C}$, the *connective K-theory spectrum* $K(\mathbf{C})$ is the connective K-theory of the corresponding Waldhausen category defined in Section 5.2 of [Ke06]. The universal *cyclic Chern character*[13] is the map

$$\mathrm{ch} : K(\mathbf{C}) \to HH(\mathbf{C}).$$

This assignment is functorial in $\mathbf{C}$.

Remark 2.15. We note two important properties of the Chern character that we use. Note that unlike in the definition of Hochschild homology, in this discussion we restrict ourselves to small categories $\mathbf{C}$ (i.e. the compact objects of a compactly generated cocompelte category).

- Via functoriality of the Chern character, for any object $X \in \mathrm{Ob}(\mathbf{C})$, the Chern character sends $[X] \in K_0(\mathbf{C}) \mapsto [X] \in HH_0(\mathbf{C})$, i.e. equivalence classes in the Grothendieck group to their characters in Hochschild homology in the sense of Definition 2.6.
- Using the lax monoidal structure of K-theory, we see that for a monoidal category $\mathbf{C}$ the Chern character defines a map of algebras (see also Theorem 1.10 of [BGT14]).

Often in applications to geometric representation theory, we are only interested in (or able to) compute the Grothendieck group K_0. In order to compare K_0 with Hochschild homology, we require certain vanishing conditions to hold. We say that $\mathbf{C}$ has a *0-truncated Chern character* if we have a factorization

$$\begin{array}{ccc} K(\mathbf{C}) & \xrightarrow{\mathrm{ch}} & HH(\mathbf{C}) \\ & \searrow \quad \nearrow & \\ & K_0(\mathbf{C}) & \end{array}$$

[13]We use this terminology to avoid overloading the word "trace."

2.1.4. *Equivariant ℓ-adic sheaves, weights, and Tate type.* In this subsection we review some standard notions concerning weights and the ℓ-adic cohomology of BG. In this section and the following one, we fix a prime power $q = p^r$ and a prime $\ell \neq p$, and will work with ℓ-adic sheaves $\mathcal{F}$ on $\overline{\mathbb{F}_q}$-schemes X. All schemes and sheaves on them that arise are defined over $\mathbb{F}_q$, i.e., X will come with a geometric Frobenius automorphism Fr and $\mathcal{F}$ with a Fr-equivariant (Weil) structure, which will be left implicit.

Fix a square root of q in $\overline{\mathbb{Q}}_\ell$, thereby defining a notion of half Tate twist (this choice can be avoided by judicious use of extended groups as in [BG14, Zh17, Ber20]). For $\mathcal{F} \in \mathrm{Sh}(X)$ where X is over $\mathbb{F}_q$, we will denote the Tate twist by $\mathcal{F}(n/2)$ for $n \in \mathbb{Z}$. For a scheme X over $\mathbb{F}_q$ with a group action G, we denote by $\mathrm{Sh}(X/G) = \mathrm{Sh}^G(X)$ the bounded derived category of finite G-equivariant $\overline{\mathbb{Q}}_\ell$-sheaves on X (see Section 1.3 of [BY13] and [BL94]). In this context, the cohomology of a sheaf $H^\bullet(X, -)$ will be understood to mean étale cohomology.

Following the Appendix of [Ga00], this notion can be extended to G-equivariant ind-schemes, where G is a pro-affine algebraic ind-group acting in a sufficiently finite way. We say a G-action on X is *nice* if the following two properties hold: (1) every closed subscheme $Z \subset X$ is contained in a closed G-stable subscheme $Z' \subset X$ such that the action of G on Z' factors through an quotient of G which is affine algebraic, and (2) G contains a pro-unipotent subgroup of finite codimension, i.e. if $G = \lim\limits_{n\to\infty} G_n$, then there is an n such that $\ker(G \to G_n)$ is a projective limit of unipotent affine algebraic groups. If G is a pro-affine group scheme acting nicely on X, and $X = \operatorname{colim}\limits_{i\to\infty} X_i$ with affine quotient G_i acting on X_i, then we define[14] $\mathrm{Sh}^G(X) = \operatorname{colim}\limits_{i\to\infty} \mathrm{Sh}^{G_i}(X_i)$.

Finally, we need a notion of Frobenius weights acting on a $\overline{\mathbb{Q}}_\ell$-vector space V, which for us will be étale cohomology groups. We will generally only be concerned with the weak notion of weights and will omit the adjective "weak" for brevity.

Definition 2.16. Let V be a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space equipped with an endomorphism F, and fix a prime power $q = p^r$. We say V is *strongly pure of weight* n if every eigenvalue of F is equal to $q^{n/2}$. We say V is *weakly pure of weight* n if every eigenvalue of F is equal to $\zeta q^{n/2}$ for varying roots of unity $\zeta \in \overline{\mathbb{Q}}_\ell$. If V is a (cohomologically) graded vector space with finite-dimensional homogeneous parts V^k, then we say V is *strongly (resp. weakly) pure of weight* n if V^k is strongly (resp. weakly) pure of weight $n + k$.

Finally we recall the ℓ-adic cohomology ring of BG, whose description we repeat for convenience following [Vi15] (in the Hodge-theory context).

Proposition 2.17. *Let G be a pro-affine group scheme with split reductive quotient over k. Then, $H^\bullet(BG, \overline{\mathbb{Q}}_\ell)$ is polynomial, generated in even degrees, and pure of weight 0. In particular, $H^{2k}(BG, \overline{\mathbb{Q}}_\ell)$ has weight $2k$.*

Proof. First, since G is pro-affine, there is a reductive (finite type) algebraic group G_0 such that the kernel $\ker(G \to G_0)$ is pro-unipotent. By Theorem 3.4.1(ii) in [BL94] we may assume that G is reductive (and finite type).

It is a standard calculation that $H^\bullet(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) = H^0(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) \oplus H^1(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ with H^0 of weight 0 and H^1 of weight 2. By Corollary 10.4 of [LO08], $H^\bullet(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell[u]$ where u has cohomological degree $|u| = 2$ and weight 2. In particular, by the Kunneth formula (Theorem 11.4 in *op. cit.*) we have that for a split torus T, $H^\bullet(BT; \overline{\mathbb{Q}}_\ell)$ is pure of weight 0 and polynomial in even degrees. Thus, the claim is true when $G = T$ is a torus. Now, assume T is a split torus inside a reductive group G, and B is a Borel subgroup with $T \subset B \subset G$. Applying Theorem 3.4.1(ii) of [BL94] again, we have $H^\bullet(BB; \overline{\mathbb{Q}}_\ell) \simeq H^\bullet(BT; \overline{\mathbb{Q}}_\ell)$. By Theorem 1.1 of [Vi16], $H^\bullet(BG; \overline{\mathbb{Q}}_\ell)$ is a polynomial subring of $H^\bullet(BB; \overline{\mathbb{Q}}_\ell) \simeq H^\bullet(BT; \overline{\mathbb{Q}}_\ell)$, completing the claim. □

[14]This definition is independent of the choice of presentation, since by [BL94] Theorem 3.4.1(ii) if $G_i \to G_j$ is a surjection with unipotent kernel, then $\mathrm{Sh}^{G_j}(Y) \to \mathrm{Sh}^{G_i}(Y)$ is an equivalence for any Y on which G_j acts. See also Section A.4 of [Ga00].

2.2. **Automorphic and spectral realizations of the affine Hecke category.** We follow the set-up of Bezrukavnikov in [Bez16], except that we view the group on the automorphic side as dual to a chosen group on the spectral side for ease of notation. Let G be a fixed reductive algebraic group over $\overline{\mathbb{Q}}_\ell$ on the spectral side of Langlands duality, and let $G^\vee$ be its dual group.

Choose a form of $G^\vee$ split over $\overline{\mathbb{F}_q}$. Let $F = \overline{\mathbb{F}_q}((t))$ and $O = \overline{\mathbb{F}_q}[[t]]$. We denote $\mathbf{G} := G^\vee(F)$ to be its dual group with coefficients in F, which we consider as an ind-group scheme over $\overline{\mathbb{F}_q}$, and its subgroup $\mathbf{G}_0 := G^\vee(O)$, a pro-affine group scheme over $\overline{\mathbb{F}_q}$. The *Iwahori subgroup* of $\mathbf{G}$ is $\mathbf{I} := \mathbf{G}_0 \times_{G^\vee(\overline{\mathbb{F}_q})} B^\vee(\overline{\mathbb{F}_q})$, which inherits its structure as a closed subgroup and is therefore also a pro-affine group. We let $\mathbf{I}^0 := \mathbf{G}_0 \times_{G^\vee(\overline{\mathbb{F}_q})} U^\vee(\overline{\mathbb{F}_q})$ denote its pro-unipotent radical.

On the automorphic side, we are interested in equivariant $\overline{\mathbb{Q}}_\ell$-sheaves on the *affine flag variety* $\mathfrak{Fl} = \mathbf{G}/\mathbf{I}$, an ind-proper ind-scheme constructed in the Appendix of [Ga00]. It carries a left action of $\mathbf{I}$ whose orbits are of finite type and naturally indexed the affine Weyl group W_a for the group $G^\vee$. For $w \in W$, we denote by $\mathfrak{Fl}^w$ the corresponding orbit. Denote by $j_w : \mathfrak{Fl}^w \hookrightarrow \mathfrak{Fl}$ the inclusion of the corresponding $\mathbf{I}$-orbit. Let $\ell : W_a \to \mathbb{Z}^{\geqslant 0}$ denote the length function on the affine Weyl group.

On the spectral side, the stacks that appear are defined over $\overline{\mathbb{Q}}_\ell$. Recall the derived Steinberg variety $\mathcal{Z} = \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ and the classical non-reduced Steinberg variety $\mathcal{Z}' = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ (see Section 1.7.3). The following is Theorem 1 of [Bez16], while the Frobenius property of Φ appears as Proposition 53.

Theorem 2.18 (Bezrukavnikov)**.** *There are equivalences of categories Φ and Φ' and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Sh}^{\mathbf{I}^0}(\mathfrak{Fl}) & \xrightarrow[\simeq]{\Phi'} & \mathrm{Coh}(\mathcal{Z}'/G) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathrm{Sh}^{\mathbf{I}}(\mathfrak{Fl}) & \xrightarrow[\simeq]{\Phi} & \mathrm{Coh}(\mathcal{Z}/G) \end{array}$$

where $\pi : \mathbf{I}^0\backslash\mathfrak{Fl} \to \mathbf{I}\backslash\mathfrak{Fl}$ is the quotient map and $i : \mathcal{Z}/G \hookrightarrow \mathcal{Z}'/G$ is the inclusion. Moreover the functors admits the following natural structures:

- *Φ is naturally an equivalence of* monoidal *categories, and*
- *Φ and Φ' intertwine the action of Frobenius on* $\mathrm{Sh}^{\mathbf{I}}(\mathfrak{Fl})$ *(resp.* $\mathrm{Sh}^{\mathbf{I}^0}(\mathfrak{Fl})$*) with the action of $q \in \mathbb{G}_m$ on* $\mathrm{Coh}(\mathcal{Z}/G)$ *(resp.* $\mathrm{Coh}(\mathcal{Z}'/G)$*).*

We point out certain distinguished sheaves in $\mathrm{Sh}^{\mathbf{I}}(\mathfrak{Fl})$ and $\mathrm{Sh}^{\mathbf{I}^0}(\mathfrak{Fl})$ (computed explicitly for $G = SL_2, PGL_2$ in Examples 2.2.3-5 in [NY19]).

(a) Let $\lambda \in X_*(T^\vee) = X^*(T) \subset W_a$ be a character of the maximal torus of G, considered as an element of the affine Weyl group of the dual group. The *Wakimoto sheaves* J_λ are defined as follows. When λ is dominant, we take $J_\lambda = j_{\lambda,*}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^\lambda}[\langle 2\rho, \lambda\rangle]$. When λ is antidominant, we take $J_\lambda = j_{\lambda,!}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^\lambda}[\langle 2\rho, -\lambda\rangle]$. In general, writing $\lambda = \lambda_1 - \lambda_2$, we define $J_\lambda = J_{\lambda_1} * J_{-\lambda_2}$, which is independent of choices due to Corollary 1 in Section 3.2 of [AB09].

(b) For any $w \in W_a$, we define the corresponding *costandard* (resp. *standard*) object by $\nabla_w := j_{w,*}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}[\ell(w)]$ (resp. $\Delta_w := j_{w,!}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}[\ell(w)]$). They are monoidal inverses by Lemma 8 in Section 3.2 of [AB09]. By Lemma 4 of [Bez16], we have $\nabla_w * \nabla_{w'} = \nabla_{ww'}$ (and likewise for standard objects) when $\ell(w)+\ell(w') = \ell(ww')$. If $\lambda \in X_*(T^\vee) = X^*(T)$ is dominant, then the Wakimoto is costandard $J_\lambda = \nabla_\lambda$; if λ is antidominant, the Wakimoto is standard $J_\lambda = \Delta_\lambda$.

(c) Let $w_0 \in W_f \subset W_a$ be the longest element of the finite Weyl group. The *antispherical projector* or *big tilting sheaf* $\Xi \in \mathrm{Sh}^{\mathbf{I}^0}(\mathfrak{Fl})$ is defined to be the tilting extension of the constant sheaf $\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^{w_0}}$ off $\mathfrak{Fl}^{w_0}$ to $\mathfrak{Fl}$, as in Proposition 11 and Section 5 of [Bez16]. Note that this object does not descend to $\mathrm{Sh}^{\mathbf{I}}(\mathfrak{Fl})$.

We abusively use the same notation to denote sheaves in $\mathrm{Sh}^{\mathbf{I}^0}(\mathfrak{Fl})$; note that $\pi^*\Delta_w \simeq \Delta_w$ and $\pi^*\nabla_w \simeq \nabla_w$ by base change. All sheaves above are perverse sheaves, since the inclusion of strata are affine.

For our applications, we need to work not with $\mathcal{Z}/G$ but with $\mathcal{Z}/\widetilde{G}$ (recall that $\widetilde{G} = G \times \mathbb{G}_m$). The following proposition is the key technical argument we need to construct the semiorthogonal decomposition of $\mathrm{Coh}(\mathcal{Z}/\widetilde{G})$ and hence deduce results on its homological invariants – a graded lift of standards and costandards under Bezrukavnikov's theorem. It is conjectured in [Bez16] that the equivalences in Theorem 2.18 should have mixed versions, relating a mixed form of the Iwahori-equivariant category of $\mathfrak{Fl}$ with a $\mathbb{G}_m$-equivariant version of $\mathrm{Coh}(\mathcal{Z}/G)$, i.e. $\mathrm{Coh}(\mathcal{Z}/\widetilde{G})$, which would immediately give us the desired result. In particular, see Example 57 in [Bez16] for an expectation of what the sheaves $\Phi(\Delta_w)$ are explicitly and note that they have $\mathbb{G}_m$-equivariant lifts.

Proposition 2.19. *The objects $\Phi(\nabla_w), \Phi(\Delta_w) \in \mathrm{Coh}(\mathcal{Z}/G)$ have lifts to objects in $\mathrm{Coh}(\mathcal{Z}/\widetilde{G})$ for all $w \in W_a$, compatible with the action of Frobenius under the equivalence in Theorem 2.18.*

Proof. We will prove the statements for the standard objects; the statements for costandards follows similarly. Wakimoto sheaves are sent to twists of the diagonal $\Phi(J_\lambda) \simeq \mathcal{O}_\Delta(\lambda)$ by Section 4.1.1 of [Bez16], which evidently have $\mathbb{G}_m$-equivariant lifts. Convolution is evidently $\mathbb{G}_m$-equivariant, so the convolution of two sheaves with $\mathbb{G}_m$-lifts also has a $\mathbb{G}_m$-lift. Assuming that the standard objects corresponding to finite reflections have $\mathbb{G}_m$-lifts, by Lemma 4 of [Bez16] we can write the standard for the affine reflection as a convolution of Wakimoto sheaves and standard objects for finite reflections. Thus, we have reduced to showing that all standard objects $\Phi(\Delta_w)$ have $\mathbb{G}_m$-lifts for w a simple finite reflection.

By Corollary 42 of [Bez16] Φ' has the favorable property that $\mathcal{Z}'$ is a classical (non-reduced) scheme, and that it restricts to a map on abelian categories on $\mathrm{Perv}^{U^\vee}(G^\vee/B^\vee) \subset \mathrm{Perv}^{\mathbf{I}^0}(\mathfrak{Fl})$ taking values in $\mathrm{Coh}(\mathcal{Z}'/G)^\heartsuit$ (though it is not surjective). In particular, by Proposition 26 and Lemma 28 in [Bez16] it takes the tilting sheaf Ξ to $\mathcal{O}_{\mathcal{Z}'/G}$, which manifestly has a $\mathbb{G}_m$-lift.

We claim that $\mathbb{G}_m$-lifts for the $\Phi'(\Delta_w) \in \mathrm{Coh}(\mathcal{Z}'/G)$ for $w \in W_f$ induce $\mathbb{G}_m$-lifts for the $\Phi(\Delta_w) \in \mathrm{Coh}(\mathcal{Z}/G)$. Since $\mathcal{Z}$ is a derived scheme, the functor $i_* : \mathrm{Coh}(\mathcal{Z}/\widetilde{G}) \to \mathrm{Coh}(\mathcal{Z}'/\widetilde{G})$ is not fully faithful (i.e. objects on the left may have additional structure). But since $\Phi'(\Delta_w) \simeq i_*\Phi(\Delta_w)$ are in the heart and i_* is t-exact (for the standard t-structures) and conservative, we have that $\Phi(\Delta_w) \in \mathrm{Coh}(\mathcal{Z}/G)^\heartsuit$. In particular, the restriction of i_* to $\mathrm{Coh}(\mathcal{Z}/G)^\heartsuit$ is fully faithful, proving the claim. Thus, we have reduced to showing that the finite simple standard objects $\Phi'(\Delta_w) \in \mathrm{Coh}(\mathcal{Z}'/G)^\heartsuit$ have $\mathbb{G}_m$-lifts; in particular these are objects in the abelian category of coherent sheaves.

By Lemma 4.4.11 in [BY13], Ξ is a successive extension of standard objects $\Delta_w(\ell(w)/2)$ for $w \in W_f$. Thus, there is a standard object $\Delta_w(\ell(w)/2)$ and a Frobenius-equivariant surjection $\Xi \twoheadrightarrow \Delta_w(\ell(w)/2)$. This implies that the kernel $K = \ker(\Xi \twoheadrightarrow \Delta_w(\ell(w)/2))$ is a Frobenius-equivariant subobject of K. On the spectral side, using Proposition 53 in *op. cit.*, this means that $\Phi'(K) \subset \Phi'(\Xi) \simeq \mathcal{O}_{/\mathcal{Z}'/G}$ is a q-equivariant subobject with quotient $\Phi'(\Delta_w(\ell(w)/2))$. We wish to show that the quotient has a $\mathbb{G}_m$-equivariant lift, which amounts to showing that $\Phi'(K)$ is a $\mathbb{G}_m$-equivariant subobject.

Since $\Phi(K)$ is already endowed with a $\mathbb{G}_m$-equivariant structure, q-equivariance for a subobject of a $\mathbb{G}_m$-equivariant object is property, not an additional structure. We claim that for q not a root of unity, any q-closed subsheaf of a $\mathbb{G}_m$-equivariant sheaf on a quotient stack must be $\mathbb{G}_m$-closed as well (i.e. the isomorphism defining the $\mathbb{G}_m$-equivariant structure restricts to the subsheaf). Assuming this claim, and iterating the above argument replacing Ξ with the kernel K, we find that $\Phi'(\Delta_w)$ has a $\mathbb{G}_m$-equivariant lift for every $w \in W_f$ (since the big tilting object contains every Δ_w as a subquotient), completing the proof.

We now justify the claim. First, if $\mathcal{F}$ is a sheaf on a quotient stack X/G with a $\mathbb{G}_m$-action, we can forget the G-equivariance (i.e. base change to the standard atlas $X \to X/G$). Now, by reducing to an open affine $\mathbb{G}_m$-closed cover of X, we can assume X is affine. On an affine scheme $X = \mathrm{Spec}(A)$, the $\mathbb{G}_m$-action gives the structure of a $\mathbb{Z}$-grading on A, and a submodule of a graded A-module $M' \subset M$ is q-equivariant if it is a sum of q-eigenspaces, and $\mathbb{G}_m$-equivariant if it is a sum of homogeneous submodules. The claim follows from the observation that any $m \in M'$ can only have eigenvalues q^n for $n \in \mathbb{Z}$, which are distinct, so the q-eigenspaces entirely determine the $\mathbb{G}_m$-weights. □

2.3. **A semiorthogonal decomposition.** In this section, we describe an "Iwahori-Matsumoto" semiorthogonal decomposition of the category $\mathrm{Coh}(\mathcal{Z}/\widetilde{G})$, arising from the stratification of the affine flag variety $\mathfrak{Fl}$ on the automorphic side of Bezrukavnikov's equivalence Theorem 2.18 and the lifting result in Proposition 2.19. This will, in turn, induce a direct sum decomposition on Hochschild homology. First, let us establish terminology.

Definition 2.20. Let $\{\mathbf{S}_n\}_{n\in\mathbb{N}}$ denote a collection of full subcategories of a small dg category $\mathbf{C}$. We say that $\{\mathbf{S}_n\}$ defines a *semiorthogonal decomposition* of $\mathbf{C}$ if there is an exhaustive left admissible filtration $F_n\mathbf{C}$ of $\mathbf{C}$ such that $\mathbf{S}_n$ is the left orthogonal of $F_{n-1}\mathbf{C}$ inside $F_n\mathbf{C}$. In particular, in this case $\mathrm{Hom}^\bullet_{\mathbf{C}}(X_n, X_m) \simeq 0$ for $X_i \in \mathbf{S}_i$ and $n > m$.

The following result is standard.

Proposition 2.21. *Let G be a pro-affine group scheme acting nicely on an ind-scheme X. Assume that the stabilizer of each orbit is connected. Let I be an indexing set for the G-orbits X_i under the (partial) closure relation, i.e. $X_n \subset \overline{X_m}$ implies $m \geqslant n$, and let $j_n : X_n \hookrightarrow X$ denote the inclusion. Then, $\langle j_{n!}\overline{\mathbb{Q}}_{\ell X_n}\rangle$ defines a semiorthogonal decomposition of $\mathrm{Sh}^G(X)$, where the ordering is given by any choice of extension of the partial order to a total order.*

Proof. It is standard that stratifications of stacks give rise to semi-orthogonal decompositions on categories of ℓ-adic sheaves. We note that each orbit is equivariantly equivalent BH where H is the stabilizer (connected by assumption), and $\mathrm{Sh}(BH)$ is generated by the constant sheaf $\overline{\mathbb{Q}}_\ell$ when H is connected. □

Corollary 2.22. *Fix a Bruhat ordering of the affine Weyl group W_a. The standard objects $\langle \nabla_w = j_{n!}\overline{\mathbb{Q}}_{\ell X_n}\rangle$ give a semiorthogonal decomposition of $\mathrm{Sh}^G(\mathfrak{Fl})$.*

Remark 2.23. The costandard objects $\Delta_w = j_{n*}\overline{\mathbb{Q}}_{\ell X_n}$ define a semiorthogonal decomposition in the reverse order.

We would like to lift the above semiorthogonal decomposition of $\mathrm{Coh}(\mathcal{Z}/G)$ to $\mathrm{Coh}(\mathcal{Z}/\widetilde{G})$. We do so by applying Lemma 2.10 to the $\mathbb{G}_m$-equivariant lifts of the objects $\Phi(\Delta_w)$ from Proposition 2.19. Adopting the notation in Section 2.1.1 (but replacing the group with H), we take:

$$\mathbf{C} = \mathbf{H}^{\mathrm{m}} = \mathrm{Coh}(\mathcal{Z}/\widetilde{G}), \qquad \mathbf{C}^{\mathrm{deq}} = \mathbf{H} = \mathrm{Coh}(\mathcal{Z}/G), \qquad H = \mathbb{G}_m = \mathrm{Spec}\, k[z, z^{-1}].$$

Corollary 2.24. *Let k be a field of characteristic zero, H a group-scheme over k, and $\mathbf{C}$ a compactly generated cocomplete $\mathrm{Rep}(H)$-module dg category, and let $(-)^{\mathrm{deq}} : \mathbf{C} \to \mathbf{C}^{\mathrm{deq}} = \mathbf{C} \otimes_{\mathrm{Rep}(H)} \mathbf{Vect}_k$ denote the de-equivariantization functor. Let $\{E_n \in \mathbf{C} \mid n \in \mathbb{N}\}$ be a linearly ordered set of objects such that $\langle E_n^{\mathrm{deq}}\rangle$ defines a semiorthogonal decomposition of $\mathbf{C}^{\mathrm{deq}}$. Denote by $A_n = \underline{\mathrm{End}}_{\mathbf{C}^{\mathrm{deq}}}(E_n^{\mathrm{deq}})^{op}$ the $\mathrm{Rep}(H)$-algebras from Lemma 2.10. Then, we have*

$$HH(\mathbf{C}) \simeq \bigoplus_{\alpha} HH(A_n\text{-mod}_{\mathrm{Rep}(H)}).$$

Proof. Let $\mathbf{C}'_n := \langle E_n^{\mathrm{deq}}\rangle$ be the category generated by E_n^{deq}, and let $\mathbf{C}_n$ be the preimage under $(-)^{\mathrm{deq}}$. We have a semiorthogonal decomposition of $\mathbf{C}$ by the categories $\mathbf{C}_n$. Hochschild homology is a localizing invariant in the sense of [BGT13], and in particular takes semiorthogonal decompositions to direct sums (this can also easily be seen directly via the dg model for Hochschild homology). Thus we have an equivalence $HH(\mathbf{C}) = \bigoplus_{n\in\mathbb{Z}} HH(\mathbf{C}_n)$. Applying the Lemma 2.10, we find $HH(\mathbf{C}) = \bigoplus_{n\in\mathbb{Z}} HH(A_n\text{-mod}_{\mathrm{Rep}(\mathbb{G}_m)})$. □

We now compute the endomorphism algebras A_w as algebras in $\mathrm{Rep}(\mathbb{G}_m)$, using the graded lifts from Proposition 2.19 and the semiorthogonal decomposition in Corollary 2.22.

Proposition 2.25. *Let E_w denote the $\mathbb{G}_m$-lifts of $\Phi(\Delta_w)$ constructed in Proposition 2.19, and $A_w = \underline{\mathrm{End}}_{\mathrm{Coh}(\mathcal{Z}/\tilde{G})}(E_w^{\mathrm{deq}})$. We have a quasi-isomorphism $A_w \simeq \mathrm{Sym}_{\overline{\mathbb{Q}}_\ell}\,\mathfrak{h}^*[-2]$ where $\mathfrak{h}^*[-2]$ is the universal dual Cartan shifted into cohomological degree 2 with $\mathbb{G}_m$-weight 1. In particular, A_w is formal.*

Proof. Recall that the pullback along multiplication by q corresponds under Φ to the Frobenius automorphism, i.e. Frobenius acts on the nth homogeneous graded piece of $T_{w!}$ by multiplication by q^n. Since q is not a root of unity, we can determine $\mathbb{G}_m$-weights by (necessarily integral) Frobenius weights as in the proof of Proposition 2.19.

Further, since Φ is an equivalence of categories we can compute A_w on the automorphic side. The unit map $\mathcal{F} \to j^! j_! \mathcal{F}$ is an equivalence for j a locally closed immersion, so that

$$A_w = \mathrm{Hom}(j_{w,!}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}, j_{w,!}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}) = \mathrm{Hom}(\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}, j_w^! j_{w,!}\underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}) \simeq R\Gamma(\mathbf{I}\backslash\mathfrak{Fl}^w, \underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w})).$$

Since $\mathfrak{Fl}^w$ is an $\mathbf{I}$-orbit, letting $\mathbf{I}^w$ denote its stabilizer for a choice of base point in $\mathfrak{Fl}^w$, we find that $A_w \simeq C^\bullet(B\mathbf{I}^w; \overline{\mathbb{Q}}_\ell)$ is the equivariant cohomology chain complex for $B\mathbf{I}^w$ with $\overline{\mathbb{Q}}_\ell$-coefficients under the cup product. The reductive quotient (i.e. by the pro-unipotent radical) of $\mathbf{I}^w$ is T, so $A_w \simeq C^\bullet(BT; \overline{\mathbb{Q}}_\ell)$. By Proposition 2.17, the Frobenius weight is equal to the cohomological degree, and the Frobenius weight is equal to twice[15] the $\mathbb{G}_m$-weight, proving the claim regarding $\mathbb{G}_m$-weights.

Finally, we need to show formality of A_w as an algebra. By purity, any cohomological degree $2n$ class in $C^\bullet(BT; \overline{\mathbb{Q}}_\ell)$ has $\mathbb{G}_m$-weight n (or Frobenius weight $2n$). By a standard weight-degree shearing argument, this implies formality. □

We now apply Corollary 2.24 to the set-up in the above proposition. We will see that since Hochschild homology is insensitive to field extensions and all our stacks of interest are defined over $\mathbb{Q}$, the following results hold for any field k of characteristic 0 (i.e. not just $k = \overline{\mathbb{Q}}_\ell$).

Corollary 2.26. *Let k be any field of characteristic 0. The isomorphism from above induces an isomorphism of $k[z, z^{-1}]$-modules*

$$HH(\mathbf{H}^{\mathrm{m}}/k) = kW_a \otimes_k k[z, z^{-1}].$$

In particular, we have that

1. *the Hochschild homology $HH(\mathbf{H}^{\mathrm{m}}/k)$ is cohomologically concentrated in degree zero,*
2. *the Chern character $K(\mathbf{H}^{\mathrm{m}}) \to HH(\mathbf{H}^{\mathrm{m}}/k)$ factors through $K_0(\mathbf{H}^{\mathrm{m}})$,*
3. *the map $K_0(\mathbf{H}^{\mathrm{m}}) \otimes_{\mathbb{Z}} k \to HH(\mathbf{H}^{\mathrm{m}}/k)$ is an equivalence,*
4. *$\mathbf{H}^{\mathrm{m}}$ satisfies Hochschild-to-cyclic degeneration, i.e. $HN(\mathbf{H}^{\mathrm{m}}/k) \simeq HH(\mathbf{H}^{\mathrm{m}}/k)[[u]]$.*

Proof. Fix a Bruhat order on W_a, extended to a total order. Let us first prove the case $k = \overline{\mathbb{Q}}_\ell$. Applying Corollary 2.24 in the case $\mathbf{C} = \mathbf{H}^{\mathrm{m}} = \mathrm{Coh}(\mathcal{Z}/\tilde{G})$, $\mathbf{C} = \mathbf{H} = \mathrm{Coh}(\mathcal{Z}/G)$, and $H = \mathbb{G}_m$, we have a canonical equivalence

$$HH(\mathbf{H}^{\mathrm{m}}/\overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell W_a \otimes_{\overline{\mathbb{Q}}_\ell} HH(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)}/\overline{\mathbb{Q}}_\ell)$$

where $A = \mathrm{Sym}^\bullet_{\overline{\mathbb{Q}}_\ell}\,\mathfrak{h}^*[-2] \simeq A_w$ is the algebra from Proposition 2.25 (which does not depend on $w \in W_a$).

Let us briefly consider the case of general k of characteristic 0, and let $A = \mathrm{Sym}^\bullet_k\,\mathfrak{h}^*[-2]$. The Hochschild homology of of A-$\mathrm{perf}_{\mathrm{Rep}(\mathbb{G}_m)}$ is computed by the Block-Getzler complex of Definition 2.11, which we can compute explicitly. Its terms are $(A^{\otimes n+1} \otimes k[z, z^{-1}])^{\mathbb{G}_m}$, and since z has $\mathbb{G}_m$-weight 0, there is an isomorphism $(A^{\otimes n+1} \otimes k[z, z^{-1}])^{\mathbb{G}_m} \simeq (A^{\otimes n+1})^{\mathbb{G}_m} \otimes k[z, z^{-1}]$ and we observe that $(A^{\otimes n+1})^{\mathbb{G}_m} = k$ since each A is generated over k by positive weights. Thus, the natural map $BG_\bullet(k) \to BG_\bullet(A)$ is a quasi-isomorphism, so the first claim claim follows. Factorization through K_0 follows since the Hochschild homology is coconnective.

[15] Often, e.g. in Remark 1 of [AB09], the $\mathbb{G}_m$-scaling action is defined to have geometric weight -2; under this differing convention, the Frobenius weight is equal to the $\mathbb{G}_m$-weight.

To show that the map $K_0(A\text{-mod}_{\mathrm{Rep}(\mathbb{G}_m)}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \to HH(A\text{-mod}_{\mathrm{Rep}(\mathbb{G}_m)}/\overline{\mathbb{Q}}_\ell)$ is an equivalence, first note that since $HH(A\text{-mod}_{\mathrm{Rep}(\mathbb{G}_m)}/\overline{\mathbb{Q}}_\ell)$ is concentrated in degree zero, the Chern character factors through K_0, i.e. we have a commuting diagram for each summand

$$\begin{array}{ccccc} K(\mathrm{Rep}(\mathbb{G}_m)) \otimes_{\mathbb{Z}} k & \longrightarrow & K_0(\mathrm{Rep}(\mathbb{G}_m)) \otimes_{\mathbb{Z}} k & \xrightarrow{\simeq} & HH(\mathrm{Rep}(\mathbb{G}_m)/\overline{\mathbb{Q}}_\ell) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ K(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)}) \otimes_{\mathbb{Z}} k & \longrightarrow & K_0(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)}) \otimes_{\mathbb{Z}} k & \longrightarrow & HH(A\text{-mod}_{\mathrm{Rep}(\mathbb{G}_m)}/\overline{\mathbb{Q}}_\ell). \end{array}$$

By Remark 2.15, the map $K_0(\mathrm{Rep}(\mathbb{G}_m)) \to K_0(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)})$ is an equivalence, since both sides are freely generated by $K_0(\mathrm{Rep}(\mathbb{G}_m)) = HH(\mathrm{Rep}(\mathbb{G}_m))$ by the character of a single object $[A]$, i.e. the free object. Using the semiorthogonal decomposition, these equivalences induce an equivalence $K_0(\mathbf{H}^{\mathrm{m}}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell \simeq HH(\mathbf{H}^{\mathrm{m}}/\overline{\mathbb{Q}}_\ell)$, which is an equivalence of algebras by Remark 2.15.

Next, to prove the equivalence for general fields k, note that all stacks and algebras in question are well-defined over $\mathbb{Q}$. Consider the field extension $\mathbb{Q} \subset \overline{\mathbb{Q}}_\ell$. To conclude the result for $k = \mathbb{Q}$, we need to show that the $\mathbb{Q}$-subspaces

$$HH(\mathrm{Coh}(\mathcal{Z}_k/\widetilde{G}_k)/\mathbb{Q}) \subset HH(\mathrm{Coh}(\mathcal{Z}_{\overline{\mathbb{Q}}_\ell}/\widetilde{G}_{\overline{\mathbb{Q}}_\ell})/\overline{\mathbb{Q}}_\ell), \qquad kW_a \otimes_{\mathbb{Q}} \mathbb{Q}[z,z^{-1}] \subset \overline{\mathbb{Q}}_\ell W_a \otimes_{\overline{\mathbb{Q}}_\ell} \overline{\mathbb{Q}}_\ell[z,z^{-1}]$$

coincide under the equivalence; this follows from the calculation of $HH(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)})$ via the Block-Getzler complex, i.e. on each summand coming from the semiorthogonal decomposition, the map $K_0(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)}) = \mathbb{Z}[z,z^{-1}] \to HH(A\text{-perf}_{\mathrm{Rep}(\mathbb{G}_m)}/\overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[z,z^{-1}]$ is an injection, with $[A] = 1$ on both sides.

Now, let $\mathbb{Q} \subset k$ be a field extension. By the change of rings formula in Hochschild homology, we have a canonical equivalence

$$HH(\mathrm{Coh}(\mathcal{Z}_k/\widetilde{G}_k)/k) \simeq HH(\mathrm{Coh}(\mathcal{Z}_{\mathbb{Q}}/\widetilde{G}_{\mathbb{Q}})/\mathbb{Q}) \otimes_{\mathbb{Q}} K \simeq kW_a \otimes_k k[z,z^{-1}].$$

Thus the result holds for k. Since every field of characteristic 0 is an extension of $\mathbb{Q}$, the result holds for any field k of characteristic 0. □

We also have the following result for the non-$\mathbb{G}_m$-equivariant version.

Corollary 2.27. *The map of algebras $K(\mathrm{Coh}(\mathcal{Z}/G)) \to HH(\mathrm{Coh}(\mathcal{Z}/G))$ factors through K_0 and we have an isomorphism as dg k-modules*

$$HH(\mathrm{Coh}(\mathcal{Z}/G)) \simeq kW_a \otimes_k \mathrm{Sym}^\bullet_k(\mathfrak{h}^*[-1], \mathfrak{h}^*[-2]).$$

Furthermore, the Connes B-differential is given by the extending identity map $\mathfrak{h}^*[-2] \to \mathfrak{h}^*[-1]$, so that applying the Tate construction we have an isomorphism of modules

$$kW_a \otimes_k k((u)) \simeq K_0(\mathrm{Coh}(\mathcal{Z}/G)) \otimes_k k((u)) \to HP(\mathrm{Coh}(\mathcal{Z}/G)).$$

Proof. Essentially the same as the previous corollary, along with a direct calculation of the Hochschild homology of the formal dg ring

$$HH(\mathrm{Sh}^T(\mathrm{pt})) = HH(k[\mathfrak{h}[-2]]\text{-mod}).$$

□

2.4. **Hochschild and cyclic homology of the affine Hecke category.** Recall the notation $\widetilde{G} = G \times \mathbb{G}_m$, and that $\mathbf{H}^{\mathrm{m}} = \mathrm{Coh}(\mathcal{Z}/\widetilde{G})$ denotes the mixed affine Hecke category, while $\mathbf{H} = \mathrm{Coh}(\mathcal{Z}/G)$ denotes the affine Hecke category. In this section, we will show that their trace decategorifications are the affine Hecke algebra $\mathcal{H}$ and a derived variant of the group algebra of the extended affine Weyl group kW_a. We assume that G has simply connected derived subgroup until Section 2.4.2, where we remove the assumption.

We begin by quoting the following celebrated theorem by Ginzburg, Kazhdan and Lusztig.

Theorem 2.28 (Ginzburg-Kazhdan-Lusztig)**.** *Let k be an algebraically closed field of characteristic 0, and assume that G has simply connected derived subgroup. Then there is an equivalence of associative algebras $\mathcal{H} \to K_0(\mathbf{H}^{\mathrm{m}}) \otimes_{\mathbb{Z}} k$, compatibly with an identification of the center with $K_0(\mathrm{Rep}(\widetilde{G})) \otimes_{\mathbb{Z}} k$. Likewise, there is an equivalence of associative algebras $kW_a \simeq K_0(\mathbf{H}) \otimes_{\mathbb{Z}} k$ with center $K_0(\mathrm{Rep}(G))$.*

Proof. The only difference between our statement and that in [KL87] [CG97] is their Steinberg stack is the classical stack $\pi_0(\mathcal{Z})/\widetilde{G}$, which has no derived structure. On the other hand, we are interested in $\mathcal{Z}/\widetilde{G}$ which has better formal properties. The statement follows from the fact that the Grothendieck group is insensitive to derived structure, i.e. the ideal sheaf for the embedding $\pi_0(\mathcal{Z})/\widetilde{G} \hookrightarrow \mathcal{Z}/\widetilde{G}$ acts nilpotently on any coherent complex. Finally, note that while the statement of Theorem 3.5 of [KL87] and Theorem 7.2.5 in [CG97] are made for $k = \mathbb{C}$, the proofs do not employ topological methods and apply to any algebraically closed field of characteristic zero. □

We combine the above theorem with Corollary 2.26 to arrive at the following main theorem. We will remove the simply connectedness assumption in Section 2.4.2.

Theorem 2.29. *Assume that G has simply connected derived subgroup over an algebraically closed field k of characteristic 0. There is an equivalence of algebras, and an identification of the center:*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\simeq} & HH(\mathbf{H}^{\mathrm{m}}) \\ \uparrow & & \uparrow \\ k[G]^G \otimes_k k[q, q^{-1}] & \xrightarrow{\simeq} & HH(\mathrm{Rep}(G \times \mathbb{G}_m)). \end{array}$$

Proof. That the map is an isomorphism is a combination of Theorem 1.2 and Corollary 2.26. □

The following may also be of interest, and is the analogue to Corollary 2.27. Note that in this case, the map to Hochschild homology is not an equivalence, though it does induce an equivalence on HH_0 and on periodic cyclic homology HP.

Corollary 2.30. *With the assumptions above, there is a commuting diagram of algebras:*

$$\begin{array}{ccc} kW_a \otimes_k \mathrm{Sym}_k^{\bullet}(\mathfrak{h}^*[-1] \oplus \mathfrak{h}^*[-2]) & \xrightarrow{\simeq} & HH(\mathbf{H}) \\ \uparrow & & \uparrow \\ k[G]^G & \xrightarrow{\simeq} & HH(\mathrm{Rep}(G)). \end{array}$$

Taking the Tate construction, there is an equivalence of $k((u))$-algebras, and an identification of the center:

$$\begin{array}{ccc} kW_a((u)) & \xrightarrow{\simeq} & HP(\mathbf{H}) \\ \uparrow & & \uparrow \\ k[G]^G((u)) & \xrightarrow{\simeq} & HP(\mathrm{Rep}(G)). \end{array}$$

Proof. By Corollary 2.27, the Hochschild homology $HH(\mathrm{Coh}(\mathcal{Z}/G))$ is coconnective, so the Chern character from $K(\mathrm{Coh}(\mathcal{Z}/G))$ factors through $K_0(\mathrm{Coh}(\mathcal{Z}/G)) \otimes_{\mathbb{Z}} k = kW_a$. Thus we have a map of algebras $kW_a \to HH(\mathrm{Coh}(\mathcal{Z}/G))$ which induces an equivalence on H^0. Next, note that the subcategory $\mathrm{Sh}^{\mathbf{I}}(\mathfrak{Fl})$ generated by the monoidal unit (i.e. the skyscraper sheaf δ_e), which is closed under the monoidal structure, is in the center of $\mathrm{Coh}(\mathcal{Z}/G)$, so that the subalgebra $HH(\langle \delta_e \rangle) \simeq \mathrm{Sym}_k^{\bullet}(\mathfrak{h}[-1] \oplus \mathfrak{h}[-2]) \subset HH(\mathrm{Coh}(\mathcal{Z}/G))$ is central. This defines a map of algebras $HH(\langle \delta_e \rangle)$-mod $\to HH(\mathrm{Coh}(\mathcal{Z}/G))$, which defines a map of algebras out of the tensor product $HH(\langle \delta_e \rangle) \otimes_k kW_a \to HH(\mathrm{Coh}(\mathcal{Z}/G))$ which is an equivalence when restricted to each tensor factor; thus we can calculate that it is an equivalence. □

2.4.1. *q-specializations of the affine Hecke algebra.* Let $q : \mathcal{Z}/G \to \mathcal{Z}/G$ be the action by $q \in \mathbb{G}_m$ under our conventions, i.e. multiplying by q^{-1}. In this section we compute the trace of the functor[16] q_* on the category $\mathbf{H} = \mathrm{Coh}(\mathcal{Z}/G)$. First, we make the general observation that if F is an automorphism of a category $\mathbf{C}$ and $\mathcal{E} \in \mathbf{C}$, then an F-equivariant structure on $\mathcal{E}$ induces an automorphism of the dg algebra $A = \mathrm{End}_{\mathbf{C}}(\mathcal{E})$, and thus an automorphism of the category A-mod, which we will abusively also denote F.

Proposition 2.31. *Let $q \neq 1$ and let A_w denote the algebras from Proposition 2.25. Then, $HH(A_w, q_*) = k$.*

Proof. First, observe that the functor q_* induces the automorphism on the algebra $A_w \simeq \mathrm{Sym}_k \mathfrak{h}^*[-2]$ arising via the q-scaling map on $\mathfrak{h}$ (in particular, $\mathfrak{h}^*$ has weight -1). The claim is a direct calculation using the complex $C_q(A_w, \mathbb{G}_m)$ from Definition 2.11 via Koszul resolutions: $C_q(A_w, \mathbb{G}_m)$ is the derived tensor product $A_w \otimes^L_{A_w \otimes A_w} A_w$ where A_w is the diagonal bimodule for one factor and is twisted by q_* on the other factor.

Rather than a direct calculation, we give a geometric argument. First, note that q_* preserves the $\mathbb{G}_m$-weights of $A_w \simeq \mathrm{Sym}^\bullet_k \mathfrak{h}^*[-2]$ (i.e. since $q \in \mathbb{G}_m$ is central). We apply a Tate shearing (i.e. sending bidegree (a, b) to $(a - 2b, b)$) to the algebra $\mathrm{Sym}_k \mathfrak{h}^*[-2]$ to obtain the algebra $\mathcal{O}(\mathfrak{h}) = \mathrm{Sym}^\bullet_k \mathfrak{h}^*$. Note that $HH(\mathrm{Perf}(\mathfrak{h}), q_*) = \mathcal{O}(\mathfrak{h}^q)$, i.e. functions on the derived fixed points of action by q. When $q \neq 1$ we have $\mathfrak{h}^q = \{0\}$, so $HH(\mathrm{Perf}(\mathfrak{h}), q_*) = k$. Undoing the shearing, we find that the natural map $HH(A_w, q_*) \to HH(k, q_*)$ is an equivalence. □

Corollary 2.32. *Let $\mathcal{H}_q$ denote the specialization of the affine Hecke algebra at $q \in \mathbb{G}_m$. If $q \neq 1$, we have an equivalence of algebras*

$$HH(\mathbf{H}, q_*) \simeq \mathcal{H}_q.$$

Proof. The calculation in Proposition 2.31 shows that specialization at $q \in \mathbb{G}_m$ induces an equivalence on Block-Getzler complexes (viewing A_w as an algebra in $\mathrm{Rep}(\mathbb{G}_m)$):

$$BG^\bullet(A_w) \otimes_{k[z,z^{-1}]} k_q \to \underline{BG}^\bullet(A_w) \otimes_{k[z,z^{-1}]} k_q \to BG^\bullet_q(A_w)$$

inducing an equivalence $HH(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})) \otimes_{k[z,z^{-1}]} k_q \simeq HH(\mathrm{Coh}(\mathcal{Z}/G), q_*)$, since the trace of an endofunctor F on a category $\mathbf{C}$ takes semiorthogonal decompositions preserved by F to direct sums. Consequently, under the identification of algebras $HH(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})) \simeq \mathcal{H}$, specialization at q defines an equivalence $HH(\mathrm{Coh}(\mathcal{Z}/G), q_*) \simeq \mathcal{H}_q$. □

Remark 2.33. The above corollary is evidently untrue for $q = 1$, since $\mathcal{H}$ is flat over $k[z, z^{-1}]$ but $HH(\mathbf{H})$ has derived structure by Corollary 2.30.

Remark 2.34. Our methods also allow for an identification of the following monodromic variants of the affine Hecke category introduced in [Bez16] (where $\mathcal{Z}' = \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ and $\mathcal{Z}^\wedge$ is the formal completion of $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$ along $\mathcal{Z}$):

$$HH(\mathrm{Coh}(\mathcal{Z}'/\widetilde{G})) \simeq HH(\mathrm{Coh}(\mathcal{Z}^\wedge/\widetilde{G})) \simeq \mathcal{H},$$

$$HH(\mathrm{Coh}(\mathcal{Z}'/G), q_*) \simeq \mathcal{H}_q,$$

$$HH(\mathrm{Coh}(\mathcal{Z}^\wedge/G), q_*) \simeq \begin{cases} kW_a \otimes_k \mathrm{Sym}^\bullet_k(\mathfrak{h}^* \oplus \mathfrak{h}^*[-1]) & q = 1 \\ \mathcal{H}_q & q \neq 1, \end{cases}.$$

Note that the category $\mathrm{Coh}(\mathcal{Z}'/\widetilde{G})$ is not monoidal, so it does not make sense to ask that it is identified with $\mathcal{H}$ as an algebra. However, it is equivalent to $\mathcal{H}$ as a (right) module for $HH(\mathrm{Coh}(\mathcal{Z}/\widetilde{G})) \simeq \mathcal{H}$.

On the other hand, the category $\mathrm{Coh}(\mathcal{Z}^\wedge/\widetilde{G})$ does not have a monoidal unit, and the monoidal product is poorly behaved (for example, when $G = T$ is a torus, the product is zero on Hochschild homology). However, one expects that there is a renormalization of $\mathrm{Coh}(\mathcal{Z}^\wedge/\widetilde{G})$ whose Hochschild homology satisfies the same good properties as (and in fact, is Koszul dual to) $\mathbf{H}^{\mathrm{m}}$.

---

[16]Note that our q_* corresponds to $\mathbf{q}^*$ in [AB09].

The only difference in these cases is that the generating objects $E_w = \underline{\overline{\mathbb{Q}}_\ell}_{\mathfrak{Fl}^w}$ for the semiorthogonal decomposition live in different categories on the automorphic side, so the resulting endormorphism algebra A_w may differ (i.e. as in Proposition 2.25). Recall that for $\mathcal{Z}$, we had $A_w = \mathrm{Sym}^\bullet_{\overline{\mathbb{Q}}_\ell} \mathfrak{h}^*[-2]$. For $\mathcal{Z}'$, the derived category of sheaves on each orbit is equivalent to $D(\mathrm{pt})$, so $A_w = \overline{\mathbb{Q}}_\ell$. For $\mathcal{Z}^\wedge$, the category is equivalent to $D^u(H/_{\!\!\!,}H) \simeq \overline{\mathbb{Q}}_\ell[\mathfrak{h}]\text{-mod}_{\mathfrak{h}-nil}$, the category of weakly H-equivariant sheaves on H with unipotent monodromy, and $A_w = \mathrm{Sym}^\bullet_{\overline{\mathbb{Q}}_\ell} \mathfrak{h}^*[-1]$.

2.4.2. *Groups of non-simply connected type.* In this section we will remove the simply connectedness assumptions from earlier theorems. We work in the following set-up. Let G be a reductive algebraic group with simply connected derived subgroup, and $\phi : G \to G'$ a central isogeny with kernel Z (i.e. a quotient by a finite subgroup Z of the center). Following Section 1.5 of [Re02], this induces a Z-action on $\mathcal{H}_G$ via the formula

$$z \cdot (T_w \otimes e^\lambda) = \lambda(z)(T_w \otimes e^\lambda), \qquad w \in W_f, \lambda \in X^*(T), z \in Z$$

and an injection of affine Hecke algebras

$$\mathcal{H}_{G'} \simeq \mathcal{H}_G^Z \hookrightarrow \mathcal{H}_G.$$

We define an analogous action on Hochschild homology in the following general set-up.

Definition 2.35. Let $Z \subset G$ be central, and $G' = G/Z$, and let $\mathbf{C}$ be a $\mathrm{Rep}(G)$-module category equipped with a *Z-trivialization*, i.e. a $\mathrm{Rep}(G')$-linear category $\mathbf{C}'$ and an equivalence $\mathbf{C} \simeq \mathbf{C}' \otimes_{\mathrm{Rep}(G')} \mathrm{Rep}(G)$ (see also Definition 5.12). In this setting, we have a natural action of Z on the Hom-spaces of $\mathrm{Rep}(G)$ (using that Z is central), compatible with the $\mathrm{Rep}(G')$-action. This induces a Z-action on the Hom-spaces of $\mathbf{C}' \otimes_{\mathrm{Rep}(G')} \mathrm{Rep}(G)$, and thus a Z-action on $HH(\mathbf{C}, F)$.

Proposition 2.36. *There is a functorial equivalence for $\mathbf{Rep}(G)$-categories $\mathbf{C}$ equipped with a Z-trivialization*

$$HH(\mathbf{C}') \simeq HH(\mathbf{C})^Z.$$

Proof. The Z-trivialization defines an equivalence between the $\mathrm{Rep}(G)$ de-equivariantization of $\mathbf{C}$ with the $\mathrm{Rep}(G')$ de-equivariantization of $\mathbf{C}'$:

$$\mathbf{C}^{\mathrm{deq}} := \mathbf{C} \otimes_{\mathrm{Rep}(G)} \mathbf{Vect}_k \simeq \mathbf{C}' \otimes_{\mathrm{Rep}(G)'} \mathrm{Rep}(G) \otimes_{\mathrm{Rep}(G)} \mathbf{Vect}_k \simeq \mathbf{C}' \otimes_{\mathrm{Rep}(G)'} \mathbf{Vect}_k.$$

Thus we have explicit models

$$HH(\mathbf{C}, F) = BG^\bullet(\mathbf{C}^{\mathrm{deq}}, F; G), \qquad HH(\mathbf{C}', F) = BG^\bullet(\mathbf{C}^{\mathrm{deq}}, F; G').$$

Tracing through the identifications in Proposition 2.12, one can identify the Z-action on the Block-Getzler complex $BG^\bullet(\mathbf{C}^{\mathrm{deq}}, F; G)$ as follows: level-wise, it only acts non-trivially on the tensor factor $k[G]$ by

$$(z \cdot f)(g) = f(zg) = f(gz), \qquad z \in Z, f \in k[G], g \in G.$$

The result is now immediate from the observation that $k[G]^Z = k[G']$ for the above action. $\square$

It remains to show that these Z-actions agree, which we do so in the following.

Proposition 2.37. *The identification $\mathcal{H} \simeq HH(\mathbf{H}^{\mathrm{m}})$ intertwines the Z-actions above.*

Proof. The action of Z on $\mathcal{H}$ defined in [Re02] decomposes into eigenspaces indexed by W_f double cosets $W_f \lambda W_f \subset W_a$ for $\lambda \in X^\bullet(T)$, spanned by Iwahori-Matsumoto basis elements T_w for $w \in W_f \lambda W_f$, with eigenvalue $\lambda|_Z$. This claim can be directly verified, e.g. using the Bernstein relations in Section 7.1 of [CG97]. Thus, it suffices to show that Z acts by the same eigenvalues on Hochschild homology, with basis given by $\{[\mathrm{id}_{\Phi(\Delta_w)}] \mid w \in W_a\}$, i.e. identity maps for the spectral-side standard objects $\Phi(\Delta_w)$ described in Section 2.2.

By functoriality, for any functor $F : \mathbf{C} \to \mathbf{D}$ of categories in our set-up, if $[\mathrm{id}_X] \in HH_0(\mathbf{C})$ is a λ-eigenvector for Z, then $[\mathrm{id}_{F(X)}] \in HH_0(\mathbf{D})$ is as well; the converse is true if F is faithful on the homotopy category (i.e. $H^0(\mathrm{Hom}^\bullet(X, X)) \to H^0(\mathrm{Hom}^\bullet(F(X), F(X))$ is injective). We will use this fact repeatedly. In particular, since the forgetful functor $\mathrm{Coh}(\mathcal{Z}/\widetilde{G}) \to \mathrm{Coh}(\mathcal{Z}/G)$ is faithful, we can forget $\mathbb{G}_m$-equivariance, and since the Z-action is compatible with convolution, it suffices to check our statement for finite reflections and the lattice.

For the lattice, we have $\Phi(\Delta_\lambda) \simeq \Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda) = \Delta_* p^* V_\lambda \in \mathrm{Coh}(\mathcal{Z}/G)$, where $p : \widetilde{\mathcal{N}}/G \to BB$ is the projection. The eigenvalue for the identity map of $V_\lambda \in \mathrm{Coh}(BB)$ is evidently $\lambda|_Z$. For finite simple reflections, since i_* is fully faithful on the homotopy category we may instead consider the equivalence Φ'. Here, the spectral-side object corresponding to the automorphic big tilting object is $\mathcal{O}_{\mathcal{Z}'/G}$. By applying functoriality to the pullback from a point we see that the identity on any structure sheaf has trivial Z-eigenvalue, and therefore any subquotient does, thus $\Phi'(\Delta_w)$ and $\Phi(\Delta_w)$ do. □

Corollary 2.38. *The statements of Theorem 2.29, Corollary 2.30 and Corollary 2.32 hold without the assumption that G has simply connected derived subgroup.*

Proof. By Theorem 2.29, we have an identification $HH(\mathbf{H}^{\mathrm{m}}_G) \simeq \mathcal{H}_G$. Since Z acts on $\mathcal{Z}$ and $\mathcal{Z}'$ trivially, the categories $\mathrm{Coh}(\mathcal{Z}/G)$ and $\mathrm{Coh}(\mathcal{Z}'/G)$ come equipped with natural Z-trivializations, and thus their Hochschild homologies have Z-actions as defined above. By Proposition 2.37 the two Z-actions coincide under our equivalence, proving the claim. □

3. Traces of representations of convolution categories

We have seen in Theorem 2.29 that the affine Hecke algebra $\mathcal{H}$ is identified with the Hochschild homology of the (mixed) affine Hecke category $\mathbf{H}^{\mathrm{m}} = \mathrm{Coh}(\mathcal{Z}/\widetilde{G})$. In this section we describe a general theory of categorical traces in derived algebraic geometry to explain why this is a useful realization. Namely, as an application we will see in Section 4 that the geometric realization of Hochschild homology via derived loop spaces implies a realization of the affine Hecke algebra as endomorphisms of a coherent sheaf on the loop space of the stacky nilpotent cone, the *coherent Springer sheaf*, and hence a localization description of its category of modules as a category of coherent sheaves.

3.1. **Traces of monoidal categories.**

In this section we present the two different trace decategorifications for a monoidal category and their relation. See [BFN10, HSS17, CP19, BN19, GKRV20] for detailed exposition.

Definition 3.1. Let $(\mathbf{A}, *)$ denote an E_1-monoidal compactly generated cocomplete k-linear dg category and F a monoidal endofunctor. There are two notions of its Hochschild homology or trace. See definitions in Section 2.1.1.

- The naive or *vertical trace* (or *Hochschild homology*) is a chain complex $\mathrm{tr}(\mathbf{A}, F) = HH(\mathbf{A}, F)$. Via functoriality of traces, and under the assumptions that the multiplication functor $* : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ preserves compact objects and that the monoidal unit is compact, it has the additional structure of an associative (or E_1-)algebra $(HH(\mathbf{A}), *)$.
- The 2-categorical or *horizontal trace* (or *monoidal/categorical Hochschild homology*) is a dg category[17] $\mathbf{Tr}((\mathbf{A}, *), F) = \mathbf{A} \otimes_{\mathbf{A} \otimes \mathbf{A}^{rv}} \mathbf{A}_F$ where $\mathbf{A}_F$ is the monoidal category whose left action is twisted by F.[18] Via functoriality of traces, the horizontal trace is the tautological receptacle for characters in $\mathbf{A}$:

$$[-] : \mathbf{A} \to \mathbf{Tr}((\mathbf{A}, *), F).$$

  The monoidal unit $1_{\mathbf{A}}$ itself defines an object $[1_{\mathbf{A}}] \in \mathbf{Tr}((\mathbf{A}, *), F)$, i.e. $\mathbf{Tr}((\mathbf{A}, *), F)$ is a pointed (or E_0-)category.

We sometimes omit the monoidal product $*$ from the notation, and when $F = \mathrm{id}_{\mathbf{C}}$ we also sometimes omit it from the notation. Both traces admit S^1 actions.

We define the notion of characters in horizontal traces more precisely and generally below.

[17] The category $\mathbf{A}^{rv}$ is obtained by reversing the monoidal product, not taking opposite morphisms.

[18] More generally, the horizontal trace may take as an input an $\mathbf{A}$-bimodule category $\mathbf{Q}$.

Definition 3.2. One can view the horizontal trace as a trace decategorification in the sense of Definition 2.1 in the following way, following Section 3.6 of [GKRV20]. We consider the symmetric monoidal "Morita" category $\mathbf{Mor}_k$, whose objects are $\mathbf{A}$-**mod**, i.e. 2-categories of module categories for a monoidal category $\mathbf{A}$, and whose 1-morphisms are given by bimodule categories. Then, for a monoidal endofunctor $F : \mathbf{A} \to \mathbf{A}$, we have $\mathrm{tr}(\mathbf{A}\text{-}\mathbf{mod}, F) = \mathbf{Tr}(\mathbf{A}, \mathbf{A}_F)$.

We can apply Definition 2.6 to obtain the following more general notion of character map for the horizontal trace (see Section 3.8.2 in [GKRV20]). That is, the horizontal trace $\mathbf{Tr}(\mathbf{A}, F)$ can be viewed as the tautological receptacle for characters $[(\mathbf{M}, F_{\mathbf{M}})]$ of left $\mathbf{A}$-module categories $\mathbf{M}$ equipped with an F-semilinear endofunctor $F_{\mathbf{M}}$, i.e. a map of $\mathbf{A}$-module categories $F_{\mathbf{M}} : \mathbf{M} \to \mathbf{M}_F := \mathbf{A}_F \otimes_{\mathbf{A}} \mathbf{M}$.[19]

The trace $[A]$ of objects $A \in \mathbf{A}$ in Definition 3.1 above is a special case in the following way: consider $\mathbf{M} := \mathbf{A}$ as the usual (left) regular $\mathbf{A}$-module category; for $A \in \mathrm{Ob}(\mathbf{A})$, we define $F_A(-) := F(-) * A$. In this case, we have $[A] = [\mathbf{A}, F_A]$. In particular, the trace of the monoidal unit[20] is $[1_{\mathbf{A}}] = [\mathbf{A}, F]$, i.e. the trace of the regular representation.

Moreover, the categorical trace provides a "delooping" of the naive trace. To make the relationship between the two traces precise, we first recall the notion of a rigid monoidal category (see Definition 9.1.2 and Lemma 9.1.5 in [GR17]).

Definition 3.3. Let $\mathbf{A}$ be a compactly generated stable monoidal ∞-category, with multiplication $\mu : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$. We say $\mathbf{A}$ is *rigid* if the monoidal unit is compact, μ preserves compact objects, and if every compact object of $\mathbf{A}$ admits a left and right (monoidal) dual.

We have the following relationship between vertical and horizontal traces of [GKRV20], which may be interpreted via Theorem 1.1 of [CP19] as a compatibility of iterated traces. Let $\mathbf{A}$ be a monoidal category, and F a monoidal endofunctor. We denote by $(\mathbf{A}, F)$-**mod** the 1-category (i.e. forget the 2-morphisms) of $\mathbf{A}$-module categories with F-semilinear endofunctors as in Definition 3.2.

Theorem 3.4 (Theorem 3.8.5 [GKRV20], Theorem 1.1 [CP19])**.** *Assume that* $\mathbf{A}$ *is compactly generated and rigid monoidal, and* F *a monoidal endofunctor. Then, there is an equivalence of algebras*

$$HH(\mathbf{A}, F) \simeq \mathrm{End}_{\mathbf{Tr}(\mathbf{A},F)}([\mathbf{A}, F])^{op},$$

More generally, there is an equivalence of functors:

$$HH(-) \simeq \mathrm{Hom}_{\mathbf{Tr}(\mathbf{A},F)}([\mathbf{A}, F], [-]) : (\mathbf{A}, F)\text{-}\mathbf{mod} \longrightarrow HH(\mathbf{A}, F)\text{-mod}\,.$$

In particular, assuming that $[\mathbf{A}, F]$ *is a compact object, then the left adjoint to the functor* $\mathrm{Hom}_{\mathbf{Tr}(\mathbf{A},F)}([\mathbf{A}, F], -)$ *defines a fully faithful embedding which preserves compact objects, whose essential image is the category generated by* $[\mathbf{A}, F]$*:*

$$\begin{array}{ccc} HH(\mathbf{A}, F)\text{-mod} & \underset{\mathrm{Hom}([\mathbf{A},F],-)}{\overset{[\mathbf{A},F]\otimes_{\mathrm{End}([\mathbf{A},F])}-}{\rightleftarrows}} & \mathbf{Tr}(\mathbf{A}, F) \\ & \searrow^{\simeq} \quad \nearrow & \\ & \langle [\mathbf{A}, F] \rangle. & \end{array}$$

3.2. **Traces in geometric settings.** The geometric avatar for Hochschild homology is the derived loop space (or more generally, the derived fixed points of a self-map), see [BN19, BN12] for extended discussions.

Definition 3.5. Let X be a derived stack.

[19] Roughly, this is the data of $F_{\mathbf{M}} \in \mathbf{End}(\mathbf{M})$ with natural compatibility isomorphisms $F_{\mathbf{M}}(A * M) \simeq F(A) * F_{\mathbf{M}}(M)$ for $A \in \mathbf{A}, M \in \mathbf{M}$, i.e. for a functor to be $\mathbf{A}$-linear is a structure, not merely a property.

[20] There is a natural F-equivariant structure on $1_{\mathbf{A}}$ encoded by the structure of F being a monoidal endofunctor, corresponding to the F-semilinear endofunctor being F itself.

- We define the *derived loop space* $\mathcal{L}X$ (or derived inertia stack) to be
$$\mathcal{L}X = \mathrm{Map}_{\mathbf{DSt}_k}(S^1, X) \simeq X \underset{X\times X}{\times} X$$
i.e. the derived mapping stack from a circle, or more concretely the derived self-intersection of the diagonal.
- More generally, if $\phi : X \to X$ is a self-map, we define the *derived fixed points* or *ϕ-twisted loop space* $\mathcal{L}_\phi X$ to be the fiber product
$$\begin{array}{ccc} \mathcal{L}_\phi X & \longrightarrow & X \\ \big\downarrow{\scriptstyle \mathrm{ev}} & & \big\downarrow{\scriptstyle \Gamma_\phi} \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$
i.e. the derived intersection of the diagonal with the graph $\Gamma_\phi = \mathrm{id}_X \times \phi$ of ϕ. Note that the derived fixed points of the identity is the derived loop space, i.e. $\mathcal{L}_{\mathrm{id}_X} X = \mathcal{L}X$.
- We fix a base point on the circle, e.g. the identity, and denote by $\mathrm{ev} : \mathcal{L}_\phi X \to X$ the evaluation at this base point.[21]
- The formation of derived loop spaces and derived fixed points are functorial, i.e. if $f : X \to Y$ is map of derived stacks, and ϕ_X, ϕ_Y are compatible self-maps, then we have a map of derived stacks $\mathcal{L}_\phi f : \mathcal{L}_\phi X \to \mathcal{L}_\phi Y$.

Example 3.6. For X a scheme we have that the derived loop space $\mathcal{L}X \simeq \mathbb{T}_X[-1]$ is the total space of the shifted tangent complex to X, while for $X = \mathrm{pt}/G$ we have $\mathcal{L}X = G/G \simeq \mathrm{Loc}_G(S^1)$, i.e. the classical inertia stack. For a general stack the loop space is a combination of the shifted tangent complex with the inertia stack.

Example 3.7. For us, the proper self-maps above will arise via a proper action of a group G on X, i.e. for $g \in G(k)$ we obtain a proper map $g : X \to X$. Then, we have the relationship $\mathcal{L}_g X = \mathcal{L}(X/G) \times_{\mathcal{L}(BG)} \{g\}$.

Note the parallel between the loop space, which is the self-intersection of the diagonal (the identity self-correspondence from X) and Hochschild homology (the trace of the identity on a category). As a result the push-pull functoriality of categories of sheaves under correspondences implies an immediate relation between their Hochschild homology and loop spaces. Since QC is functorial under $*$-pullbacks and $\mathrm{QC}^!$ under !-pullbacks, this produces the following answers, both of which hold in particular for QCA stacks (see Corollary 4.2.2 of [DG13], [BN19], and Example 2.2.10 in [Ch20a]):
$$HH(\mathrm{QC}(X), \phi_*) \simeq \Gamma(\mathcal{L}_\phi X, \mathcal{O}_{\mathcal{L}_\phi X}), \qquad HH(\mathrm{QC}^!(X), \phi_*) \simeq \Gamma(\mathcal{L}_\phi X, \omega_{\mathcal{L}_\phi X}).$$
In other words, taking $\phi = \mathrm{id}_X$, the Hochschild homology of $\mathrm{QC}(X)$ (respectively $\mathrm{QC}^!(X)$) is given by functions (respectively volume forms) on the derived loop space. For $X = \mathrm{Spec}(R)$ a smooth affine scheme this recovers the Hochschild-Kostant-Rosenberg identification of Hochschild homology of R-mod with differentials on R,
$$HH(R\text{-mod}) = \mathcal{O}(\mathcal{L}X) = \mathcal{O}(\mathbb{T}_X[-1]) = \mathrm{Sym}^\bullet(\Omega^1_R[1]) = \Omega_R^{-\bullet}.$$

Example 3.8 (Quasicoherent sheaves under tensor product)**.** Let X be a perfect stack in the sense of [BFN10]. Then, $\mathrm{QC}(X)$ has a monoidal structure via tensor product of sheaves. We have that $HH(\mathrm{QC}(X)) = \mathcal{O}(\mathcal{L}X)$ is an algebra object via the shuffle product, and the universal trace $\mathrm{QC}(X) \to \mathbf{Tr}(\mathrm{QC}(X)) = \mathrm{QC}(\mathcal{L}X)$ given by pullback along evaluation at the identity. Furthermore, the monoidal unit is $\mathcal{O}_X \in \mathrm{QC}(X)$ with trace $[\mathcal{O}_X] = \mathcal{O}_{\mathcal{L}X} \in \mathrm{QC}(\mathcal{L}X)$. Finally, we have
$$\mathcal{O}(\mathcal{L}X)\text{-mod} \simeq \langle \mathcal{O}_{\mathcal{L}X} \rangle \subset \mathrm{QC}(\mathcal{L}X)$$
where the fully faithful inclusion is an equivalence if X is affine.

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21 For any other two choices of base point s_1, s_2, it is possible to consistently identify the maps $\mathrm{ev}_{s_1} \simeq \mathrm{ev}_{s_2}$ by choice of path in the circle.

We now establish a certain Calabi-Yau property of derived fixed points of smooth stacks (or more generally, smooth maps). In our arguments it will be useful to factor the loop space of a map $\mathcal{L}f : \mathcal{L}X \to \mathcal{L}Y$ through the following intermediate derived stack, which we define in three equivalent ways.

Definition 3.9. Let $f : X \to Y$ be a map of derived stacks with compatible self-maps ϕ_X, ϕ_Y, and define $Z := X \times_Y X$. We define $\mathcal{L}_\phi Y_X$ via the pullback diagrams:

$$\begin{array}{ccc} \mathcal{L}_\phi Y_X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Gamma_\phi} \\ Z & \longrightarrow & X \times X \end{array} \qquad \begin{array}{ccc} \mathcal{L}_\phi Y_X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f\times\phi_X} \\ X & \xrightarrow[f\times \mathrm{id}_X]{} & Y \times X \end{array} \qquad \begin{array}{ccc} \mathcal{L}_\phi Y_X & \longrightarrow & \mathcal{L}_\phi Y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Roughly, this is the derived moduli stack of paths in X mapping to loops in Y.

The following lemma is a straightforward verification of the depicted diagrams, which we leave to the reader.

Lemma 3.10. *The above three presentations are canonically equivalent, and we have a canonical factorization*

$$\mathcal{L}_\phi X \xrightarrow{\delta} \mathcal{L}_\phi Y_X \xrightarrow{\pi} \mathcal{L}_\phi Y$$

where the maps are realized via the base change

$$\begin{array}{ccccc} \mathcal{L}_\phi X & \xrightarrow{\delta} & \mathcal{L}_\phi Y_X & \longrightarrow & X \\ {\scriptstyle \mathrm{ev}_X}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Gamma_{\phi_X}} \\ X & \xrightarrow[\Delta_f = \Delta_{X/Y}]{} & Z & \longrightarrow & X \times X \end{array} \qquad \begin{array}{ccccc} \mathcal{L}_\phi Y_X & \xrightarrow{\pi} & \mathcal{L}_\phi Y & \longrightarrow & Y \\ {\scriptstyle \mathrm{ev}_{X/Y}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Gamma_{\phi_Y}} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[\Delta_Y]{} & Y \times Y. \end{array}$$

i.e. δ is a base change of the relative diagonal for f, and π is a base change of f itself.

Example 3.11. When ϕ is the identity and $Y = \mathrm{pt}$, the factorization above is just $\mathcal{L}X \to X \to \mathrm{pt}$.

When X is a smooth stack, there is an equivalence of categories $\mathrm{Perf}(X) = \mathrm{Coh}(X)$. Thus, by the above we expect that $\mathcal{O}(\mathcal{L}X) \simeq \omega(\mathcal{L}X)$. It turns out that this equivalence on global sections comes from a map on the underlying sheaves themselves. We now establish the following Calabi-Yau property of derived fixed points of smooth stacks, which we will use repeatedly in our arguments. We refer the reader to Section 8 of [AG14] for discussion of quasi-smoothness for derived Artin stacks.

Lemma 3.12. *Let X, Y be derived Artin stacks equipped with proper self-maps ϕ_X, ϕ_Y, and let $f : X \to Y$ be a compatible smooth relative Artin 1-stack.*[22] *Then, there is a canonical equivalence of functors*

$$\mathcal{L}_\phi f^! \simeq \mathcal{L}_\phi f^* : \mathrm{QC}^!(\mathcal{L}_\phi Y) \longrightarrow \mathrm{QC}^!(\mathcal{L}_\phi X).$$

In particular, if X is a smooth Artin 1-stack with a proper self-map ϕ, then $\omega_{\mathcal{L}_\phi X} \simeq \mathcal{O}_{\mathcal{L}_\phi X}$.

Proof. Following the notation and factorization in Lemma 3.10, we have canonical identifications:

$$\omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y_X} \simeq \mathrm{ev}_X^* \omega_{X/Z}, \qquad \omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y} \simeq \mathrm{ev}_{X/Y}^* \omega_{X/Y}.$$

Furthermore, after choosing[23] one of the projections $Z = X \times_Y X \to X$, the usual exact triangle for cotangent complexes for the composition $X \to Z \to X$ gives a canonical equivalence

$$\omega_{X/Z} \simeq \Delta_{X/Y}^* \omega_{Z/X}^{-1} \simeq \omega_{X/Y}^{-1}.$$

[22] By this we mean such that the relative cotangent complex is perfect of Tor amplitude $[0,1]$, i.e. the fibers are are allowed to be stacky, and in particular, this map does not need to be representable by schemes.

[23] The definition of Hochschild homology implicitly requires us to choose an orientation on the circle S^1. We make one such choice, once and for all, which forces a particular choice here (i.e. a choice of sign).

Thus, we have a canonical equivalence

$$\omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y} \simeq \mathrm{ev}_X^* \omega_{X/Y}^{-1} \otimes \delta^* \mathrm{ev}_{X/Y}^* \omega_{X/Y} \simeq \mathcal{O}_{\mathcal{L}_\phi X}.$$

By assumption the contangent complex $\mathbb{L}_f$ is perfect in degrees $[0,1]$, so the relative cotangent complex $\mathbb{L}_{\Delta_{X/Y}}$ is perfect in degrees $[-1,0]$; in particular, $\Delta_{X/Y}$ is representable by schemes and quasi-smooth and thus we have a canonical equivalence (see Proposition 7.3.8 of [Ga13]) $\mathcal{L}_\phi f^! \simeq \mathcal{L}_\phi f^* \otimes \omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y} \simeq \mathcal{L}_\phi f^*$ as desired. ☐

Furthermore, by functoriality of Hochschild homology, for a map of stacks $f : X \to Y$ we expect that the pullback and pushforward functors define maps of global functions or volume forms $HH(f^*) : \mathcal{O}(\mathcal{L}Y) \to \mathcal{O}(\mathcal{L}X)$ and (if f is proper) $HH(f_*) : \omega(\mathcal{L}X) \to \omega(\mathcal{L}Y)$. We identify this map with the global sections of a natural map on the underlying sheaves in two cases of concern (see Appendix A.1 for the proof).

Definition 3.13. Let $f : X \to Y$ be a map of QCA stacks, and ϕ_X, ϕ_Y compatible proper self-maps.

- If f is proper, then we have a pushforward map $\omega(\mathcal{L}_\phi f_*) : \omega(\mathcal{L}_\phi X) \to \omega(\mathcal{L}_\phi Y)$ of global volume forms. That is, by Remark 4.6 in [BN19], since f is proper, $\mathcal{L}_\phi f : \mathcal{L}_\phi X \to \mathcal{L}_\phi Y$ is proper; $\omega(\mathcal{L}_\phi f_*)$ is the global sections of the counit of the adjunction $(\mathcal{L}_\phi f_*, \mathcal{L}_\phi f^!)$ applied to $\omega_{\mathcal{L}_\phi Y}$.
- If f is smooth, then we have a "Gysin" pullback $\omega(\mathcal{L}_\phi f^*) : \omega(\mathcal{L}_\phi Y) \to \omega(\mathcal{L}_\phi X)$ of global volume forms. That is, by Proposition 3.12, if f is smooth then $\mathcal{L}_\phi f$ is Calabi-Yau; passing through this equivalence, $\omega(\mathcal{L}_\phi f^*)$ is the global sections of the unit of the adjunction $(\mathcal{L}_\phi f^*, \mathcal{L}_\phi f_*)$ applied to $\omega_{\mathcal{L}_\phi Y}$.

Proposition 3.14. *Let $f : X \to Y$ be map of QCA stacks with compatible proper self-maps ϕ_X, ϕ_Y. We consider the following functors, which preserve compact objects.*

- *There are canonical identifications*

$$HH(\mathrm{QC}^!(X), \phi_*) \simeq \omega(\mathcal{L}_\phi X).$$

- *Suppose f is proper, and consider $f_* : \mathrm{QC}^!(X) \to \mathrm{QC}^!(Y)$. Then, the map $HH(f_*, \phi_*)$ is canonically identified with the map on global volume forms $\omega(\mathcal{L}_\phi f_*)$.*
- *Suppose that f is smooth, and consider $f^* : \mathrm{QC}^!(Y) \to \mathrm{QC}^!(X)$. Then, the map $HH(f^*, \phi_*)$ is canonically identified with the map on volume forms $\omega(\mathcal{L}_\phi f^*)$.*

3.3. **Convolution patterns in Hochschild homology.** Convolution patterns in Borel-Moore homology and algebraic K-theory play a central role in the results of [CG97]. We now describe a similar pattern which appears in Hochschild homology.

Definition 3.15. We will work with the following general setup (see Section 1.5 of [BNP17b]).

- $f : X \to Y$ is a proper morphism of smooth, QCA stacks over k, and $Z = X \times_Y X$.
- $\phi_X : X \to X$ and $\phi_Y : Y \to Y$ are (representable) proper self-maps commuting with f, inducing a proper self-map $\phi : Z \to Z$.

We refer to any Z arising from the set-up above a *convolution space*, and call the category $\mathrm{QC}^!(Z)$ a *convolution category.*

In this setup the category $\mathrm{QC}^!(Z)$ carries a monoidal structure under convolution[24], and ϕ_* is a monoidal endofunctor. The convolution monoidal structure restricts to the compact objects $\mathrm{Coh}(Z)$ thanks to the smoothness of X (hence finite Tor-dimension of the diagonal of X) and the properness of f; furthermore, since ϕ is proper, ϕ_* has a continuous right adjoint, and preserves $\mathrm{Coh}(Z)$.

[24] As explained in Remark 3.0.7 and Lemma 3.0.8 of [BNP17a], on the compact objects $\mathrm{Coh}(Z)$ there are two monoidal products, given by $*$- or !-convolution, intertwined by Grothendieck duality. We will default to the !-version, which is amenable to the ind-completed category $\mathrm{QC}^!(Z)$.

By Theorem 1.1.3 of [BNP17a], there is an equivalence of small monoidal categories[25]

$$(\mathrm{Coh}(X \times_Y X), *) \simeq (\mathrm{Fun}^{ex}_{\mathrm{Perf}(Y)}(\mathrm{Coh}(X), \mathrm{Coh}(X)), \circ).$$

Moreover, we will argue in Theorem 3.25 that $(\mathrm{QC}^!(Z), *)$ is rigid monoidal. The monoidal unit is the dualizing sheaf of the relative diagonal $\omega_\Delta := \iota_* \omega_X$, where $\iota : X \to X \times_Y X$.

Recall (from Section 2.1.1) that the Hochschild homology of $\mathrm{Coh}(Z)$ (or equivalently of its large variant $\mathrm{QC}^!(Z)$ by Remark 2.2.11 of [Ch20a]) for a stack Z is given geometrically by volume forms on the loop space, or in the case of the trace of ϕ_* the derived fixed points:

$$HH(\mathrm{QC}^!(Z), \phi_*) \simeq \Gamma(\mathcal{L}_\phi Z, \omega_{\mathcal{L}_\phi Z}).$$

Thus the vertical trace of the monoidal category $\mathrm{Coh}(Z)$ defines an algebra structure on global distributions $\Gamma(\mathcal{L}_\phi Z, \omega_{\mathcal{L}_\phi Z})$.

We want to relate this convolution structure on sheaves its decategorified version involving volume forms on the corresponding loop spaces. Thus we consider the loop map $\mathcal{L}_\phi f : \mathcal{L}_\phi X \to \mathcal{L}_\phi Y$ to f, whose self-fiber product is $\mathcal{L}_\phi Z \simeq \mathcal{L}_\phi X \simeq_{\mathcal{L}_\phi Y} \mathcal{L}_\phi X$. Note that $\mathcal{L}_\phi f$ is a proper map of quasismooth derived stacks. In particular, $\omega_{\mathcal{L}_\phi X}$ is coherent (a compact object in $\mathrm{QC}^!(\mathcal{L}_\phi X)$) and $\mathcal{L}_\phi f_*$ preserves coherence. We thus define our main object of interest:

Definition 3.16. We define the *universal trace sheaf*

$$\mathcal{S}_{X/Y,\phi} := \mathcal{L}_\phi f_* \omega_{\mathcal{L}_\phi X} \simeq \mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X} \in \mathrm{Coh}(\mathcal{L}_\phi Y).$$

The latter isomorphism follows since the loop space of smooth stacks are naturally Calabi-Yau (see Lemma 3.12).

The endomorphisms of the universal trace sheaf have a close relationship to volume forms on the loop space of the convolution space. Namely, we have a canonical equivalence

$$\omega(\mathcal{L}_\phi Z) \simeq \mathrm{End}_{\mathcal{L}_\phi Y}(\mathcal{S}_{X/Y,\phi}).$$

Furthermore, these equivalences are functorial; on the left, this was discussed in Definition 3.13. On the right, the functoriality arises via the following functoriality of the universal trace sheaf.

Definition 3.17. Let (X, Y, f, ϕ) and (X', Y', f', ϕ') as in Definition 3.15 (with convolution spaces Z, Z'). Suppose we have maps $\alpha_X : X \to X'$ and $\alpha_Y : Y \to Y'$ commuting with f, f', inducing $\alpha_Z : Z \to Z'$. Then, we have the following due to base change.

- Suppose that $X = X'$ and that α_Y is proper. Then, there is a canonical equivalence $\mathcal{L}\alpha_{Y*}\mathcal{S}_{X/Y,\phi} \simeq \mathcal{S}_{X'/Y',\phi'}$, and the functor $\alpha_{Z*} : \mathrm{Coh}(Z) \to \mathrm{Coh}(Z')$ is monoidal.
- Suppose that α_Y is smooth and f is base-changed from f', i.e. $X = X' \times_{Y'} Y$. Then there is a canonical equivalence $\mathcal{L}\alpha_Y^! \mathcal{S}_{X'/Y',\phi'} \simeq \mathcal{S}_{X/Y,\phi}$, and the functor $\alpha_Z^! : \mathrm{Coh}(Z') \to \mathrm{Coh}(Z)$ is monoidal.

The functorialities on the two sides of the equivalence are compatible. We summarize our above discussion in the following.

Proposition 3.18. *We let $p : Z \to Y$ denote the structure map. In the set-up of Definition 3.15, we have canonical equivalences*

$$\zeta : \mathcal{L}_\phi p_* \omega_{\mathcal{L}_\phi Z} \simeq \mathcal{E}nd_{\mathcal{L}_\phi Y}(\mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X})$$

such that if $\alpha : Y \to Y'$ is proper and $X = X'$, we have commuting squares

$$\begin{array}{ccc} \mathcal{L}_\phi \alpha_* \mathcal{L}_\phi p_* \omega_{\mathcal{L}_\phi Z} & \xrightarrow[\simeq]{\mathcal{L}_\phi \alpha_*(\zeta)} & \mathcal{L}_\phi \alpha_* \mathcal{E}nd_{\mathcal{L}_\phi Y}(\mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X}) \\ \downarrow {\scriptstyle \mathrm{Def.3.13}} & & \downarrow {\scriptstyle \mathrm{Def.3.17}} \\ \mathcal{L}_\phi p'_* \omega_{\mathcal{L}_\phi Z'} & \xrightarrow[\simeq]{\zeta'} & \mathcal{E}nd_{\mathcal{L}_\phi Y'}(\mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X'}). \end{array}$$

[25]Via the discussion in Section 4.7 of [Lur18], endofunctor categories naturally possess the structure of an associative monoidal ∞-category. Theorem 1.1.3 in [BNP17a] identifies the underlying categories, with convolution corresponding to composition object-by-object. Thus we can simply define the monoidal structure (with all its higher coherence compatibilities) on the left by transporting it from the right.

while if $\alpha : Y \to Y'$ *is smooth and* $X = X' \times_{Y'} Y$*, we have commuting squares*

$$\begin{array}{ccc} \mathcal{L}_\phi p'_* \omega_{\mathcal{L}_\phi Z'} & \xrightarrow[\simeq]{\zeta'} & \mathcal{E}nd_{\mathcal{L}_\phi Y'}(\mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X'}) \\ \Big\downarrow {\scriptstyle \text{Def.3.13}} & & \Big\downarrow {\scriptstyle \text{Def.3.17}} \\ \mathcal{L}_\phi \alpha_* \mathcal{L}_\phi p_* \omega_{\mathcal{L}_\phi Z} & \xrightarrow[\simeq]{\mathcal{L}_\phi \alpha_*(\zeta)} & \mathcal{L}_\phi \alpha_* \mathcal{E}nd_{\mathcal{L}_\phi Y}(\mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X}). \end{array}$$

Proof. Application of Proposition A.1, noting that if f is smooth then $\mathcal{L}_\phi f$ is Calabi-Yau by Proposition 3.12. □

Remark 3.19 (Convolution of volume forms and endomorphisms of $\mathcal{S}_{X/Y}$). Applying the above proposition to $\mathcal{L}_\phi f : \mathcal{L}_\phi X \to \mathcal{L}_\phi Y$, i.e. if we sheafify over $\mathcal{L}_\phi Y$, we can identify this algebra structure more concretely as convolution of volume forms on $\mathcal{L}_\phi Z$. That is, $\mathcal{L}_\phi Z = \mathcal{L}_\phi X \times_{\mathcal{L}_\phi Y} \mathcal{L}_\phi X$ has the structure of proper monoid in stacks over $\mathcal{L}_\phi Y$, from which one deduces the structure of algebra object in $(\mathrm{QC}^!(\mathcal{L}_\phi Y), \otimes^!)$ on the pushforward of $\omega_{\mathcal{L}_\phi Z}$. One can also use proper descent for $\mathcal{L}_\phi f : \mathcal{L}_\phi X \to \mathcal{L}_\phi Y$ to identify this sheaf of algebras with the internal endomorphism sheaf of $\mathcal{S}_{X/Y}$ – an analog, in the setting of derived categories of coherent sheaves on derived stacks, of the standard proof (see e.g. [CG97]) that self-Ext of the Springer sheaf is identified with Borel-Moore homology of Z. It would be interesting to see how these arguments globalize over $\mathcal{L}_\phi Y$ to give the isomorphism $\Gamma(\mathcal{L}_\phi Z, \omega_{\mathcal{L}_\phi Z}) \simeq \mathrm{End}_{\mathrm{QC}^!(\mathcal{L}_\phi Y)}(\mathcal{S}_{X/Y})$ of Theorem 3.25.

3.3.1. *Horizontal trace of convolution categories.* Recall that Theorem 3.4 identifies the vertical trace $HH(\mathrm{QC}^!(Z)), *)$ as the endomorphism algebra of the distinguished object in the horizontal trace $\mathbf{Tr}(\mathrm{QC}^!(Z), *)$, under the assumption that this distinguished object is compact (and a rigidity condition to be addressed in Proposition 3.25). In this section we discuss this horizontal trace in the context of convolution spaces following [BNP17b], slightly generalizing the main theorem of *op. cit.*

For this we require a discussion of singular supports; we summarize the main points and refer the reader to [AG14, BNP17b] for details. Note that singular supports do not appear in our main application Theorem 4.12, since the singular support condition there is actually a classical support condition (see Remark 4.14).

Definition 3.20. Let $f : X \to Y$ be a representable map of quasi-smooth stacks.

- We define the *scheme of singularities* or (classical) *odd cotangent bundle* to be
$$\mathbb{T}^{*[-1]}_X := \mathrm{Spec}_X \mathrm{Sym}^\bullet_X H^1(\mathcal{T}_X) = \mathrm{Spec}_X \mathrm{Sym}^\bullet_X H^0(\mathcal{T}_X[1])$$
where $\mathcal{T}_X$ denotes the tangent complex of X, i.e. the $\mathcal{O}_X$-linear dual of the cotangent complex.
- Any ind-coherent sheaf $\mathcal{F} \in \mathrm{QC}^!(X)$ has a closed conical *singular support* $\mathrm{SS}(\mathcal{F}) \subset \mathbb{T}^{*[-1]}_X$. To any subset $\Lambda \subset \mathbb{T}^{*[-1]}_X$ we can associate the full category $\mathrm{QC}^!_\Lambda(X) \subset \mathrm{QC}^!(X)$ consisting of sheaves with the specified singular support.
- Let $\Lambda_X \subset \mathbb{T}^{*[-1]}_X$ and $\Lambda_Y \subset \mathbb{T}^{*[-1]}_Y$. One can push forward $f_*\Lambda_X$ and pull back $f^!\Lambda_Y$ singular support conditions in a compatible way with the pullback and pushforward functors:
$$f_* : \mathrm{QC}^!_{\Lambda_X}(X) \to \mathrm{QC}^!_{f_*\Lambda_X}(Y), \qquad f^! : \mathrm{QC}^!_{\Lambda_Y}(Y) \to \mathrm{QC}^!_{f^!\Lambda_Y}(X).$$

Example 3.21. If X is smooth, then $\mathbb{T}^{*[-1]}_X = X$, i.e. there are no possible singular codirections to consider. In particular, the nontrivial fibers of the map $\mathbb{T}^{*[-1]}_X \to X$ live over the singular locus of X.

When $\Lambda = \mathbb{T}^{*[-1]}_X$, we have $\mathrm{QC}^!_\Lambda(X) = \mathrm{QC}^!(X)$. At the opposite extreme, when $\Lambda = \{0\}_X$ is the zero section, we have $\mathrm{QC}^!_\Lambda(X) = \mathrm{QC}(X)$. If $Z \subset X$ is a closed subscheme and $\Lambda = Z \times_X \mathbb{T}^{*[-1]}_X$, then $\mathrm{QC}^!_\Lambda(X) = \mathrm{QC}^!_Z(X)$, i.e. the full subcategory of ind-coherent sheaves with classical support at $Z \subset X$. If instead we take $\Lambda = Z \times \{0\}_X$, then $\mathrm{QC}^!_\Lambda(X) = \mathrm{QC}_Z(X)$.

The following singular support condition appears when taking traces of convolution categories.

Definition 3.22. Recall the notation from Definition 3.5 and Definition 3.9. We have the following *trace correspondence*:

$$Z = X \times_Y X \xleftarrow{\delta} \mathcal{L}_\phi Y_X = Z \underset{X \times X}{\times} X \simeq X \underset{Y \times X}{\times} X \xrightarrow{\pi} \mathcal{L}_\phi Y.$$

We define a singular support condition $\Lambda_{X/Y,\phi} := \pi_* \delta^! \mathbb{T}_Z^{*[-1]}$.

We now give a description of the horizontal trace. The following statement is more general than the statement of Theorem 3.3.1 in [BNP17b], but follows from the same argument in the proof with the definitions given above; the proof is in Appendix A.2.

Theorem 3.23. *There is a canonical identification of the horizontal trace (i.e. the monoidal Hochschild homology)*

$$\mathbf{Tr}((\mathrm{QC}^!(Z), *), \phi_*) \simeq \mathrm{QC}^!_{\Lambda_{X/Y,\phi}}(\mathcal{L}_\phi Y),$$

with the universal trace given by[26]

$$[-] = \pi_* \delta^! : \mathrm{QC}^!(X \times_Y X) \to \mathrm{QC}^!_{\Lambda_{X/Y,\phi}}(\mathcal{L}_\phi Y).$$

Next we identify the universal trace sheaf (i.e. coherent Springer sheaf) as the trace of the monoidal unit (which is a compact object of the trace category) or regular representation:

Lemma 3.24. *There is a natural equivalence* $\mathcal{S}_{X/Y,\phi} \simeq [\omega_\Delta] = \pi^* \delta^! \omega_\Delta$ *in* $\mathrm{Coh}(\mathcal{L}_\phi Y)$.

Proof. The calculation of $\delta^! \omega_\Delta = \delta^! \Delta_* \omega_X$ arises via base change along the diagram

$$\begin{array}{ccc} \mathcal{L}_\phi X & \longrightarrow & \mathcal{L}_\phi Y_X = Z \times_{X \times X} X \\ \downarrow & & \downarrow \\ X & \xrightarrow[\Delta]{} & Z = X \times_Y X \end{array}$$

and the statement follows. □

3.3.2. *Trace delooping in convolution categories.* We now deduce the main structural relation between universal trace sheaves (see Definition 3.16) and iterated categorical traces of convolution categories.

Theorem 3.25. *Let* $f : X \to Y$ *be as in Definition 3.15. Then, the convolution category* $\mathrm{QC}^!(X \times_Y X)$ *is rigid. In particular, the statements of Theorem 3.4 apply: the vertical trace of the convolution category* $(\mathrm{QC}^!(Z), *)$ *is identified as an algebra with the endomorphisms of the universal trace sheaf*

$$HH(\mathrm{QC}^!(X \times_Y X), \phi_*) \simeq \mathrm{End}_{\mathrm{QC}^!(\mathcal{L}_\phi Y)}(\mathcal{L}_\phi f_* \omega_{\mathcal{L}_\phi X})$$

compatibly with the natural S^1*-actions (from the cyclic trace and loop rotation, respectively).*

Proof. We need to verify that $\mathrm{QC}^!(Z)$ is rigid monoidal. Standard arguments show that integral transforms arising via coherent sheaves preserve compact objects; this statement is also contained within Theorem 1.1.3 in [BNP17a]; one further immediately observes that the monoidal unit $\Delta_* \omega_X$ is a compact object, i.e. coherent, since the diagonal is a closed embedding. It remains to verify that the right and left duals of coherent sheaves $\mathcal{K} \in \mathrm{QC}^!(Z)$ are again coherent. Using *loc. cit.*, it suffices to show that the right and left adjoints of the corresponding integral transform $F_\mathcal{K} : \mathrm{QC}(X) \to \mathrm{QC}(X)$ preserve compact objects. We note that since the projection maps $p : Z \to X$ are quasi-smooth, the functors $p^!$ and p^* differ by a shifted line bundle. By Lemma 3.0.8 in *op. cit.* we can consider equivalently either the $*$ or !-transforms up to twisting by Grothendieck duality. For convenience we will consider the $*$-transform.

[26] Note that our trace functor is given by $\delta^!$ rather than the δ^* in [BNP17b], since we employ the !-transform rather than the $*$-transform.

To see the claim, note that we can write the $*$-integral transform $F_{\mathcal{K}}$ as a composition:

$$\mathrm{QC}(X) \xrightarrow{p^*} \mathrm{QC}(Z) \xrightarrow{-\otimes\mathcal{K}} \mathrm{QC}^!(Z) \xrightarrow{p_*} \mathrm{QC}^!(X).$$

We claim that the right adjoint preserves compact objects. The claim for the left adjoint follows similarly by replacing p^* with a twist of $p^!$ by a shifted line bundle. The right adjoints define a sequence of functors

$$\mathrm{QC}(X) \xleftarrow{p_*} \mathrm{QC}(Z) \xleftarrow{\mathcal{H}om_{\mathrm{QC}^!(Z)}(\mathcal{K},-)} \mathrm{QC}^!(Z) \xleftarrow{p^!=p^*\otimes\mathcal{L}} \mathrm{QC}^!(X).$$

The functor $\mathcal{H}om_{\mathrm{QC}^!(Z)}(\mathcal{K},-) : \mathrm{QC}^!(Z) \to \mathrm{QC}(Z)$ is defined as follows. Given $\mathcal{G} \in \mathrm{QC}^!(Z)$, we may write $\mathcal{G} = \mathrm{colim}_i\, \mathcal{G}_i$ with $\mathcal{G}_i \in \mathrm{Coh}(Z)$. Since $\mathcal{K}$ is compact, we may define:

$$\mathcal{H}om_{\mathrm{QC}^!(Z)}(\mathcal{K},\mathcal{G}) := \lim_i \mathcal{H}om_Z(\mathcal{K},\mathcal{G}_i) \in \mathrm{QC}(Z)$$

where the internal Hom on the right is taken inside $\mathrm{Coh}(Z) \subset \mathrm{QC}(Z)$ as usual. Let us justify the claim that this functor is a right adjoint to tensoring with $\mathcal{K}$. Let $\mathcal{F} \in \mathrm{QC}(Z)$, and write $\mathcal{F} = \mathrm{colim}_j\, \mathcal{F}_j$ with $\mathcal{F}_j \in \mathrm{Perf}(Z)$. Then, by the usual adjunction in $\mathrm{QC}(Z)$, and using the facts that the $\mathcal{F}_j$ are compact in $\mathrm{QC}(Z)$ and that $\mathcal{F}_j \otimes \mathcal{K} \in \mathrm{Coh}(Z)$ are compact in $\mathrm{QC}^!(Z)$ since $\mathcal{F}_j$ are perfect, we have:

$$\mathrm{Hom}_{\mathrm{QC}(Z)}(\mathcal{F}, \mathcal{H}om_{\mathrm{QC}(Z)}(\mathcal{K},\mathcal{G})) \simeq \mathrm{Hom}_{\mathrm{QC}(Z)}(\mathrm{colim}_j\, \mathcal{F}_j, \lim_i \mathcal{H}om_Z(\mathcal{K},\mathcal{G}_i))$$

$$\simeq \lim_{i,j} \mathrm{Hom}_{\mathrm{QC}(Z)}(\mathcal{F}_j, \mathcal{H}om_Z(\mathcal{K},\mathcal{G}_i)) \simeq \lim_{i,j} \mathrm{Hom}_{\mathrm{QC}^!(Z)}(\mathcal{F}_j \otimes \mathcal{K}, \mathcal{G}_i) \simeq \mathrm{Hom}_{\mathrm{QC}^!(Z)}(\mathcal{F} \otimes \mathcal{K}, \mathcal{G}).$$

Finally, we verify that $\mathcal{H}om_{\mathrm{QC}^!(Z)}(\mathcal{K},-)$ sends $\mathrm{Perf}(Z)$ to $\mathrm{Coh}(Z)$, which implies that the sequence of right adjoints above preserves compact objects. The Grothendieck dual $\mathbb{D}(\mathcal{K}) = \mathcal{H}om_Z(\mathcal{K},\omega_Z)$ is coherent, and since Z is quasi-smooth, ω_Z is a line bundle, so we have for $\mathcal{E} \in \mathrm{Perf}(Z)$:

$$\mathcal{H}om_{\mathrm{QC}^!(Z)}(\mathcal{K},\mathcal{E}) = \mathcal{H}om_{\mathrm{QC}^!(Z)}(\mathcal{K},\omega_Z) \otimes_{\mathcal{O}_Z} \omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{E} \simeq \mathbb{D}(\mathcal{K}) \otimes_{\mathcal{O}_Z} \omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{E}$$

which is coherent. □

3.4. **Trace of the standard categorical representation.** In Lemma 3.24, we have computed the trace of the regular representation $\mathrm{QC}^!(Z)$ of $\mathrm{QC}^!(Z)$ to be the universal trace sheaf, i.e. $[\mathrm{QC}^!(Z),\phi_*] \simeq \mathcal{S}_{X/Y,\phi} := \mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X}$. Our convolution set-up comes equipped with another natural module category: the the *standard representation*, i.e. the module category $\mathrm{QC}^!(X)$. In this section we compute the trace of this categorical representation, and relate it to the trace of the regular representation in a special case. We first note a degenerate example.

Example 3.26. Consider the case when $X = Y = Z$ is smooth. In this case, $\mathrm{QC}^!(Y) = \mathrm{QC}(Y)$, and the trace correspondence of Definition 3.22 is simply given by pullback along the evaluation $\mathrm{ev} : \mathcal{L}_\phi Y \to Y$:

$$Y \xleftarrow{\mathrm{ev}} \mathcal{L}_\phi Y = \mathcal{L}_\phi Y.$$

In this case, the standard representation is the regular representation, and by Theorem 3.3.1 of [BNP17b] (and Proposition 3.12), the trace of the regular representation is

$$[\mathrm{QC}^!(Y),\phi_*] = [\omega_Y] = \omega_{\mathcal{L}_\phi Y} \simeq \mathcal{O}_{\mathcal{L}_\phi Y}$$

and the corresponding singular support condition $\Lambda_{Y/Y,\phi} = \{0\}_{\mathcal{L}_\phi Y}$ is the zero section, i.e. we have $\mathbf{Tr}(\mathrm{QC}^!(Y),\phi_*) = \mathrm{QC}(\mathcal{L}_\phi Y)$ (see Corollary 5.2 of [BFN10]).

We recall a few notions from Section 2.3 of [BNP17b]. The following functors allow us to pass between categories with different singular supports.

Definition 3.27. For a pair (X, Λ_X), there is an adjoint pair of functors (see Definition 2.3.2 of [BNP17b]):

$$\iota_\Lambda : \ \mathrm{QC}^!_\Lambda(X) \rightleftarrows \mathrm{QC}^!(X) \ : \Gamma_\Lambda$$

where ι_Λ is the the natural inclusion, and Γ_Λ is the corresponding colocalization.[27]

We need an identification of the relative tensor product of convolution categories, with specified support. We work in the set-up of Definition 3.15: let X_i be smooth QCA stacks over k, proper over Y, and let $Z_{ij} = X_i \times_Y X_j$.

Definition 3.28. Let $\Lambda_{12} \subset \mathbb{T}^{*[-1]}_{Z_{12}}$ and $\Lambda_{23} \subset \mathbb{T}^{*[-1]}_{Z_{23}}$. Consider the diagram

$$Z_{12} \times Z_{23} \xleftarrow{\ \delta\ } X_1 \times_Y X_2 \times_Y X_3 \xrightarrow{\ \pi\ } Z_{13}.$$

We define the *convolution of singular supports*

$$\Lambda_{12} * \Lambda_{23} = \pi_* \delta^! (\Lambda_{12} \boxtimes \Lambda_{23}).$$

We say that Λ_{ij} is Z_{ii}*-stable* if $\mathbb{T}^{*[-1]}_{Z_{ii}} * \Lambda_{ij} \subset \Lambda_{ij}$.

Remark 3.29. The trace singular support condition $\Lambda_{X/Y}$ of Definition 3.22 can be viewed as the convolution of $\mathbb{T}^{*[-1]}_Z$ with itself "in a circle."

We immediately observe that the convolution action restricts to an action of $\mathrm{QC}^!_{\Lambda_{ii}}(Z_{ii})$ on $\mathrm{QC}^!_{\Lambda_{ij}}(Z_{ij})$ if and only if Λ_{ij} is Λ_{ii}-stable. In particular, we have the following identification, which we prove in Appendix A.2; a proof will also appear in [CD21].

Proposition 3.30. *In the set-up above, let $\Lambda_{12} \subset \mathbb{T}^{*[-1]}_{Z_{12}}$ and $\Lambda_{23} \subset \mathbb{T}^{*[-1]}_{Z_{23}}$ be Z_{22}-stable. Define $\Lambda_{13} := \Lambda_{12} * \Lambda_{23}$. Then convolution defines an equivalence of categories:*

$$\mathrm{QC}^!_{\Lambda_{12}}(Z_{12}) \otimes_{\mathrm{QC}^!(Z_{22})} \mathrm{QC}^!_{\Lambda_{23}}(Z_{23}) \xrightarrow{\ \simeq\ } \mathrm{QC}^!_{\Lambda_{13}}(Z_{13}).$$

Furthermore, we have the following functoriality of supports: let $\Lambda_{i,i+1} \subset \Lambda'_{i,i+1}$ be another singular support condition on $Z_{i,i+1}$ (for $i = 1, 2$) with $\Lambda'_{13} := \Lambda'_{12} * \Lambda'_{23}$. Then, $\Lambda_{13} \subset \Lambda'_{13}$, and the following squares commute:

$$\begin{array}{ccc}
\mathrm{QC}^!_{\Lambda_{12}}(Z_{12}) \underset{\mathrm{QC}^!(Z_{22})}{\otimes} \mathrm{QC}^!_{\Lambda_{23}}(Z_{23}) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\Lambda_{13}}(Z_{13}) \\
{\scriptstyle \iota_{\Lambda_{12}} \otimes \iota_{\Lambda_{23}}} \Big\downarrow \Big\uparrow {\scriptstyle \Gamma_{\Lambda_{13}} \otimes \Gamma_{\Lambda_{23}}} & & {\scriptstyle \iota_{\Lambda_{13}}} \Big\downarrow \Big\uparrow {\scriptstyle \Gamma_{\Lambda_{13}}} \\
\mathrm{QC}^!_{\Lambda'_{12}}(Z_{12}) \underset{\mathrm{QC}^!(Z_{22})}{\otimes} \mathrm{QC}^!_{\Lambda'_{23}}(Z'_{23}) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\Lambda'_{13}}(Z_{13}).
\end{array}$$

We now compute the the trace of the categorical representation, which arises via functoriality of horizontal traces (see Section 3.5 of [BN19] for details). Namely, consider the "renormalized (ind-coherent) Morita invariance" functor

$$T(-) := \mathrm{QC}^!(X) \otimes_{\mathrm{QC}(Y)} - : \mathrm{QC}(Y)\textbf{-mod} = \mathrm{QC}^!(Y)\textbf{-mod} \longrightarrow \mathrm{QC}^!(Z)\textbf{-mod}.$$

Note that the $\mathrm{QC}(Y)$ action on $\mathrm{QC}(X) = \mathrm{QC}^!(X)$ via pullback commutes with the $\mathrm{QC}^!(Z)$-action by convolution. This functor defines a functor on horizontal traces (note that, as discussed in Example 3.21, $\mathrm{QC}^!_{\{0\}_{\mathcal{L}_\phi Y}}(\mathcal{L}_\phi Y) = \mathrm{QC}(\mathcal{L}_\phi Y)$):

$$\mathbf{Tr}(T, \phi_*) : \mathbf{Tr}(\mathrm{QC}(Y), \phi_*) = \mathrm{QC}^!_{\{0\}_{\mathcal{L}_\phi Y}}(\mathcal{L}_\phi Y) \longrightarrow \mathbf{Tr}(\mathrm{QC}^!(Z), \phi_*) = \mathrm{QC}^!_{\Lambda_{X/Y,\phi}}(\mathcal{L}_\phi Y).$$

[27] I.e. a "projection" functor to the subcategory $\mathrm{QC}^!_\Lambda(X)$, which we view as a singular support analogue of local cohomology. Note the abusive notation, i.e. the local cohomology functor usually refers to the functor $\iota_\Lambda \circ \Gamma_\Lambda$.

There is a canonical endofunctor $\phi_* : \mathrm{QC}^!(X) \to \mathrm{QC}^!(X)$ for which the actions above are canonically ϕ_*-semilinear. By definition,

$$[\mathrm{QC}^!(X), \phi_*] = \mathbf{Tr}(T, \phi_*)([\mathrm{QC}(Y), \phi_*]) = \mathbf{Tr}(T, \phi_*)(\mathcal{O}_{\mathcal{L}_\phi Y}).$$

A variant of this functor for quasi-coherent sheaves, and in the setting where $f : X \to Y$ is surjective, was studied in [BFN12]. Note that unlike in their setting, this functor T is not an equivalence since we are considering ind-coherent sheaves. Furthermore, the failure of f to be surjective in our setting requires the application of local cohomology in the calculation of its trace. We now identify the trace of the standard representation.

Proposition 3.31. *Define the singular support condition* $\{0\}_{f(X)} := \{0\}_{\mathcal{L}_\phi Y} \cap \Lambda_{X/Y,\phi}$. *There is a canonical identifications of functors*

$$\mathbf{Tr}(T, \phi_*) \simeq \iota_{\{0\}_{f(X)}} \circ \Gamma_{\{0\}_{f(X)}} : \mathrm{QC}^!_{\{0\}_{\mathcal{L}_\phi Y}}(\mathcal{L}_\phi Y) \to \mathrm{QC}^!_{\Lambda_{X/Y,\phi}}(\mathcal{L}_\phi Y).$$

Furthermore, letting $\mathrm{ev}^{-1} f(X) \subset \mathcal{L}_\phi Y$ *corresponding to* $\{0\}_{f(X)}$*, we have*

$$[\mathrm{QC}^!(X), \phi_*] \simeq \Gamma_{\mathrm{ev}^{-1} f(X)}(\omega_{\mathcal{L}_\phi Y}).$$

Proof. For simplicity, we will prove the statement where ϕ is the identity; the general case follows similarly. We claim that the right dual to T is

$$T^R(-) := \mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Z)} - : \mathrm{QC}^!(Z)\textbf{-mod} \to \mathrm{QC}^!(Y)\textbf{-mod}$$

where $\mathrm{QC}^!(X)$ here is considered as *right* $\mathrm{QC}^!(Z)$-module, so that we have

$$T^R \circ T(-) = (\mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Z)} \mathrm{QC}^!(X)) \otimes_{\mathrm{QC}^!(Y)} - \simeq \mathrm{QC}^!_{f(X)}(Y) \otimes_{\mathrm{QC}(Y)} -,$$

$$T \circ T^R(-) = (\mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Y)} \mathrm{QC}^!(X)) \otimes_{\mathrm{QC}^!(Z)} - \simeq \mathrm{QC}^!_{\{0\}_Z}(Z) \otimes_{\mathrm{QC}^!(Z)} -.$$

Note that the convolution $\mathrm{QC}(Y)$-action can be re-interpreted as the usual pullback and tensor product, while the $\mathrm{QC}^!(Z)$-action is by convolution. The first isomorphism is due to Proposition 3.30, whereby

$$\mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Z)} \mathrm{QC}^!(X) \simeq \mathrm{QC}^!_{f(X)}(Y)$$

i.e. the full subcategory of $\mathrm{QC}^!(Y) = \mathrm{QC}(Y)$ with classical support on the closed subset $f(X)$ (since Y is smooth there are no possible singular codirections). The second isomorphism is due to Theorem 4.7 of [BFN10], i.e. we have $\mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Y)} \mathrm{QC}^!(X) = \mathrm{QC}(Z) = \mathrm{QC}^!_{\{0\}_Z}(Z)$.

To establish duality, we need to write down unit and counit maps

$$\eta : \mathrm{QC}^!(Y) \longrightarrow \mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Z)} \mathrm{QC}^!(X) \simeq \mathrm{QC}^!_{f(X)}(Y),$$

$$\epsilon : \mathrm{QC}^!_{\{0\}_Z}(Z) \simeq \mathrm{QC}^!(X) \otimes_{\mathrm{QC}^!(Y)} \mathrm{QC}^!(X) \to \mathrm{QC}^!(Z)$$

satisfying the usual "Zorro's identities". We define $\eta := \Gamma_{f(X)}$ to be the local cohomology functor, and $\epsilon = \iota_{\{0\}_Z}$ to be the fully faithful inclusion. The verification of Zorro's identities is immediate from the observation that tensoring η or ϵ with $\mathrm{id}_{\mathrm{QC}^!(X)}$ (on either side) gives rise to the identity functor, i.e. that the following diagrams commute:

$$\begin{array}{ccc} \mathrm{QC}^!(X) \underset{\mathrm{QC}^!(Y)}{\otimes} \mathrm{QC}^!(Y) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!(X) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{QC}^!(X)} \otimes \eta} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{QC}^!(X)}} \\ \mathrm{QC}^!(X) \underset{\mathrm{QC}^!(Y)}{\otimes} \mathrm{QC}^!_{f(X)}(Y) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!(X) \end{array} \qquad \begin{array}{ccc} \mathrm{QC}^!_{\{0\}_Z}(Z) \underset{\mathrm{QC}^!(Z)}{\otimes} \mathrm{QC}^!(X) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!(X) \\ \Big\downarrow{\scriptstyle \epsilon \otimes \mathrm{id}_{\mathrm{QC}^!(X)}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{QC}^!(X)}} \\ \mathrm{QC}^!(Z) \otimes_{\mathrm{QC}^!(Z)} \mathrm{QC}^!(X) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!(X) \end{array}$$

This follows by Proposition 3.30 and the singular support calculations (note that X is smooth and thus $\mathbb{T}_X^{*[-1]}$ has no singular codirections):

$$\{0\}_X * f(X) = \{0\}_X, \qquad \{0\}_Z * \{0\}_X = \{0\}_X.$$

This establishes the duality of (T, T^R).

Now, we compute the map on traces, using the functoriality described in Section 3.5 of [BN19]. There is a canonical commuting structure $\psi : T \circ \phi_{Y*} \to \phi_{Z*} \circ T$, which for us is an equivalence (thus induces an equivalence on traces). We let $f(X) \subset \mathbb{T}_Y^{*[\text{-}1]} = Y$ denote the (necessarily, since Y is smooth) classical support condition, and define $\Lambda := \mathrm{ev}^!(f(X))$, i.e. the loops with base points classically supported over $f(X) \subset Y$ and no singular codirections. We have $\{0\}_{\mathcal{L}Y} \supset \Lambda \subset \Lambda_{X/Y}$.

$$
\begin{array}{ccc}
\mathbf{Tr}(\mathrm{QC}^!(Y), \phi_{Y*}) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\{0\}_{\mathcal{L}_\phi Y}}(\mathcal{L}_\phi Y) \\
{\scriptstyle \mathbf{Tr}(\mathrm{QC}^!(Y), \eta \circ \mathrm{id}_{\phi_*})}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_\Lambda \circ \iota_{\{0\}} = \Gamma_\Lambda} \\
\mathbf{Tr}(\mathrm{QC}^!(Y), T^R \circ T \circ \phi_{Y*}) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\mathrm{ev}^! f(X)}(\mathcal{L}_\phi Y) \\
{\scriptstyle \mathbf{Tr}(\mathrm{QC}^!(Y), \mathrm{id}_{T^R} \circ \psi)}\big\downarrow{\scriptstyle \simeq} & & \big\| \\
\mathbf{Tr}(\mathrm{QC}^!(Y), T^R \circ \phi_{Z*} \circ T) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\mathrm{ev}^! f(X)}(\mathcal{L}_\phi Y) \\
\big\downarrow{\scriptstyle \simeq} & & \big\| \\
\mathbf{Tr}(\mathrm{QC}^!(Z), \phi_{Z*} \circ T \circ T^R) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\delta_* \pi^! \{0\}_Z}(\mathcal{L}_\phi Y) \\
{\scriptstyle \mathbf{Tr}(\mathrm{QC}^!(Z), \mathrm{id}_{\phi_*} \circ \epsilon)}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_{\Lambda_{X/Y}} \circ \iota_\Lambda = \iota_\Lambda} \\
\mathbf{Tr}(\mathrm{QC}^!(Z), \phi_{Z*}) & \xrightarrow{\ \simeq\ } & \mathrm{QC}^!_{\Lambda_{X/Y}}(\mathcal{L}_\phi Y)
\end{array}
$$

The top and bottom isomorphisms are given by Theorem 3.3.1 in [BNP17b]. We argue the middle isomorphisms. A combination of the arguments of Propositions 3.23 and 3.30 gives rise to identifications

$$\mathbf{Tr}(\mathrm{QC}^!(Z), T \circ T^R \circ \phi_{Y*}) = \mathrm{QC}^!(Y) \otimes_{\mathrm{QC}^!(Y \times Y)} \mathrm{QC}^!_{f(X)}(Y) \simeq \mathrm{QC}^!_{\mathrm{ev}^!(f(X))}(\mathcal{L}_\phi Y),$$

$$\mathbf{Tr}(\mathrm{QC}(Y), \phi_{Z*} \circ T^R \circ T) = \mathrm{QC}^!(Z) \otimes_{\mathrm{QC}^!(Z \times Z)} \mathrm{QC}^!_{\{0\}_Z}(Z) \simeq \mathrm{QC}^!_{\delta_* \pi^! \{0\}_Z}(\mathcal{L}_\phi Y),$$

where $\delta_* \pi^! \{0\}_Z$ is the pull-push of $\{0\}_Z$ along the correspondence in Theorem 3.23. We note that $\delta_* \pi^! \{0\}_Z = \delta_* \{0\}_{\mathcal{L}_\phi Y_X} = \mathrm{ev}^! f(X) = \{0\}_{f(X)}$ (where $\{0\}_{f(X)}$ is as defined in the theorem statement). The identification of the vertical functors follows via the functoriality of supports in Proposition 3.30 applied to the setting of Proposition 3.23, and the observation that $\{0\}_{\mathcal{L}_\phi Y} \supset \Lambda \subset \Lambda_{X/Y,\phi}$. This establishes the first statement of the theorem.

For the second statement, note that $\omega_{\mathcal{L}_\phi Y}$ is perfect (since $\mathcal{L}Y$ is quasi-smooth), i.e. has no singular codirections. Note that in general, for singular support conditions $\Lambda_1, \Lambda_2 \subset \mathbb{T}_X^{*[\text{-}1]}$, we have $\Gamma_{\Lambda_2} \circ \iota_{\Lambda_1} \circ \Gamma_{\Lambda_1} = \Gamma_{\Lambda_1 \cap \Lambda_2}$. Now, take $\Lambda_1 = \{0\}_{\mathcal{L}Y}$ (i.e. no singular codirections with unrestricted classical support) and $\Lambda_2 = \mathrm{ev}^{-1} f(X) \times_{\mathcal{L}_\phi Y} \mathbb{T}^{*[\text{-}1]}_{\mathcal{L}_\phi Y}$ (i.e. all singular codirections with restricted classical support). The second statement follows, since $\Gamma_{\Lambda_1}(\omega_{\mathcal{L}_\phi Y}) = \omega_{\mathcal{L}_\phi Y}$ and Γ_{Λ_2} is the classical local cohomology functor with support $\mathrm{ev}^{-1} f(X)$. □

Corollary 3.32. *The functor*

$$\mathrm{Hom}(\mathcal{S}_{X/Y,\phi}, -) : \mathbf{Tr}(\mathrm{QC}^!(Z), \phi_*) \simeq \mathrm{QC}^!_{\mathrm{ev}^{-1} f(X)}(\mathcal{L}_\phi Y) \longrightarrow \mathrm{End}(\mathcal{S}_{X/Y,\phi})\text{-mod}$$

takes $\Gamma_{\mathrm{ev}^{-1} f(X)}(\mathcal{O}_{\mathcal{L}_\phi Y})$ *to the* $HH(\mathrm{QC}^!(Z), \phi_*)$*-module* $HH(\mathrm{QC}^!(X), \phi_*)$.

Proof. By Theorem 3.4, it suffices to identify the trace of the $\mathrm{QC}^!(Z)$-module category $\mathrm{QC}^!(X)$. By the above theorem, $[\mathrm{QC}^!(X), \phi_*] \simeq \Gamma_{\mathrm{ev}^{-1} f(X)}(\omega_{\mathcal{L}_\phi Y}) \simeq \Gamma_{\mathrm{ev}^{-1} f(X)}(\mathcal{O}_{\mathcal{L}_\phi Y})$ (the latter isomorphism by Proposition 3.12). □

3.4.1. *Splitting the universal trace sheaf.* There is a canonical map

$$[\mathrm{QC}^!(Z), \phi_*] = \mathcal{S}_{X/Y,\phi} = \mathcal{L}_\phi f_* \omega_{\mathcal{L}_\phi X} \longrightarrow \Gamma_{\mathrm{ev}^{-1} f(X)}(\omega_{\mathcal{L}_\phi Y}) = [\mathrm{QC}^!(X), \phi_*]$$

arising via the pushforward of volume forms. In this section we investigate when this map splits, realizing the trace of the standard representation as a summand of the trace of the regular representation. Our goal is to prove the following.

Proposition 3.33. *Let $f : X \to Y$ be a proper morphism of smooth QCA stacks, with compatible self-maps ϕ_X, ϕ_Y. Assume that:*

- *ϕ_{Y*} is a monoidal endofunctor of $\mathrm{QC}(Y)$ and ϕ_{X*} is a ϕ_{Y*}-semilinear endofunctor of $\mathrm{QC}(X)$ (e.g. ϕ_X, ϕ_Y are automorphisms),*
- *$f_*\mathcal{O}_X \simeq \mathcal{O}_{Y_0}$ for $Y_0 \subset Y$ a (possibly singular) closed substack, i.e. f is a resolution of rational singularities,*
- *the closed substack Y_0 has finite Tor dimension, and*
- *$\mathcal{L}_\phi Y_0 = \mathcal{L}_\phi Y$ as derived stacks.*

Then, $[\mathrm{QC}^!(Z), \phi_*] = \mathcal{S}_{X/Y} \simeq \mathcal{L}_\phi f_* \omega_{\mathcal{L}_\phi X}$ contains $[\mathrm{QC}^!(X), \phi_*] \simeq \Gamma_{\mathrm{ev}^{-1} f(X)}(\omega_{\mathcal{L}_\phi Y})$ as a summand, i.e. the map defined above splits. In particular, we have the following converse to Corollary 3.32: the fully faithful map of Theorem 3.4

$$HH(\mathrm{QC}^!(Z), \phi_*)\text{-mod} \hookrightarrow \mathrm{QC}^!_{\Lambda_{X/Y,\phi}}(\mathcal{L}_\phi Y)$$

takes $HH(\mathrm{QC}^!(X), \phi_*) \longmapsto \omega_{\mathcal{L}_\phi Y} \simeq \mathcal{O}_{\mathcal{L}_\phi Y}$.

To prove the above result, we require a discussion of enhanced vertical traces, i.e. the realization of vertical traces of module categories for a monoidal category as characters in the horizontal trace of the monoidal category.

Definition 3.34. Let us fix a monoidal dg category $\mathbf{A}$, and a monoidal endofunctor F. For any $\mathbf{A}$-module category $\mathbf{C}$ equipped with a commuting structure $F_{\mathbf{M}}$ for F (see Definitions 2.6 and 3.2), we define the *enhanced Hochschild homology* to be

$$\underline{HH}(\mathbf{C}, F_{\mathbf{M}}) := [\mathbf{C}, F_{\mathbf{M}}] \in \mathbf{Tr}(\mathbf{A}, F).$$

By Theorem 3.4, the usual Hochschild homology can be recovered by applying the functor $\mathrm{Hom}_{\mathbf{Tr}(\mathbf{A},F)}([\mathbf{A}, F], -)$.

Remark 3.35. We have seen examples of this enhanced Hochschild homology in Section 3.2, namely that in geometric settings Hochschild homology and maps induced by functoriality often sheafify, i.e. arise as global objects via local ones by taking global sections. The category $\mathrm{QC}(Y)$ is monoidal, and for any module category $\mathbf{C}$ the Hochschild homology $HH(\mathbf{C}) := [\mathbf{C}] \in \mathbf{Vect}_k$ has an enhancement $\underline{HH}(\mathbf{C}) \in \mathbf{Tr}(\mathrm{QC}(Y)) = \mathrm{QC}(\mathcal{L}Y)$. Though we do not need or prove it, the enhanced Block-Getzler complex in Definition 2.11 is also an example of this phenomenon, where we view the Hochschild homology of a $\mathrm{Rep}(G)$-module category as an object of $\mathbf{Tr}(\mathrm{Rep}(G)) = \mathrm{QC}(G/G)$.

We now compute the enhanced trace in an example of interest; see Appendix A.2 for a proof.

Proposition 3.36. *Let $f : X \to Y$ be a map of QCA (or more generally, perfect) stacks, and ϕ_X, ϕ_Y compatible self-maps such that $\phi_{Y*} : \mathrm{QC}(Y) \to \mathrm{QC}(Y)$ is monoidal and $\phi_{X*} : \mathrm{QC}(X) \to \mathrm{QC}(X)$ is ϕ_{Y*}-semilinear. Consider $\mathrm{QC}(X)$ as a $\mathrm{QC}(Y)$-module category. Then, we have*

$$\underline{HH}(\mathrm{QC}(X), \phi_{X*}) = [\mathrm{QC}(X), \phi_{X*}] \simeq \mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X} \in \mathbf{Tr}(\mathrm{QC}(Y), \phi_{Y*}) = \mathrm{QC}(\mathcal{L}_\phi Y).$$

We now prove the result via the following mild generalization.

Proposition 3.37. *Let $f : X \to Y$ be a morphism of QCA stacks, with compatible self-maps ϕ_X, ϕ_Y such that $\phi_{Y*} : \mathrm{QC}(Y) \to \mathrm{QC}(Y)$ is monoidal and $\phi_{X*} : \mathrm{QC}(X) \to \mathrm{QC}(X)$ is ϕ_{Y*}-semilinear. Further assume that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$, and that f_* sends $\mathrm{Perf}(X)$ to $\mathrm{Perf}(Y)$. Then, $\mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X}$ contains $\mathcal{O}_{\mathcal{L}_\phi Y}$ as a summand.*

Proof. To prove the claim, we need to produce a splitting. First note that $HH(\mathrm{QC}(Y), \phi_*) = \mathcal{O}(\mathcal{L}_\phi Y)$ has the structure of an algebra object given by multiplication of functions. The structure sheaf $\mathcal{O}_Y$ is the monoidal unit and thus has a canonical ϕ_*-equivariant structure, and thus its trace $[\mathcal{O}_X]$ is the monoidal unit in $HH(\mathrm{QC}(Y), \phi_*)$. Thus, $[\mathcal{O}_Y] = 1_{\mathcal{L}_\phi Y}$ (and similarly for

X). Now, consider the diagram

$$\begin{array}{ccccccc}
\underline{HH}(\mathrm{QC}(Y),\phi_*) & = & \mathcal{O}_{\mathcal{L}_\phi Y} & \ni & 1_{\mathcal{L}_\phi Y} & = & [\mathcal{O}_Y] \\
\Big\downarrow {\scriptstyle \underline{HH}(f^*,\phi_*)} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\underline{HH}(\mathrm{QC}(X),\phi_*) & = & \mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X} & \ni & 1_{\mathcal{L}_\phi X} & = & [f^*\mathcal{O}_Y] = [\mathcal{O}_X] \\
\Big\downarrow {\scriptstyle \underline{HH}(f_*,\phi_*)} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\underline{HH}(\mathrm{QC}(Y),\phi_*) & = & \mathcal{O}_{\mathcal{L}_\phi Y} & \ni & 1_{\mathcal{L}_\phi Y} & = & [f_*\mathcal{O}_X] = [\mathcal{O}_Y]
\end{array}$$

Note that f^* always preserves compact objects, and f_* preserves compact (i.e. perfect) objects by assumption, giving us the functoriality on the left following Proposition 3.36. To see that the composition is the identity, note that a map $\mathcal{O}_{\mathcal{L}_\phi Y} \to \mathcal{O}_{\mathcal{L}_\phi Y}$ is determined by where the constant function maps; in particular, since it maps to itself, the map is the identity. □

Proof of Proposition 3.33. Since X and Y are smooth, $\omega_{\mathcal{L}_\phi X} \simeq \mathcal{O}_{\mathcal{L}_\phi X}$ and $\mathcal{O}_{\mathcal{L}_\phi Y} \simeq \omega_{\mathcal{L}_\phi Y}$. Since f surjects onto $\mathcal{L}_\phi Y_0$, which is equal to $\mathcal{L}_\phi Y$, we have that $\Gamma_{\mathrm{ev}^{-1} f(X)}(\omega_{\mathcal{L}_\phi Y}) = \omega_{\mathcal{L}_\phi Y} \simeq \mathcal{O}_{\mathcal{L}_\phi Y} \simeq \mathcal{O}_{\mathcal{L}_\phi Y_0}$, and the result follows by applying Proposition 3.37 to the map $X \to Y_0$ (note that since f is proper and X is smooth, f_* sends $\mathrm{Perf}(X) = \mathrm{Coh}(X)$ to $\mathrm{Coh}(Y_0)$, and since we require Y_0 to have finite Tor dimension we have $\mathrm{Perf}(Y_0) = \mathrm{Coh}(Y_0)$). □

4. The affine Hecke algebra and the coherent Springer sheaf

We now specialize the discussion of Section 3 to our Springer theory setting. We are interested in the following special cases.

Definition 4.1 (Coherent Springer sheaves)**.** Recall that $\widetilde{G} = G \times \mathbb{G}_m$, and the set-up in Definition 3.15 and the universal trace sheaf of Definition 3.16.

- We take
$$f = \mu : X = \widetilde{\mathcal{N}}/\widetilde{G} \longrightarrow \widehat{\mathcal{N}}/\widetilde{G} \hookrightarrow Y = \mathfrak{g}/\widetilde{G}$$
to be the scaling-equivariant Springer resolution (with codomain in the Lie algebra rather than the nilpotent cone). We call the resulting sheaf $\mathcal{S}$ on $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ (or equivalently, on $\mathcal{L}(\mathfrak{g}/\widetilde{G})$ supported over $\mathcal{N}$) the *coherent Springer sheaf.*
- We take
$$f = \mu : X = \widetilde{\mathcal{N}}/G \longrightarrow \widehat{\mathcal{N}}/G \hookrightarrow Y = \mathfrak{g}/G$$
to be the above Springer resolution without $\mathbb{G}_m$-equivariance, and $\phi := q$ to be multiplication by $q \in \mathbb{G}_m(k)$. Then we have the derived q-fixed points:
$$\mathcal{L}_q(\widehat{\mathcal{N}}/G) \simeq \mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}) \times_{\mathcal{L}(B\mathbb{G}_m)} \{q\}.$$
This is the stack $\mathbb{L}^u_{q,G}$ from the introduction. We call the sheaf $\mathcal{S}_q$ on $\mathcal{L}_q(\widehat{\mathcal{N}}/G)$ the *coherent q-Springer sheaf.*

We note the following convenient presentation of the stacks $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})$ and $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$.

Remark 4.2. We realize $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ as the formal completion of $\mathcal{L}(\mathfrak{g}/\widetilde{G}) \to \mathfrak{g}/\widetilde{G}$ over the nilpotent cone. By Proposition 2.1.8 of [Ch20a], we can write $\mathcal{L}(\mathfrak{g}/\widetilde{G})$ as the pullback

$$\begin{array}{ccccc}
\mathcal{L}(\mathfrak{g}/\widetilde{G}) & \longrightarrow & \mathfrak{g}/\widetilde{G} & \longrightarrow & \{0\}/\widetilde{G} \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \Delta} & & \Big\downarrow \\
(\mathfrak{g} \times \widetilde{G})/\widetilde{G} & \xrightarrow{a \times p} & (\mathfrak{g} \times \mathfrak{g})/\widetilde{G} & \xrightarrow{-} & \mathfrak{g}/\widetilde{G}
\end{array}$$

where the bottom right map is given by subtraction in $\mathfrak{g}$, a is the action map, p the projection, and Δ the diagonal. Explicitly, the map $\mathfrak{g} \times \widetilde{G} \to \mathfrak{g}$ is given by $(x, g, q) \mapsto q^{-1}\mathrm{Ad}_g(x) - x$. We also have a version for fixed q:

$$\begin{array}{ccccc} \mathcal{L}_q(\mathfrak{g}/G) & \longrightarrow & \mathfrak{g}/G & \longrightarrow & \{0\}/\widetilde{G} \\ \downarrow & & \downarrow \Delta & & \downarrow \\ (\mathfrak{g}\times G)/G & \xrightarrow{a_q \times p} & (\mathfrak{g}\times\mathfrak{g})/\widetilde{G} & \xrightarrow{-} & \mathfrak{g}/\widetilde{G}. \end{array}$$

where a_q is the q-twisted action map. There is a similar description for $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) = \mathcal{L}(\mathfrak{n}/\widetilde{B})$:

$$\begin{array}{ccccc} \mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) & \longrightarrow & \widetilde{\mathcal{N}}/\widetilde{G} & \longrightarrow & (G/B)/\widetilde{G} \\ \downarrow & & \downarrow \Delta & & \downarrow \\ (\widetilde{\mathcal{N}}\times \widetilde{G})/\widetilde{G} & \xrightarrow{a \times p} & (\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})/\widetilde{G} & \xrightarrow{-} & \widetilde{\mathcal{N}}/\widetilde{G}. \end{array}$$

We record the following mild generalization and direct consequence of Proposition 4.2 in [H20] and Proposition 2.1 in [He20] (also proven for q a prime power in Proposition 3.1.5 of [Zh20]).

Proposition 4.3. *If q is not a root of unity, then $\mathcal{L}_q(\widehat{\mathcal{N}}/G)$ is a classical stack, i.e. has trivial derived structure and is supported at the nilpotent cone.*

Proof. We first argue that $\mathcal{L}_q(\mathfrak{g}/G)$ is supported over the nilpotent cone, thus $\mathcal{L}_q(\mathfrak{g}/G) = \mathcal{L}_q(\widehat{\mathcal{N}}/G)$. The formation of (twisted) loop spaces commutes with products; note the Cartesian square

$$\begin{array}{ccc} \mathcal{N}/G & \longrightarrow & \mathfrak{g}/G \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathfrak{h}/\!/W. \end{array}$$

The morphisms are $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\mathfrak{h}$ by weight 1, and thus on $\mathfrak{h}/\!/W$ by weights $\geqslant 2$. Thus if q is not a root of unity, then the (derived and classical) q-fixed points of $\mathfrak{h}/\!/W$ is precisely $\{0\}$. Thus the map on the bottom is an equivalence, and the claim follows. The vanishing of derived structure follows by Proposition 4.2 in [He20] and in view of Remark 2.2(b) of *op. cit.* ☐

Remark 4.4. It is necessary to exclude roots of unity; when $G = \mathrm{SL}_2$, the weight of $\mathfrak{h}/\!/W$ is 2, so the argument fails for $q = \pm 1$. When $G = \mathrm{SL}_2$, the weights of $\mathfrak{h}/\!/W$ are 2 and 3, so the argument fails for $q = \pm 1$ and any cubic root of unity.

We now give an alternative characterization of the coherent Springer sheaf (and likewise for the q-version) via coherent parabolic induction.

Definition 4.5. Consider the parabolic induction correspondence

$$\widehat{\mathcal{N}}/\widetilde{G} \xleftarrow{\mu} \widehat{\mathfrak{n}}/\widetilde{B} \xrightarrow{\nu} \widehat{\{0\}}/\widetilde{H}.$$

We define the *coherent Springer sheaf* by applying the loop space of the above correpsondence to the reduced structure sheaf of $\mathcal{L}(\{0\}/\widetilde{H})$:

$$\mathcal{S} := \mathcal{L}\mu_* \mathcal{O}_{\widetilde{\mathcal{N}}/\widetilde{G}} = \mathcal{L}\mu_* \mathcal{L}\nu^* \mathcal{O}_{\mathcal{L}(\{0\}/\widetilde{H})} \in \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

We define the *coherent q-Springer sheaf* analogously, or equivalently we can take $\mathcal{S}_q := \iota_q^* \mathcal{S}$, where $\iota_q : \mathcal{L}_q(\mathcal{N}/G) \to \mathcal{L}(\mathcal{N}/G)$ is the closed immersion.

Remark 4.6. Note that a priori, one could define $\mathcal{S}_q$ via either the $*$ or !-pullback. However, the map ι_q is base-changed from the map $i_q : \{q\} \to \mathbb{G}_m/\mathbb{G}_m$. Since $\{q\} \subset \mathbb{G}_m$ has trivial normal bundle and i_q has relative dimension zero, we have a canonical equivalence $\iota_q^! \simeq \iota_q^*$, i.e. it did not matter which definition we took. Likewise, since derived loop spaces of smooth stacks (or smooth morphisms) are Calabi Yau by Proposition 3.12, we have an equivalence $\mathcal{L}\nu^* \simeq \mathcal{L}\nu^!$ and can use either.

For number theory applications, we will be interested in specializing at q a prime power. There are the algebraic specializations of the affine Hecke algebra, which have no derived structure since $\mathcal{H}$ is flat over $k[z, z^{-1}]$.

Definition 4.7. We define the *Iwahori-Hecke algebra* by

$$\mathcal{H}_q := \mathcal{H} \otimes_{k[z,z^{-1}]} k[z, z^{-1}]/\langle z - q\rangle.$$

A potentially different algebra arises when specializing geometrically, i.e. taking endomorphisms of a q-specialized Springer sheaf. We introduce the following unmixed version of the affine Hecke algebra, which is obtained by taking G-equivariant endomorphisms of the Springer sheaf *without* taking $\mathbb{G}_m$-invariants, i.e. by passing to the base changed stack $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}) \times_{B\mathbb{G}_m} \mathrm{pt}$.

Definition 4.8. We define the *unmixed affine Hecke algebra* and its specialization by

$$\mathcal{H}^{un} := \mathrm{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})\times_{B\mathbb{G}_m}\mathrm{pt}}(\mathcal{S}), \qquad \mathcal{H}^{un}_q := \mathcal{H}^{un} \otimes^L_{k[z,z^{-1}]} k[z, z^{-1}]/\langle z - q\rangle.$$

The algebra $\mathcal{H}^{un}$ has the additional structure of a $\mathbb{G}_m$-representation, i.e. a weight grading.

The unmixed affine Hecke algebra arises naturally when considering the trace by pullback by various $q \in \mathbb{G}_m$ acting on the affine Hecke category $\mathbf{H} = \mathrm{Coh}(\mathcal{Z}/G)$ (as opposed to the mixed affine Hecke category $\mathbf{H}^{\mathrm{m}} = \mathrm{Coh}(\mathcal{Z}/\widetilde{G})$).

Proposition 4.9. *There is a natural equivalence of algebras*

$$\mathcal{H}^{un}_q \simeq HH(\mathbf{H}, q_*) \simeq \mathrm{End}_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}(\mathcal{S}_q).$$

That is,

$$\mathcal{H}^{un}_q \simeq \begin{cases} kW_a \otimes_k \mathrm{Sym}_k(\mathfrak{h}^*[-1] \oplus \mathfrak{h}^*[-2]) & \text{when } q = 1, \\ \mathcal{H}_q & \text{when } q \neq 1. \end{cases}$$

Proof. We adopt the shorthand notation $\mathcal{L}^{un}(\widehat{\mathcal{N}}/\widetilde{G}) := \mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}) \times_{B\mathbb{G}_m} \mathrm{pt}$, and $\mathcal{S}^{un}$ for the corresponding coherent Springer sheaf. Let $\iota_q : \mathcal{L}_q(\widehat{\mathcal{N}}/G) \hookrightarrow \mathcal{L}^{un}(\widehat{\mathcal{N}}/\widetilde{G})$ be the base change along the closed immersion $\{q\} \hookrightarrow \mathbb{G}_m$. Consider the forgetful functor for the natural map of algebras

$$\mathcal{H}^{un} = \mathrm{End}_{\mathcal{L}^{un}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S}^{un}) \to \mathrm{Hom}_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}(\iota_q^*\mathcal{S}^{un}, \iota_q^*\mathcal{S}^{un}) = HH(\mathbf{H}, q_*).$$

obtained via functoriality (Proposition 2.12). Using the $(\iota_q^*, \iota_{q,*})$ adjunction, we have $\iota_{q,*}\iota_q^*\mathcal{F} = \mathrm{cone}(q : \mathcal{F} \to \mathcal{F})$, and an equivalence of complexes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S}, \iota_{q,*}\iota_q^*\mathcal{S}) & \xleftarrow{\simeq} & \mathrm{Hom}_{\mathcal{L}^{un}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S}^{un}, \mathcal{S}^{un}) = \mathcal{H}^{un} \\ & & \uparrow q \\ & & \mathrm{Hom}_{\mathcal{L}^{un}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S}^{un}, \mathcal{S}^{un}) = \mathcal{H}^{un}. \end{array}$$

The equivalence is an equivalence of dg algebras, so $HH(\mathbf{H}, q_*) \simeq \mathcal{H}^{un}_q$, proving the claim. ☐

Remark 4.10. The algebra $\mathcal{H}^{un}$ can be recovered as the $\mathbb{G}_m$-enhanced Hochschild homology of $\mathbf{H}^{\mathrm{m}}$ discussed in [GKRV20] and Section 3.4.1. In particular, take coordinates $\mathcal{O}(\mathbb{G}_m) = k[z, z^{-1}]$, let $\mathfrak{h}^*[-n]$ denote the shifted dual Cartan algebra in cohomological-weight bidegree $(n, 1)$, and define the graded $k[z, z^{-1}]$ algebra

$$A^{[-n]} := \mathcal{O}(\mathcal{L}(\mathfrak{h}[n]/\mathbb{G}_m) = \mathrm{Sym}^\bullet_{\mathcal{O}(\mathbb{G}_m)}(\mathfrak{h}^*[-n] \otimes_k \mathcal{O}(\mathbb{G}_m))/\langle x(z-1) \mid x \in \mathfrak{h}^*[-n]\rangle.$$

One can compute (in a similar manner as Theorem 2.30 and Corollary 2.32) that

$$\mathcal{H}^{un} = \underline{HH}^{\mathbb{G}_m}(\mathbf{H}^{\mathrm{m}}) = \mathcal{H} \otimes_{\mathcal{O}(\mathbb{G}_m)} A^{[-2]}$$

recovering the above proposition on specialization at various $z = q$. One can do the same for the variants in Remark 2.34, i.e.

$$\underline{HH}^{\mathbb{G}_m}(\mathrm{Coh}(\mathcal{Z}'/\widetilde{G})) = \mathcal{H}, \qquad \underline{HH}^{\mathbb{G}_m}(\mathrm{Coh}(\mathcal{Z}^\wedge/\widetilde{G})) = \mathcal{H} \otimes_{\mathcal{O}(\mathbb{G}_m)} A^{[-1]}.$$

Note that Theorem 4.4.4 in *op. cit.* establishes a relationship similar to this one.

Remark 4.11. One can similarly argue that $\mathcal{H}_q$ can be realized as the endomorphisms of the restriction of $\mathcal{S}$ along the base change of the inclusion $\{q\}/\mathbb{G}_m \hookrightarrow \mathcal{L}(B\mathbb{G}_m)$, i.e. where we retain $\mathbb{G}_m$-equivariance.

Our main result is the following theorem.

Theorem 4.12. *Assume that $q \neq 1$. The dg algebra of endomorphisms of the coherent Springer sheaf is concentrated in degree zero and is identified with the affine Hecke algebra,*

$$\mathrm{End}_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}(\mathcal{S}) \simeq \mathcal{H}, \qquad \mathrm{End}_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}(\mathcal{S}_q) \simeq \mathcal{H}_q.$$

In particular, $\mathcal{S}$ generates full embeddings, the Deligne-Langlands functors*:*

$$\mathrm{DL} : \mathcal{H}\text{-mod} \hookrightarrow \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})), \qquad \mathrm{DL}_q : \mathcal{H}_q\text{-mod} \hookrightarrow \mathrm{QC}^!(\mathcal{L}_q(\widehat{\mathcal{N}}/G)).$$

On the anti-spherical modules $M^{\mathrm{asp}} := \mathrm{Ind}_{\mathcal{H}^f}^{\mathcal{H}}(\mathrm{sgn})$ and $M_q^{\mathrm{asp}} := \mathrm{Ind}_{\mathcal{H}_q^f}^{\mathcal{H}_q}(\mathrm{sgn})$, these functors take values

$$\mathrm{DL}(M^{\mathrm{asp}}) \simeq \mathrm{pr}_{\mathcal{S}}(\omega_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}), \qquad \mathrm{DL}_q(M_q^{\mathrm{asp}}) \simeq \mathrm{pr}_{\mathcal{S}_q}(\omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}),$$

where $\mathrm{pr}_{\mathcal{S}} = \mathrm{DL} \circ \mathrm{DL}^R$ (resp. $\mathrm{pr}_{\mathcal{S}_q} = \mathrm{DL}_q \circ \mathrm{DL}_q^R$), i.e. the composition of the Deligne-Langlands functor with its right adjoint. When q is not a root of unity,

$$\mathrm{DL}_q(M_q^{\mathrm{asp}}) \simeq \mathrm{pr}_{\mathcal{S}_q}(\omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}) = \omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)} \simeq \mathcal{O}_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}.$$

Furthermore, these embeddings are compatible with parabolic induction, i.e. for a parabolic $P \supset B$ with quotient Levi M, we have commuting diagrams

$$\begin{array}{ccc} \mathcal{H}_M\text{-mod} & \hookrightarrow & \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}_M/\widetilde{M})) \\ \downarrow{\scriptstyle \mathcal{H}_G \otimes_{\mathcal{H}_M} -} & & \downarrow{\scriptstyle \mathcal{L}\mu_* \circ \mathcal{L}\nu^*} \\ \mathcal{H}_G\text{-mod} & \hookrightarrow & \mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}_G/\widetilde{G})) \end{array} \qquad \begin{array}{ccc} \mathcal{H}_{q,M}^{un}\text{-mod} & \hookrightarrow & \mathrm{QC}^!(\mathcal{L}_q(\widehat{\mathcal{N}}_M/M)) \\ \downarrow{\scriptstyle \mathcal{H}_{q,G} \otimes_{\mathcal{H}_{q,M}} -} & & \downarrow{\scriptstyle \mathcal{L}_q\mu_* \circ \mathcal{L}_q\nu^*} \\ \mathcal{H}_{q,G}\text{-mod} & \hookrightarrow & \mathrm{QC}^!(\mathcal{L}_q(\widehat{\mathcal{N}}_G/G)). \end{array}$$

That is, the parabolic induction functor is the pull-push along the correspondence obtained by applying $\mathcal{L}$ or $\mathcal{L}_q$ to the usual correspondence

$$\widehat{\mathcal{N}}_M/\widetilde{M} \xleftarrow{\mu} \widehat{\mathcal{N}}_P/\widetilde{P} \xrightarrow{\nu} \widehat{\mathcal{N}}_G/\widetilde{G}.$$

Proof. The first claim of the theorem is a combination of Theorems 2.29 and Theorem 3.25, Corollaries 2.32 and 2.30, and Proposition 4.9, for both general q and specific q. It remains to prove the claims regarding the anti-spherical module and compatibility with parabolic induction.

We first address the claim regarding anti-spherical modules. By Corollary 3.32, we have an equivalence as $\mathrm{End}(\mathcal{S}) \simeq HH(\mathrm{Coh}(\mathcal{Z}/\widetilde{G}))$-modules

$$\mathrm{Hom}(\mathcal{S}, \omega_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}) \simeq HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})).$$

Thus, it follows that $\mathrm{pr}_{\mathcal{S}}(\omega_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}) \simeq HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})$ as $HH(\mathrm{Coh}(\mathcal{Z}/\widetilde{G}))$-modules (and similarly for special q). Thus, we need to compute the module $HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))$ (and likewise for special q), and we need to identify the projection for q not a root of unity.

We first produce an isomorphism $HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})) \simeq M^{\mathrm{asp}}$ as $HH(\mathrm{Coh}(\mathcal{Z}/\widetilde{G}))$-modules, and isomorphisms $HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/G), q_*) \simeq M_q^{\mathrm{asp}}$ as $HH(\mathrm{Coh}(\mathcal{Z}/G), q_*)$-modules. The first isomorphism follows via the identification of $K_0(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))$ as the anti-spherical module for $K_0(\mathrm{Coh}(\mathcal{Z}/\widetilde{G}))$ in Section 7.6 of [CG97][28] once we establish an equivalence $K_0(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})) \simeq HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))$ as $K_0(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})) \simeq HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))$-modules, and the second would follow from an equivalence $HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/G, q_*) \simeq HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})) \otimes_{k[\mathbb{G}_m]} k_q$ (similar to the identification in Proposition 4.9).

[28]In our convention, $K_0(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))$ is identified with the anti-spherical module, while $K_0(\mathrm{Coh}_{\mathcal{B}/\widetilde{G}}(\widetilde{\mathcal{N}}/\widetilde{G}))$ is identified with the spherical module.

To see this, note that $\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G})$ has a semiorthogonal decomposition indexed by $\lambda \in X^\bullet(H)$ characters of the quotient torus $H = B/[B,B]$, where each subcategory $\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))_\lambda$ is generated over $\mathrm{Rep}(\mathbb{G}_m)$ by the line bundle $\mathcal{O}_{\widetilde{\mathcal{N}}/\widetilde{G}}(\lambda)$. Computing via the Block-Getzler complex of Definition 2.11 (see also Corollary 2.24), and noting that $\mathrm{End}_{\widetilde{\mathcal{N}}/\widetilde{G}}(\mathcal{O}_{\widetilde{\mathcal{N}}/\widetilde{G}}(\lambda)) = k$ we have that the specialization at q map is:

$$\begin{array}{ccccc} HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))_\lambda) & \longrightarrow & \underline{HH}^{\mathbb{G}_m}(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))_\lambda) & \longrightarrow & HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/G))_\lambda, q_*) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \mathcal{O}(\mathbb{G}_m) & = & \mathcal{O}(\mathbb{G}_m) & \longrightarrow & k_q. \end{array}$$

The equivalence on the left induces an equivalence $K_0(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))_\lambda) \simeq HH(\mathrm{Coh}(\widetilde{\mathcal{N}}/\widetilde{G}))_\lambda)$. Summing over each subcategory in the semiorthogonal deomposition, this establishes both claims.

It remains to compute the projection $\mathrm{pr}_{\mathcal{S}_q}(\omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/\widetilde{G})})$ for q not a root of unity. We wish to apply Proposition 3.33 to show that $\omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/\widetilde{G})}$ is a summand of $\mathcal{S}_q$, but to do so we need to replace the formal completion $\widehat{\mathcal{N}} \subset \mathfrak{g}$ with the reduced nilpotent cone $\mathcal{N} = \mathfrak{g} \times_{\mathfrak{h}//W} \{0\}$. Since derived fixed points commutes with fiber products, the diagram

$$\begin{array}{ccc} \mathcal{L}_q(\mathcal{N}/G) & \longrightarrow & \mathcal{L}_q(\mathfrak{g}/G) \\ \downarrow & & \downarrow \\ \mathcal{L}_q(\{0\}) & \longrightarrow & \mathcal{L}_q(\mathfrak{h}//W) \end{array}$$

is Cartesian. When q is not a root of unity, by Proposition 4.3 we have $\mathcal{L}_q(\{0\}) = \mathcal{L}_q(\mathfrak{h}//W)$. Thus, $\mathcal{L}_q(\mathcal{N}/G) = \mathcal{L}_q(\mathfrak{g}/G)$, and $\omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)} \simeq \omega_{\mathcal{L}_q(\mathcal{N}/G)}$, so it suffices to show that $\omega_{\mathcal{L}_q(\mathcal{N}/G)} \simeq \mathcal{O}_{\mathcal{L}_q(\mathcal{N}/G)}$ is a summand of $\mathcal{S}_q$. Since $f_*\mathcal{O}_{\widetilde{\mathcal{N}}} \simeq \mathcal{O}_{\mathcal{N}}$, we may apply Proposition 3.33 to establish the splitting.

We now address compatibility with parabolic induction. First, note that by Proposition 3.12 we have $\mathcal{L}\nu^* = \mathcal{L}\nu^*$, since ν is smooth. Let $H = B/U$, fix a parabolic $P \supset B$ with quotient Levi M, and let $B_M \subset B$ denote the Borel subgroup defined to be the image of $B \subset P$ under the quotient. Consider the correspondence

$$\mathcal{Z}_G/\widetilde{G} := \mathfrak{n}/\widetilde{B} \times_{\mathfrak{g}/\widetilde{G}} \mathfrak{n}/\widetilde{B} \xleftarrow{\ i\ } \mathcal{Z}_P/\widetilde{P} := \mathfrak{n}/\widetilde{B} \times_{\mathfrak{p}/\widetilde{P}} \mathfrak{n}/\widetilde{B} \xrightarrow{\ p\ } \mathcal{Z}_M/\widetilde{M} := \mathfrak{n}_M/\widetilde{B_M} \times_{\mathfrak{m}/M} \mathfrak{n}_M/\widetilde{B_M}.$$

Note that the correspondence satisfies the conditions of Proposition 3.17, i.e. since $\mathfrak{n}/B = \mathfrak{b}/B \times_{\mathfrak{h}/H} \{0\}/H$ (and similarly for B_M), and the formation of loop spaces commutes with fiber products, we have via base change that $\mathcal{S}_G = \mathcal{L}\mu_*\mathcal{O}_{\mathcal{L}(\mathfrak{n}/B)} \simeq \mathcal{L}\mu_*\mathcal{L}\nu^*\mathcal{O}_{\mathcal{L}(\{0\}/H)}$, and similar formulas hold for $\mathcal{S}_M$. That is, the coherent Springer sheaf is the parabolic induction of the structure sheaf of $\mathcal{L}(\{0\}/H)$. Thus, we have a Cartesian diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{L}(\mathfrak{b}/B) & & & & \\ & & & \swarrow & & \searrow & & & \\ & & \mathcal{L}(\mathfrak{b}_M/B_M) & & & & \mathcal{L}(\mathfrak{p}/P) & & \\ & \swarrow & & \searrow & & \swarrow \nu & & \searrow \mu & \\ \mathcal{L}(\mathfrak{h}/H) & & & & \mathcal{L}(\mathfrak{m}/M) & & & & \mathcal{L}(\mathfrak{g}/G) \end{array}$$

thus $\mathcal{L}\mu_*\mathcal{L}\nu^*\mathcal{S}_M \simeq \mathcal{S}_G$ by base change. By the commuting diagram

$$\begin{array}{ccccc} HH(\mathrm{Coh}(\mathcal{Z}_M/\widetilde{M})) & \xrightarrow[\simeq]{\text{Prop. 3.14}} & \omega(\mathcal{L}(\mathcal{Z}_M/\widetilde{M})) & \xrightarrow[\simeq]{\text{Prop. 3.18}} & \mathrm{End}(\mathcal{S}_M) \\ \downarrow HH(i_*p^*) & & \downarrow \text{Def. 3.13} & & \downarrow \text{Def. 3.17} \\ HH(\mathrm{Coh}(\mathcal{Z}_G/\widetilde{G})) & \xrightarrow[\simeq]{\text{Prop. 3.14}} & \omega(\mathcal{L}(\mathcal{Z}_G/\widetilde{G})) & \xrightarrow[\simeq]{\text{Prop. 3.18}} & \mathrm{End}(\mathcal{S}_G), \end{array}$$

it remains to check that the map $HH(\mathrm{Coh}(\mathcal{Z}_M/\widetilde{M})) \to HH(\mathrm{Coh}(\mathcal{Z}_G/\widetilde{G}))$ induces the parabolic induction map on affine Hecke algebras. By Corollary 2.26 we can argue for K_0 instead, i.e. we show that the map

$$\mathcal{H}_M \simeq K_0(\mathrm{Coh}(\mathcal{Z}_M/\widetilde{M})) \longrightarrow K_0(\mathrm{Coh}(\mathcal{Z}_G/\widetilde{G})) \simeq \mathcal{H}_G$$

agrees with the natural parabolic induction map of affine Hecke algebras $\mathcal{H}_M \to \mathcal{H}_G$ which takes $T_{M,w} \mapsto T_{G,w}$ where $w \in W_{a,M}$ (in the notation of Section 7.1 of [CG97]). We will assume G has simply connected derived subgroup, but the general case follows by passing to invariants of finite central subgroups (i.e. as in Section 2.4.2). It suffices to show that they agree for finite simple reflections and on the lattice. Via the proof of Theorem 7.2.5 in [CG97], it is clear that the map is as claimed on the lattice; we argue that parabolic induction on K_0 sends $[\mathcal{Q}_{M,s}] \mapsto [\mathcal{Q}_{G,s}]$ where s is a finite simple reflection of M.

Let us recall the definition of $\mathcal{Q}_{M,s}$. The underlying closed, reduced scheme of $\mathcal{Z}_M$ is a disjoint union of conormal bundles to closures of M-orbits $\overline{Y}_{M,s} \subset M/B_M \times M/B_M$; we denote these subschemes and the projection by $\pi_{M,s} : \mathcal{Z}_{M,s} \to \overline{Y}_{M,s}$ and the inclusion $\iota_{M,s} : \mathcal{Z}_{M,s} \hookrightarrow \mathcal{Z}_M$. We define $\mathcal{Q}_{M,s} := \iota_{M,s,*}\pi_{M,s}^*\Omega^1_{\overline{Y}_{M,s}/(M/B_M)^2}$.

We have a similar description of $\mathcal{Z}_{P,s} \subset \mathcal{Z}_P$. The map $p : \mathcal{Z}_P \to \mathcal{Z}_M$ is a $\mathfrak{u}/U$-fibration, base changed from the quotient the quotient map $\mathfrak{p}/P \to \mathfrak{m}/M$. In particular, $\mathcal{Z}_{P,s}$ and $\mathcal{Z}_{M,s} \times_{\mathcal{Z}_M} \mathcal{Z}_P$ are closed reduced underived subschemes of $\mathcal{Z}_P$ with the same points, and thus agree. On the other hand, we have $\overline{Y}_{P,s} = (B\backslash P/B) \times_{\overline{Y}_{M,s}} (B_M\backslash M/B_M)$, so that denoting the projection $p : \overline{Y}_{P,s} \to \overline{Y}_{M,s}$ we have $\Omega^1_{\overline{Y}_{P,s}/(P/B)^2} \simeq p^*\Omega^1_{\overline{Y}_{M,s}/(M/B_M)^2}$ and thus $p^*\mathcal{Q}_{M,s} \simeq \mathcal{Q}_{P,s}$ by base change. We have $\mathcal{Q}_{G,s} = i_*\mathcal{Q}_{P,s}$ by definition, and the claim follows. Finally, the statements for specialized q follow by Proposition 4.10, completing the proof. □

Remark 4.13. A few remarks on the theorem.

- Analogous statements hold when $q = 1$, where Hochschild homology of the Steinberg stack does not agree with the Grothendieck group, i.e. we have $\mathrm{End}_{\mathcal{L}(\mathcal{N}/G)}(\mathcal{S}_1) \simeq \mathcal{H}_1^{un} \simeq kW^a \otimes \mathrm{Sym}(\mathfrak{h}^*[-1] \oplus \mathfrak{h}^*[-2])$. However, the anti-spherical module arising via Hochschild homology agrees with that arising via K_0, i.e. $HH(\mathrm{Coh}(\mathcal{N}/G)) \simeq kW^a \otimes_{kW^f} k_{\mathrm{sgn}}$, where $\mathfrak{h}^*[-1] \oplus \mathfrak{h}^*[-2] \subset \mathcal{H}_1^{un}$ acts by zero.
- The Deligne-Langlands functor is not expected to be an equivalence before applying the Tate construction, even for GL_n. Taking $G = \mathrm{GL}_1$, the category $\mathcal{H}$-mod has a compact generator, whereas $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ contains a factor of $\mathrm{Coh}(B\,\mathrm{GL}_1)$ and therefore does not. Put another way, $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))^{S^1}$ is not a constant u-deformation but the subcategory generated by the Springer sheaf is. A very computable toy example where this occurs is $\mathrm{Coh}(\mathcal{L}(BT))^{S^1}$ (see Example 4.1.4 in [Ch20a]).
- We expect $\mathrm{pr}_{\mathcal{S}_q}(\omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}) = \omega_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}$ when q is a root of unity, and also $\mathrm{pr}_{\mathcal{S}}(\omega_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}) = \omega_{\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})}$. However, we do not prove this.
- Compatibility with parabolic induction implies that the action of the lattice on the coherent Springer sheaf. That is, $\mathcal{O}(\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}))$ is an $\mathcal{O}(\mathcal{L}(\{0\}/H)) = \mathcal{O}(H)$-module.
- If $G = T$ is a torus, then $\mathcal{L}(\mathcal{N}_T/T) \simeq \widehat{\{e\}} \times T \times BT$ and $\mathcal{S} = \mathcal{O}_{\{e\}\times T\times BT}$, and we see immediately that $\mathrm{End}(\mathcal{S}) \simeq k[T] = kX^\bullet(T)$.

Remark 4.14. We explain the absence of a singular support condition. There are two Koszul dual versions of the Steinberg variety leading to two versions of the unipotent affine Hecke algebra: our version $\mathcal{Z} = \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$ and a "global" version $\mathcal{Z}_{\mathfrak{g}} := \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$. Theorem 4.4.1 of [BNP17b] shows the trace sheaves in $\mathbf{Tr}(\mathrm{Coh}(\mathcal{Z}_{\mathfrak{g}}/\widetilde{G}))$ satisfy a nilpotent singular support condition.

We now argue that the singular support condition for $\mathbf{Tr}(\mathrm{Coh}(\mathcal{Z}/\widetilde{G}))$ is vacuous, i.e. that the singular support locus $\Lambda_{\widetilde{\mathcal{N}}/\mathfrak{g}}$ is the entire scheme of singularities $\mathrm{Sing}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$. The singular locus of $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ at a k-point $\eta = (n, z = (g,q))$ where $gng^{-1} = qn$ is the set (after identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ via a non-degenerate form $\langle -, - \rangle$):

$$\mathrm{Sing}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))_\eta = \{v \in \mathfrak{g} \mid gvg^{-1} = q^{-1}v, [n,v] = 0, \langle n,v\rangle = 0\}.$$

A calculation[29] shows that the singular support locus is given by:

$$(\Lambda_{\widetilde{\mathcal{N}}/\mathfrak{g}})_\eta = \{v \in \mathrm{Sing}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))_\eta \mid \exists \text{ Borel } B \subset G \text{ such that } n, v \in \mathfrak{b} = \mathrm{Lie}(B)\}.$$

Note that n, v generate a two-dimensional solvable Lie algebra, thus are contained in a Borel, so $\mathrm{Sing}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))_\eta = \Lambda_{\widetilde{\mathcal{N}}/\mathfrak{g}}$. In particular, the singular codirection v need not be nilpotent.

The analogous claim at specific $q \in \mathbb{G}_m$ follows by a similar argument and a calculation of the singular support locus at a point $\eta = (n, g)$ (for $gng^{-1} = qn$) as

$$\mathrm{Sing}(\mathcal{L}_q(\widehat{\mathcal{N}}/\widetilde{G}))_\eta = \{v \in \mathfrak{g} \mid gvg^{-1} = q^{-1}v, [n, v] = 0\}.$$

In the case of q not a root of unity, the argument in Proposition 4.3 shows that the singular codirection v must be nilpotent.

It is natural to conjecture that the coherent Springer sheaf is in fact a sheaf – i.e., lives in the heart of the dg category $\mathrm{Coh}(\mathcal{L}(\mathfrak{g}/\widetilde{G}))$. We prove this in the case $G = \mathrm{GL}_2, \mathrm{SL}_2$ in Proposition 4.19.

Conjecture 4.15. The Springer sheaf $\mathcal{S}$ lives in the abelian category $\mathrm{Coh}(\mathcal{L}(\mathcal{N}/\widetilde{G}))^\heartsuit$.

Remark 4.16. One consequence of the conjecture would be an explicit description of the endomorphisms of the cohrent Springer sheaf. Namely, it is easy to see that the underived parabolic induction from $\mathcal{L}(\{0\}/H)$ is generated as a module by the lattice $X^\bullet(H)$, and via the identification with K-theory and Theorem 7.2.16 of [CG97] we would obtain a description of the action of finite simple reflections in terms of Demazure operators.

Remark 4.17. A variant of Conjecture 4.15 was answered in the affirmative in Corollary 4.4.6 of [Gi12]. Namely, in *loc. cit.* it is proven that the Lie algebra version of our coherent Springer sheaf at $q = 1$ has vanishing higher cohomology.

Remark 4.18. When $\widetilde{G}$ acts on $\widetilde{\mathcal{N}}$ by finitely many orbits, then $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})$ has trivial derived structure, and the conjecture is implied by the vanishing of higher cohomology of a classical scheme $H^i(\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) \times_{B\widetilde{G}} \mathrm{pt}, \pi_0(\mathcal{O}_{\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) \times_{B\widetilde{G}} \mathrm{pt}}))$ for $i > 0$. The G-orbits in the Springer resolution are known to be finite exactly in types A_1, A_2, A_3, A_4, B_2 by [Kas90].

We discuss the relation of the Deligne-Langlands correspondence and t-structures in more detail in Section 5.3.

4.1. **Conjectures and examples for** $G = \mathrm{SL}_2, \mathrm{GL}_2, \mathrm{PGL}_2$**.** In this case, $\widetilde{G}$ acts on both $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ by finitely many orbits, the derived loop spaces $\mathcal{L}(\mathcal{N}/\widetilde{G})$ and $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G})$ are classical stacks. Recall that $\mathcal{N}$ is a formal completion; if the reader would rather do so, they may replace $\mathcal{N}$ with $\mathfrak{g}$, which is also acted on by finitely many orbits. We prove Conjecture 4.15 in these cases.

Proposition 4.19. *Conjecture 4.15 holds for* $G = \mathrm{SL}_2, \mathrm{GL}_2, \mathrm{PGL}_2$.

Proof. We give a proof for $G = \mathrm{SL}_2$; the case of $G = \mathrm{GL}_2$ is the same. In view of Remark 4.18, it suffices to forget equivariance and show vanishing of higher cohomology. Since $X := \mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) \times_{B\widetilde{G}} \mathrm{pt}$ is a closed subscheme of $\mathfrak{g} \times G/B \times G$, and $\dim(G/B) = 1$, we know that $R\Gamma^i(X, -) = 0$ for $i > 1$. To verify vanishing for $i = 1$, let $i : X \hookrightarrow \widetilde{\mathcal{N}} \times \widetilde{G}$ be the closed immersion. We have a short exact sequence of sheaves:

$$0 \to \mathcal{I} \to \mathcal{O}_{\widetilde{\mathcal{N}} \times \widetilde{G}} \to i_* \mathcal{O}_X \to 0$$

leading to a long exact sequence with vanishing H^2 terms (for the above reason). Thus, it suffices to show that $H^1(\widetilde{\mathcal{N}} \times \widetilde{G}, \mathcal{O}_{\widetilde{\mathcal{N}} \times \widetilde{G}})$. By the projection formula, we have $H^1(\widetilde{\mathcal{N}} \times \widetilde{G}, \mathcal{O}_{\widetilde{\mathcal{N}} \times \widetilde{G}}) \simeq H^1(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) \otimes_k \mathcal{O}(\widetilde{G})$, but it is well-known that $H^i(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) = 0$ for $i > 0$. □

[29] In contrast to the singular support calculation for $\mathrm{Coh}(\mathcal{Z}_\mathfrak{g}/\widetilde{G})$, it is the Lie algebra of the Borel $\mathfrak{b}$ that appears in the above condition rather than its nilradical $\mathfrak{n}$ since

$$\mathrm{Sing}(\widetilde{\mathcal{N}} \times_\mathfrak{g} \widetilde{\mathcal{N}})_{(n,B,B')} = \mathfrak{b} \cap \mathfrak{b}', \qquad \mathrm{Sing}(\widetilde{\mathfrak{g}} \times_G \widetilde{\mathfrak{g}})_{(x,B,B')} = \mathfrak{n} \cap \mathfrak{n}'.$$

Example 4.20 (Geometry of the loop space of the Springer resolution)**.** We describe the geometry of the looped Springer resolution $\mathcal{L}(\widetilde{\mathcal{N}}/\widetilde{G}) \to \mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$ for $G = \mathrm{SL}_2$. Though this example is well-known, we reproduce it for the reader's convenience. Let $A(s,n)$ denote the component group of the double stabilizer group, i.e. the component group of $\{g \in G \mid gng^{-1} = n, gs = sg\}$. Let $\mathbb{A}^1_{\mathrm{node}} = \operatorname{Spec} k[x,y]/xy$ denote the affine nodal curve, and $(-)^\nu$ the normalization.

q	n	$s = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$	$\mathcal{N}^{(s,q)} \to \widetilde{\mathcal{N}}^{(s,q)}$	$A(s,n)$	G^s
$q = 1$	$n = 0$ $n \neq 0$	$\lambda = \pm 1$	$\widetilde{\mathcal{N}} \to \mathcal{N}$	1 $\mathbb{Z}/2$	G
$q = 1$	$n = 0$	$\lambda \neq \pm 1$	$\mathrm{pt} \cup \mathrm{pt} \to \mathrm{pt}$	1	T
$q = -1$	$n = 0$ $n \neq 0$, upper triangular $n \neq 0$, lower triangular	$\lambda = i$	$\mathbb{A}^{1,\nu}_{\mathrm{node}} \to \mathbb{A}^1_{\mathrm{node}}$	1 $\mathbb{Z}/2$ $\mathbb{Z}/2$	T
$q = -1$	$n = 0$	$\lambda = \pm 1$	$\mathbb{P}^1 \to \mathrm{pt}$	1	G
$q = -1$	$n = 0$	$\lambda \neq \pm 1$	$\mathrm{pt} \cup \mathrm{pt} \to \mathrm{pt}$	1	T
$q \neq \pm 1$	$n = 0$ $n \neq 0$	$\lambda = \pm\sqrt{q}$	$\mathbb{A}^1 \cup \mathrm{pt} \to \mathbb{A}^1$	1 $\mathbb{Z}/2$	T
$q \neq \pm 1$	$n = 0$	$\lambda = \pm 1$	$\mathbb{P}^1 \to \mathrm{pt}$	1	G
$q \neq \pm 1$	$n = 0$	$\lambda \neq \pm 1, \pm\sqrt{q}$	$\mathrm{pt} \cup \mathrm{pt} \to \mathrm{pt}$	1	T

Example 4.21 (Generators and relations)**.** For $G = \mathrm{SL}_2$, with some work, one can write down generators and relations for the (underived) scheme $\mathcal{L}(\mathfrak{g}/\widetilde{G})$ and the coherent Springer sheaf $\mathcal{S}$. Let us fix coordinates

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2, \qquad N = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in \mathcal{N}_{\mathfrak{sl}_2}, \qquad q \in \mathbb{G}_m.$$

We implicitly impose the equations $ad - bc = 1$ and $x^2 + yz = 0$, and by convention we take the commuting relation $gxg^{-1} = qx$; note that this is the relation that arises when $\mathbb{G}_m$ acts on fibers by weight -1 (i.e. inversely). Then, we have that $\mathcal{S}$ is the module with generators λ^n for $n \in \mathbb{Z}$:

$$\frac{\mathcal{O}(\mathrm{SL}_2 \times \mathcal{N}_{\mathfrak{sl}_2} \times \mathbb{G}_m)[\lambda, \lambda^{-1}]}{a + d = \lambda + \lambda^{-1}, (x,y,z)(q - \lambda^2) = 0, z(\lambda - d) = ax, y(a - \lambda) = bx, x(d - \lambda) = cy, x(\lambda - a) = bz}.$$

In particular, multiplication by λ^n defines the action of the lattice, and one can verify that the Demazure operator for the anti-spherical module (see Theorem 7.2.16 of [CG97]) defines the endomorphism

$$T(\lambda^n) = \frac{\lambda^n - \lambda^{-n+2}}{\lambda^2 - 1} - q\frac{\lambda^n - \lambda^{-n}}{\lambda^2 - 1}$$

corresponding to the finite reflection. In particular, it preserves the relations in the module, and the endomorphism satisfies $(T - q)(T + 1) = 0$. For fixed q, and letting k_{sgn} denote the character of $\mathcal{H}^f$ with $T \mapsto -1$, one can verify that $\mathcal{S} \otimes_{\mathcal{H}^f} k_{\mathrm{sgn}} \simeq \mathcal{O}_{\mathcal{L}_q(\widehat{\mathcal{N}}/G)}$, i.e. amounts to imposing the relation $\lambda^2 = q$, thus identifying the structure sheaf with the anti-spherical module.

5. The coherent Springer sheaf at parameters

Completing or specializing the coherent Springer sheaf at semisimple parameters recovers classical Springer sheaves in the constructible or D-module context. This process happens in two steps: first we apply an equivariant localization pattern described in [Ch20a] to pass between the stack of unipotent Langlands parameters $\mathcal{L}(\mathcal{N}/\widetilde{G})$ to a completed or specialized version at a semisimple parameter $z = (s,q)$, and second we apply a Koszul duality equivalence of categories between S^1-equivariant sheaves at this parameter and a certain category of filtered D-modules. All results in this section take place over an algebraically closed field k of characteristic 0.

5.1. **Equivariant localization of derived loop spaces.** We now describe equivariant localization patterns in derived loop spaces. See Section 3 of [Ch20a] for an extended discussion, as well as Section 2 of *op. cit.* and Section 4 of [BN12] for a discussion of derived loop spaces. We fix a reductive group G (over an algebraically closed field k of characteristic zero). Let

$$\mathcal{L}(BG) = G/G \longrightarrow G//G$$

denote the "characteristic polynomial" map from the quotient stack of G by conjugation to the affine quotient, i.e., to the variety parametrizing semisimple conjugacy classes. For a G-variety X we have the maps

$$\mathcal{L}(X/G) \longrightarrow \mathcal{L}(\mathrm{pt}/G) = G/G \longrightarrow G//G.$$

The loop map $\mathcal{L}(X/G) \to G/G$ parametrizes fixed points of elements of G – i.e., for $g \in G$ the fiber of $\mathcal{L}(X/G)$ over $g : \mathrm{pt} \to G/G$ is the derived fixed point scheme X^g, i.e. we have two descriptions of X^g by Cartesian squares[30]

$$\begin{array}{ccc} X^g & \longrightarrow & \mathcal{L}(X/G) \\ \downarrow & & \downarrow \\ \{g\} & \longrightarrow & G/G \end{array} \qquad \begin{array}{ccc} X^g & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Gamma_g} \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

This allows us to define variants of the fixed points according to the Jordan decomposition in G. In particular we are interested in fibers of the map $\mathcal{L}(X/G) \to G//G$, i.e. loops whose semisimple part[31] is conjugate to a fixed semsimple element $g \in G$.

Let $z \in G$ denote a fixed semisimple element, with centralizer G^z. We denote by $\mathcal{O}_z \simeq BG^z \subset G/G$ its equivariant conjugacy class and $[z] \in G//G$ its class in the affine quotient. The stack $\mathcal{O}_Z$ comes equipped with a natural atlas $\mathrm{Spec}(k) = \{z\} \to \mathcal{O}_Z$.

Definition 5.1. The *z-unipotent loop space* of X, denoted $\mathcal{L}^u_z(X/G)$, is the completion of $\mathcal{L}(X/G)$ along the inverse image of the saturation $[z] \in G//G$. The *z-formal loop space* $\widehat{\mathcal{L}}_z(X/G)$ is the completion of $\mathcal{L}(X/G)$ along the orbit $\mathcal{O}_z$ and the *z-specialized loop space*[32] $\mathcal{L}'_z(X/G)$ is the (derived) fiber of $\mathcal{L}(X/G)$ over $\mathcal{O}_z$.

Remark 5.2. We have containments $\mathcal{L}'_z(X/G) \subset \widehat{\mathcal{L}}_z(X/G) \subset \mathcal{L}^u_z(X/G) \subset \mathcal{L}(X/G)$, and a map $\mathcal{L}_z(X) \to \mathcal{L}_z(X/G)$ (see Defintion 3.5).

We will state the equivariant localization theorem of [Ch20a], which is a form of Jordan decomposition for loops, describing loops in the quotient stack X/G with given semisimple part z in terms of unipotent loops on the quotient stack $X^z_\circ/G^z$ (using a natural map $X^z_\circ/G^z \hookrightarrow X/G^z \to X/G$), where $X^z_\circ$ is a slight modification of the z-fixed points of the classical (underived) fixed points by the centralizer of z. We now describe this modification $X^z_\circ$ in the setting of complete intersections.

Definition 5.3. Let $z \in G$ be a semisimple closed point. Recall that the classical z-fixed points of a G-variety can be expressed as the underlying classical scheme $\pi_0(X^z)$ of the derived fixed points.

(1) A G-variety X is said to be a G-complete intersection if X is given as a fiber product $X \simeq Y \times_Z W$ in the category of G-varieties, with Y, Z and W smooth.
(2) The *modified z-fixed points* $X^z_\circ$ for a G-complete intersection is the (derived) fiber product of the classical fixed points

$$X^z_\circ := \pi_0(Y^z) \times_{\pi_0(Z^z)} \pi_0(W^z).$$

In particular we have (derived) G^z-equivariant containments

$$\pi_0(X^z) \subset X^z_\circ \subset X^z.$$

[30]Note that $\mathcal{L}_g X = X^g$, in the notation of Definition 3.5. We use the latter notation in this section to emphasize the relationship between various fixed points.

[31]Note that the preimage of $[g] \in G//G$ in G/G is the closed substack of group elements whose semisimple part in the Jordan decomposition is conjugate to g.

[32]Note that in this notation, $\mathcal{L}'_z(X/G) = \mathcal{L}_z(X)/G$.

We consider $X^z_\circ$ with its induced structure as a G^z variety with a trivialized[33] action of z.

Remark 5.4. Note that for X a smooth G-scheme, we have that $X^z_\circ = \pi_0(X^z)$ is smooth. For X quasismooth, we have that $X^z_\circ$ is quasismooth, and in particular may have nontrivial derived structure.

Remark 5.5. As a consequence of the next theorem and the fact that formal loop spaces commute with fiber products, one can recover the derived fixed points as the derived loop space of the modified fixed points $X^z \simeq \mathcal{L}(X^z_\circ)$.

Theorem 5.6 (Equivariant localization for derived loop spaces)**.** *For X a G-complete intersection, the unipotent z-localization map*

$$\ell^u_z : \mathcal{L}^u_z(X^z_\circ/G^z) \to \mathcal{L}^u_z(X/G)$$

is an S^1-equivariant equivalence.

Proof. This is Theorem A in [Ch20a], along with the observation that derived loop spaces commute with fiber products. For a precise definition of the unipotent z-localization map, see Definition 3.1.6 of [Ch20a]. ☐

Remark 5.7. Note that it follows that the corresponding localization maps on formal and specialized loops

$$\hat{\ell}_z : \widehat{\mathcal{L}}_z(X^z_\circ/G^z) \to \widehat{\mathcal{L}}_z(X/G), \qquad \ell'_z : \mathcal{L}'_z(X^z_\circ/G^z) \to \mathcal{L}'_z(X/G)$$

are also equivalences in this setting.

5.1.1. *Central shifting.* Let $Z = Z(H)$ be the center of a group prestack H. For any H-space Y the action of Z on Y commutes with the action of H, hence defines an action on the quotient Y/H, which we denote by *shifting*

$$Z \ni z \mapsto sh_z \in \mathrm{Aut}(Y/H).$$

Passing to loop spaces, the shifting action identifies[34] the fiber of $\mathcal{L}(Y/H)$ over 1 and over z.

For example in the setting of Theorem 5.6, taking $Y = X^z_\circ$ and $H = G^z$ with its central element $z \in Z(G^z)$, we get equivalences of stacks

$$\mathcal{L}^u(X^z_\circ/G^z) \xrightarrow[\simeq]{\mathcal{L}(sh_z)} \mathcal{L}^u_z(X^z_\circ/G^z) \xrightarrow[\simeq]{\ell^u_z} \mathcal{L}^u_z(X/G)$$

by shifting by z. The left identification is however *not* S^1-equivariant for the loop rotation; we need to twist the loop rotation on one side.

Definition 5.8. We have a group structure on the classifying stack BZ of the center and a group homomorphism $BZ \to \mathrm{Aut}(BH)$ induced by the trivialization of the conjugation action of Z on H. In particular fixing $z \in Z$ we obtain a *twisting by z* action of $S^1 = B\mathbb{Z}$ on BH, which we denote $\sigma(z)$. This structure generalizes to H-spaces Y that are equipped with a trivialization of the action of z (extending the case $Y = \mathrm{pt}$ above). Namely, the trivialization of the z-action produces a lift of the twisting S^1-action on $Y/H \to BH$ which we also denote by $\sigma(z)$.

Remark 5.9. Letting $H' = H/\mathbb{Z}z$, the twisting S^1-action on Y/H can also be described using the identification $Y/H \simeq Y/H' \times_{BH'} BH$ and noting that the fiber product diagram is S^1-equivariant, where we let S^1 act trivially on Y/H' and BH', and via the z-twisting S^1-action on BH.

We can combine the twisting and shifting S^1-actions as follows. Note that the loops to the z-twisting action $\mathcal{L}(\sigma(z))$ naturally commutes with the loop rotation S^1-action on $\mathcal{L}(BH) \simeq H/H$, which we denote ρ.

[33]A *z-trivialization* of a G-scheme Y is a $G' := G/\mathbb{Z}z$-action on Y along with an identification $Y/G \simeq Y/G' \times_{BG'} BG$. These choices are canonical if Y is a classical scheme; since the $X^z_\circ$ we consider are built functorially from classical ones, there will always be a canonical choice which we suppress throughout the exposition.

[34]I.e. the shifting on $\mathcal{L}(BH) = H/H$ is given by $\mathcal{L}(sh_z)(h) = zh = hz$.

Definition 5.10. We define $\rho(z)$ to be the diagonal to the $S^1 \times S^1$-action $\rho \times \mathcal{L}(\sigma(z))$.

Thus we have the following Jordan decomposition result: shifting by z intertwines ρ with the twisted version $\rho(z) = \rho \circ \mathcal{L}(\sigma(z))$.

Corollary 5.11. *For X a G-complete intersection, the shifted localization map defines an equivalence*

$$s\ell_z^u : \mathcal{L}^u(X_\circ^z/G^z) \xrightarrow{\simeq} \mathcal{L}_z^u(X/G)$$

which is S^1-equivariant with respect to $\rho(z)$ on the source and ρ on the target, and likewise for the shifts of the completed and specialized localization maps $s\ell_z^\wedge$ and $s\ell_z'$.

5.1.2. *Neutral blocks.* In order to apply Koszul duality (as in Section 5 of [BN12]), we are interested in identifying a subcategory of various categories of sheaves on derived loop spaces over semisimple parameter z on which the z-twisting is trivial, so that the twisted rotation is equal to the untwisted rotation. This is useful since the ρ circle action on unipotent loop spaces factors through an action of $B\mathbb{G}_a$, but the twisted $\rho(z)$ action does not (since it has nontrivial semisimple part). This problem is an obstacle to applying the Koszul duality described in [BN12] to obtain an identification of $\mathrm{Coh}(\widehat{\mathcal{L}}_z(X/G))^{S^1}$ with some kind of category of D-modules. We avoid this obstacle by focusing only on the z-trivial block. For this we give a categorical interpretation of the geometric z-twisting S^1-action $\sigma(z)$ discussed above.

Definition 5.12. Let H be an affine algebraic group, $z \in H$ central and $\mathbf{C}$ a category over $\mathrm{Rep}(H)$. A *z-trivialization* of $\mathbf{C}$ is an identification of the action of z on $\mathbf{C}$ with the identity functor.[35] The category $\mathbf{C}^H$ of equivariant objects then acquires an automorphism of the identity functor (i.e., S^1-action) as the ratio of the z-trivialization and the equivariance structure for z. We define the subcategory $\mathbf{C}_z^H \subset \mathbf{C}^H$ of *z-trivial objects* to be the full subcategory on which this automorphism is trivial, i.e., on which the equivariance agrees with the z-trivialization.

We can apply this categorical notion to the categories of sheaves $\mathrm{Perf}, \mathrm{Coh}, \mathrm{QC}$, and $\mathrm{QC}^!$ on a scheme Y with trivialization of the z-action. In particular the z-twisting action on the z-trivial subcategory of equivariant sheaves in each case is trivial. Further, all sheaves on the z-specialized loop space are z-trivial, so z-triviality is only relevant for Koszul duality for stacks.

Proposition 5.13. *Let $z \in G$ be central and let $\dagger = \wedge, u, '$. There is a canonical equivalence $\rho \simeq \rho(z)$ on the z-trivial block of $\mathrm{Perf}(\mathcal{L}^\dagger(X/G))$, $\mathrm{Coh}(\mathcal{L}^\dagger(X/G))$, $\mathrm{QC}(\mathcal{L}^\dagger(X/G))$ and $\mathrm{QC}^!(\mathcal{L}^\dagger(X/G))$. When $\dagger = '$, the z-trivial block is the entire category.*

Proof. It is more or less immediate to see that the z-twisting action $\sigma(z)$ acts trivially on the z-trivial block of any $\mathrm{Rep}(G)$-category $\mathbf{C}$. Furthermore, we observe that there is a canonical identification $\mathcal{L}(\sigma_{X/G}(z)) = \sigma_{\mathcal{L}(X/G)}(z)$, and the claim follows. To see that z-triviality is an empty condition on the specialized loop space, note that the twisting $\sigma(z)$ acts trivially on the identity $e \in \mathcal{L}(BG)$, and therefore trivially on the base change $\{e\} \times_{\mathcal{L}(BG)} \mathcal{L}(X/G)$. ☐

We can see via examples that z-triviality is not an empty condition for formal loop spaces.

Example 5.14. Consider Example 4.1.6 from [Ch20a], i.e. take the z-twisted loop rotation action on $\mathcal{L}(BT) = T \times BT$. Let Λ be the character lattice of T, so that $\mathcal{O}(T)$ is spanned by t^λ for $\lambda \in \Lambda$. We have $\mathrm{Perf}(\mathcal{L}(BT))) = \bigoplus_{\lambda \in \Lambda} \mathrm{Perf}(T) \otimes \mathrm{Rep}(T)$ and therefore

$$\mathrm{Perf}(\widehat{\mathcal{L}}(BT))^{\rho(z)} = \bigoplus_{\lambda \in \Lambda} \mathrm{PreMF}(\widehat{T}, 1 - t^\lambda(z)t^\lambda) \otimes \mathrm{Rep}(T)$$

where PreMF is defined in [Pr11]. The z-trivial subcategory corresponds to the subcategory of $\mathrm{Rep}(T)$ of representations on which $z \in T$ acts trivially, i.e. if $T' = T/\mathbb{Z}z$, then

$$\mathrm{Perf}(\widehat{\mathcal{L}}(BT))_z^{\rho(z)} = \bigoplus_{\lambda \in \Lambda} \mathrm{PreMF}(\widehat{T}, 1 - t^\lambda(z)t^\lambda) \otimes \mathrm{Rep}(T').$$

[35] This can also be described as equivariance for an action of the quotient $G' = G/\mathbb{Z}z$ on $\mathbf{C}$ defined by the z-trivialization. Namely, given a G'-linear category $\mathbf{C}'$, an identification $\mathbf{C} \simeq \mathbf{C}' \otimes_{\mathrm{Rep}(G')} \mathrm{Rep}(G)$ gives an identification of the action of z on the left with the identity functor.

Example 5.15. We specialize the above example at $T = \mathbb{G}_m$ and $z = 1$. In this case, we have
$$\mathrm{Perf}(\widehat{\mathcal{L}}(B\mathbb{G}_m))^{B\mathbb{G}_a \rtimes \mathbb{G}_m} = \bigoplus_{n\in\mathbb{Z}} \mathrm{PreMF}(\widehat{\mathbb{G}_m}, 1-t^n) \simeq \mathrm{Coh}((\{0\} \times_{\mathfrak{t}} \{0\}))_{\mathbb{Z}} \oplus \bigoplus_{n\neq 0} \mathrm{Coh}(*)_{\mathbb{Z}}$$
where $\mathbb{Z}$ indicates the grading coming from $\mathbb{G}_m$-scaling on odd tangent bundles (rather than from $T = \mathbb{G}_m$-equivariance). This example will be computed in parallel in Example 5.26.

Definition 5.16. In the set-up above, for $\dagger = \wedge, u, '$, note that by passing through the equivalence $s\ell_z^\dagger$ and restricting to the z-trivial block, we have
$$\mathrm{Coh}(\mathcal{L}_z^\dagger(X/G))_z^{S^1_\rho} \simeq \mathrm{Coh}(\mathcal{L}^\dagger(X_\circ^z/G^z))_z^{S^1_{\rho(z)}} \simeq \mathrm{Coh}(\mathcal{L}^\dagger(X_\circ^z/G^z)_\rho)_z^{S^1} \simeq \mathrm{Coh}(\mathcal{L}^\dagger(X_\circ^z/G^z))_z^{B\mathbb{G}_a}$$
where the first isomorphism is $s\ell_z^\dagger$, the second arising from the Proposition 5.13, and the third by Corollary 6.10 in [BN12]. Therefore, moving the $\mathbb{G}_m$-scaling action on the right through this equivalence, we define[36] the category
$$\mathrm{Coh}(\mathcal{L}_z^\dagger(X/G))_z^{B\mathbb{G}_a \rtimes \mathbb{G}_m} := \mathrm{Coh}(\mathcal{L}^\dagger(X/G))_z^{B\mathbb{G}_a \rtimes \mathbb{G}_m}$$
as well as a forgetful functor
$$\mathrm{Coh}(\mathcal{L}_z^\dagger(X/G))_z^{B\mathbb{G}_a \rtimes \mathbb{G}_m} \to \mathrm{Coh}(\mathcal{L}^\dagger(X/G))_z^{S^1} = \mathrm{Coh}(\mathcal{L}_z^\dagger(X/G))_z^{S^1}.$$
Furthermore, these categories are covariantly functorial under pushforward by proper maps $f : X/G \to Y/G$ (compatible with z-trivialization).

5.2. **Koszul duality.**

A Koszul duality between modules for the de Rham algebra and the algebra of differential forms has been long established in the literature, e.g. in [Ka91] [BeDr91]. This Koszul duality was reinterpreted in [BN12] as an equivalence of categories between graded S^1-equivariant quasicoherent sheaves on the formal loop space and filtered D-modules on smooth quotient stacks X/G. We require a renormalized version of this equivalence from the forthcoming work [Ch21]. In this section, we state the main results and definitions of this work. We use the notion of singular support defined in [AG14] for completeness, though it does not appear in our results.

It will be convenient for us to replace the formal loop space with the equivalent odd tangent bundle, defined in Defintion 4.3 of [BN12].

Definition 5.17. Let X be a derived stack with cotangent complex Ω^1_X. The *odd tangent bundle* is defined
$$\mathbb{T}_X[-1] := \mathrm{Spec}_X \mathrm{Sym}^\bullet_X \Omega^1_X[1].$$
We sometimes use the notation $\mathbb{T}_X^{[-1]}$ to save space. We define the *formal odd tangent bundle* $\widehat{\mathbb{T}}_X[-1]$ to be the completion at the zero section. By Theorem 6.9 in [BN12], when X is a QCA stack the exponential map is an S^1-equivariant equivalence $\exp: \widehat{\mathbb{T}}_X[-1] \xrightarrow{\simeq} \widehat{\mathcal{L}}(X)$.

Before we proceed, we emphasize the main subtleties. The first sutblety is that the operation of taking S^1-invariants in the setting of presentable (large) categories often gives poorly behaved results (in particular, such categories that are always killed by the Tate construction; see the introduction to [Pr15]). This phenomenon is exhibited in the results of Section 5 in [BN12], where $\mathrm{QC}(\widehat{\mathcal{L}}X)^{S^1}$ is identified with *complete* modules for a certain completed Rees construction. One tends to rectify this by renormalization of large categories or by working with small categories throughout, i.e. applying S^1-invariants to a small category first, and then ind-completing.

In our setting, the category of compact objects of $\mathrm{QC}(\widehat{\mathcal{L}}X)$ or $\mathrm{QC}^!(\widehat{\mathcal{L}}X) = \mathrm{QC}^!(\widehat{\mathbb{T}}_X[-1])$ is still not the correct candidate. Roughly speaking, Koszul duality swaps free modules with simple modules. Let us specialize to the case $X = BG$ where G is reductive. On the D-modules side, we are interested in objects such as $\mathcal{O}_X$; this object is "simple" but not "free" (in the equivariant setting, "free" D-modules correspond to the notion of *safe* D-modules [DG13]). However, the

[36]The notation is abusive: there is not a $B\mathbb{G}_a \rtimes \mathbb{G}_m$-action on $\mathcal{L}_z^u(X/G)$, but the discussion above allows us to pretend there is one on the z-trivial block.

corresponding expected "free" object $\omega_{\widehat{\mathbb{T}}_{BG}[-1]} \in \mathrm{QC}^!(\widehat{\mathbb{T}}_{BG}[-1])$ is not compact (i.e. is not a finitely generated torsion sheaf) since $\widehat{\mathbb{T}}_X[-1]$ is an inf-stack. Thus, we define a different small subcategory of $\mathrm{QC}^!(\widehat{\mathbb{T}}_X[-1])$, which we call $\mathrm{KPerf}(\widehat{\mathbb{T}}_X[-1])$ (for Koszul-perfect). To define this notion, we first focus on an easier, degenerate form of Koszul duality.

Definition 5.18. There is a Koszul resolution of $\mathcal{O}_X \in \mathrm{QC}^!(\widehat{\mathbb{T}}_X[-1])$ by $\mathrm{Sym}^\bullet_{\mathcal{O}_{\mathbb{T}_X[-1]}} \Omega^1[2]$ with internal differential given by the identity map. Thus, $\mathcal{H}om_{\mathbb{T}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Sym}^\bullet_X \mathcal{T}[-2]$, which is coconnective and generated over $\mathcal{O}_X$ in strictly positive degrees. We denote by $\mathrm{QC}(\mathbb{T}^*_X[2])$ the dg derived category of sheaves of $\mathcal{O}_X$-quasicoherent $\mathrm{Sym}^\bullet_X \mathcal{T}[-2]$-modules on X. We denote by $\mathrm{Perf}(\mathbb{T}^*_X[2])$ the full subcategory of sheaves locally (in X) quasi-isomorphic to a finite rank semi-free complex of $\mathrm{Sym}^\bullet_X \mathcal{T}_X[-2]$-modules.[37]

The following result is a standard Koszul duality result (e.g. see [MR10]).

Proposition 5.19 (Koszul duality for formal vector bundles)**.** *The functor*

$$z^! = \mathcal{H}om_{\mathbb{T}_X[-1]}(\mathcal{O}_X, -) : \mathrm{Coh}(\widehat{\mathbb{T}}_X[-1])^{\mathbb{G}_m} \to \mathrm{Perf}(\mathbb{T}^*_X[2])^{\mathbb{G}_m}$$

is an equivalence of categories. The same is true non-equivariantly.

Definition 5.20. We define the category $\mathrm{Coh}(\mathbb{T}^*_X[2])$ to be the full dg subcategory of $\mathrm{QC}(\mathbb{T}^*[2])$ consisting of sheaves $\mathcal{M}$ of $\mathcal{O}_{\mathbb{T}^*_X[2]}$-modules such that $\mathcal{H}^\bullet(\mathcal{M})$ is (smooth) locally finitely generated as an $\mathcal{H}^\bullet(\mathcal{O}_{\mathbb{T}^*_X[2]}) = \mathcal{H}^\bullet(\mathrm{Sym}^\bullet_X \mathcal{T}[-2])$-module. We define the *category of Koszul-perfect complexes* to be the full subcategory $\mathrm{KPerf}(\widehat{\mathbb{T}}_X[-1]) \subset \mathrm{QC}^!(\widehat{\mathbb{T}}_X[-1])^+$ of sheaves $\mathcal{F}$ such that $z^!\mathcal{F} \in \mathrm{Coh}(\mathbb{T}^*_X[2])$.

We highlight four favorable properties of the category $\mathrm{KPerf}(\widehat{\mathbb{T}}_X[-1])$ from [Ch21].

1. If X is a smooth scheme, then $\mathrm{KPerf}(\widehat{\mathbb{T}}_X[-1]) = \mathrm{Coh}(\widehat{\mathbb{T}}_X[-1])$.
2. If X is a smooth Artin stack with atlas $p : U \to X$, the subcategory $\mathrm{KPerf}(\widehat{\mathbb{T}}_X[-1]) \subset \mathrm{QC}^!(\widehat{\mathbb{T}}_X[-1])$ can be characterized as objects that pull back to $\mathrm{KPerf}(\widehat{\mathbb{T}}_U[-1])$.
3. Given $f : X \to Y$ a smooth (resp. proper) map of smooth Artin stacks, the functors $\mathbb{T}[-1]^!_f : \mathrm{QC}^!(\widehat{\mathbb{T}}_Y[-1]) \to \mathrm{QC}^!(\widehat{\mathbb{T}}_X[-1])$ (resp. $\mathbb{T}[-1]_{f,*}$) restrict to KPerf.
4. Let G be a reductive group. For a smooth quotient stack X/G, and semisimple $z \in G$, the !-pullback $\iota^!_z : \mathrm{Coh}(\mathcal{L}(X/G)) \to \mathrm{QC}^!(\widehat{\mathcal{L}}_z(X/G)) = \mathrm{QC}^!(\widehat{\mathbb{T}}_{X^z_\circ/G^z})$ takes values in $\mathrm{KPerf}(\widehat{\mathbb{T}}_{X^z_\circ/G^z})$.

Before stating the Koszul duality theorem, we need a corresponding notion of z-triviality, as in Definition 5.12, in the setting of D-modules.

Definition 5.21. Let G be an affine algebraic group acting on a smooth scheme X and assume that $z \in G$ acts on X trivially; then z induces an automorphism of the identity functor of $\mathrm{QC}^G(X)$. We say a complex of G-equivariant sheaves (in particular, a weakly or strongly equivariant D-module) is *z-trivial* if this automorphism is the identity on cohomology.

Example 5.22. It is well-known that if G acts on X by finitely many orbits, then the simples in $\mathcal{D}^G(X)^\heartsuit$ are given by pairs $(\mathcal{O}, V)$ where $\mathcal{O}$ is a G-orbit and V is a representation of the component group $A(x)$ of the stabilizer of $x \in \mathcal{O}$. The z-trivial simples are subject to the additional requirement that $[z] \in A(x)$ acts on V by the identity.

The following is the main result from the forthcoming work [Ch21]. We let $F\check{\mathcal{D}}^\omega_\Lambda(X/G)_z$ denote the derived category of filtered coherent D-modules on X/G with singular support Λ.

Theorem 5.23 (Koszul duality for loop spaces of quotient stacks)**.** *Let X/G be a smooth quasiprojective quotient stack, and Λ be a conical closed subset specifying singular support. Let*

[37] Note that we never consider $\mathbb{T}^*_X[2]$ as an honest object in the category of derived stacks.

$z \in G$ be a central element[38]. *We have compatible adjoint equivalences:*

$$\begin{array}{ccc} \mathrm{KPerf}_\Lambda(\widehat{\mathbb{T}}_{X/G}[-1])_z^{B\mathbb{G}_a \rtimes \mathbb{G}_m} & \rightleftarrows & F\breve{\mathcal{D}}^\omega_\Lambda(X/G)_z \\ \downarrow & & \downarrow \mathrm{gr} \\ \mathrm{KPerf}_\Lambda(\widehat{\mathbb{T}}_{X/G}[-1])_z^{\mathbb{G}_m} & \rightleftarrows & \mathrm{Coh}_\Lambda(\mathbb{T}^*_{X/G})_z^{\mathbb{G}_m} \end{array}$$

functorial with respect to smooth pullback and proper pushforward.

Remark 5.24. There is a category of weakly G-equivariant D-modules that sits between strongly G-equivariant D-modules and non-equivariant D-modules:

$$F\breve{\mathcal{D}}^\omega(X/G) \to F\breve{\mathcal{D}}^\omega(X/'G) \to F\mathcal{D}^\omega(X).$$

Under Koszul duality, it corresponds to sheaves on specialized loops $\mathcal{L}'(X/G)$.

Remark 5.25. We observe that in the case of the coherent Springer sheaf, the equivalence of Theorem 2.29 is an equivalence *before* taking S^1-equivariant objects. Thus, in our setting we may actually use the easier *graded Koszul duality* which corresponds on the D-modules side to passing to the associated graded of a filtered D-module. However, we discuss the full theory for completeness.

The following example of a category of filtered D-modules is parallel to Example 5.15.

Example 5.26. Take $G = \mathbb{G}_m$ and $X = \mathrm{pt}$ and fix an isomorphism $C_\bullet(\mathbb{G}_m; k) \simeq k[\epsilon]$; then the category $F\breve{\mathcal{D}}^\omega(X)$ splits as a direct sum by isotypic component of the underlying $\mathbb{G}_m$-representation:

$$F\breve{\mathcal{D}}^\omega(X) = \bigoplus_{n\in\mathbb{Z}} F\breve{\mathcal{D}}^\omega(X)_n.$$

We have that

$$F\breve{\mathcal{D}}^\omega(X)_n = k[t, \epsilon t]\text{-coh}$$

with $|t| = 0$ and $|\epsilon t| = -1$ (in particular, $(\epsilon t)^2 = 0$) and internal differential $d(\epsilon t) = nt$. When $n = 0$, this dg algebra is a graded version of the usual shifted dual numbers $k[\epsilon t]$, and when $n \neq 0$, it is quasi-isomorphic to k, i.e.

$$F\breve{\mathcal{D}}^\omega(X) = k[t, \epsilon t]\text{-coh} \oplus \bigoplus_{n\neq 0} k\text{-coh}.$$

Note that if we forget the filtration, only the trivial isotypic summand survives.

5.3. **The coherent Springer sheaf at parameters.** We can now construct a variety of localization functors between the category of unipotent Langlands parameters $\mathrm{QC}^!(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ and categories of D-modules. We begin in a general setting, considering subcategories of the category $\mathrm{Coh}_\Lambda(\mathcal{L}(X/G))^{S^1}$ generated by a sheaf $\langle\mathcal{S}\rangle$ satisfying a z-triviality condition. Since Koszul duality requires us to consider an additional $\mathbb{G}_m$-equivariant structure, we will need to *choose* a graded lift of the z-localization of $\mathcal{S}$. In general there may be many choices, and choices cannot always be made globally.

Before we proceed, let us review our notation conventions. The sheaf $\mathcal{S} \in \mathrm{Coh}(\mathcal{L}(X/G))^{S^1}$ is an S^1-equivariant sheaf on the global derived loop space. For semisimple parameters z, we define its z-completion by $\mathcal{S}(\hat{z})$ and its z-specialization by $\mathcal{S}(z)$. We denote graded lifts by $\widetilde{\mathcal{S}}(\hat{z})$ and $\widetilde{\mathcal{S}}(z)$. The corresponding filtered complex of D-modules under Koszul duality are denoted $\widetilde{\mathbf{S}}(\hat{z})$ and $\widetilde{\mathbf{S}}(z)$. Forgetting the filtration, we obtain D-modules $\mathbf{S}(\hat{z})$ and $\mathbf{S}(z)$

In the following, for $\dagger = \wedge, '$, we let $f_z^\dagger : \mathcal{L}^\dagger(X_\circ^z/G^z) \to \mathcal{L}_z^\dagger(X/G) \to \mathcal{L}(X/G)$ be the composition of the "shift by z" map with the equivariant localization map; $f_z^\dagger$ is S^1-equivariant where S^1 acts via $\rho(z)$ on the source and ρ on the target. We let the undecorated $f_z : X_\circ^z/G^z \to \mathcal{L}(X/G)$ be the pre-composition with the inclusion of constant loops. Let $\widetilde{B\mathbb{G}_a} = B\mathbb{G}_a \rtimes \mathbb{G}_m$, let $\widetilde{\mathrm{Tate}}$

[38]This is used to phrase z-triviality; if the reader would prefer to ignore this technicality, they may take $z = e$ to be the identity.

be the $\mathbb{G}_m$-equivariant Tate construction[39], and recall the notation from Definition 5.16. Let $\mathbf{C}$ be a category over $\mathrm{Rep}(\mathbb{G}_m)$ and $\mathrm{oblv} : \mathbf{C} \to \mathbf{C}'$ the forgetful functor; a *graded lift* of an object $X \in \mathbf{C}'$ is an object $\widetilde{X} \in \mathbf{C}$ along with an equivalence $\mathrm{oblv}(\widetilde{X}) \simeq X$.

Proposition 5.27. *Let X/G be a smooth quasiprojective quotient stack by a reductive group, and $\Lambda \subset \mathrm{Sing}(\mathcal{L}(X/G))$ a singular support condition, with restriction $\Lambda_z = f_z^! \Lambda$. Let $\mathcal{S} \in \mathrm{Coh}_\Lambda(\mathcal{L}(X/G))^{S^1}$ be such that $\mathcal{S}(\widehat{z}) := \widehat{f}_z^! \mathcal{S}$ is in the z-trivial block and choose a graded lift $\widetilde{\mathcal{S}}(\widehat{z})$. Then, there is a commuting diagram*

$$\begin{array}{ccccccc}
\langle \mathcal{S} \rangle & \longrightarrow & \langle \mathcal{S}(\widehat{z}) \rangle & \longleftarrow & \langle \widetilde{\mathcal{S}}(\widehat{z}) \rangle & \longrightarrow & \langle \mathbf{S}(\widehat{z}) \rangle \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Coh}_\Lambda(\mathcal{L}(X/G))^{S^1} & \to & \mathrm{KPerf}_{\Lambda_z}(\widehat{\mathcal{L}}_z(X/G))^{S^1} & \leftarrow & \mathrm{KPerf}_{\Lambda_z}(\widehat{\mathbb{T}}^{[-1]}_{X^z_\circ/G^z})_z^{\widetilde{B\mathbb{G}_a}} & \to & \mathrm{KPerf}_{\Lambda_z}(\widehat{\mathbb{T}}^{[-1]}_{X^z_\circ/G^z})_z^{\widetilde{\mathrm{Tate}}} \\
 & & & & \big\downarrow \simeq & & \big\downarrow \simeq \\
 & & & & F\breve{\mathcal{D}}^\omega_{\Lambda_z}(X^z_\circ/G^z)_z & \longrightarrow & \breve{\mathcal{D}}^\omega_{\Lambda_z}(X^z_\circ/G^z)_z.
\end{array}$$

Remark 5.28. Note that aside from applying renormalized Koszul duality, the category KPerf is required for the following reason. In a general setting, if $F : \mathbf{C} \to \mathbf{D}$ is a continuous functor which preserves compact objects (i.e. a left adjoint with a continuous right adjoint) between compactly generated categories, then for $X \in \mathbf{C}$ compact we have a commuting diagram:

$$\begin{array}{ccc}
\mathrm{End}(X)^{op}\text{-mod} & \hookrightarrow & \mathbf{C} \\
{\scriptstyle -\otimes_{\mathrm{End}(X)} \mathrm{End}(F(X))} \big\downarrow & & \big\downarrow {\scriptstyle F} \\
\mathrm{End}(F(X))^{op}\text{-mod} & \hookrightarrow & \mathbf{D}.
\end{array}$$

Commutativity follows by checking the tautological commutativity of right adjoints, while compactness of X guarantees that the left adjoint to $\mathrm{Hom}(X, -)$ is fully faithful[40] (and similarly for $F(X)$). Unfortunately, the functor $\widehat{f}^! : \mathrm{QC}^!(\mathcal{L}(X/G)) \to \mathrm{QC}^!(\widehat{\mathcal{L}}(X/G))$ is a right adjoint and does not preserve compact objects. On the other hand, the renormalization $\widehat{f}^! : \mathrm{QC}^!(\mathcal{L}(X/G)) \to \mathrm{Ind}(\mathrm{KPerf}(\widehat{\mathcal{L}}(X/G)))$ preserves compact objects by construction. In particular, we have a commuting square:

$$\begin{array}{ccc}
\mathrm{End}(\mathcal{S})\text{-mod} \simeq \langle \mathcal{S} \rangle & \hookrightarrow & \mathrm{QC}^!_\Lambda(\mathcal{L}(X/G)) \\
{\scriptstyle -\otimes_{\mathrm{End}(\mathcal{S})} \mathrm{End}(\mathcal{S}(\widehat{z}))} \big\downarrow & & \big\downarrow {\scriptstyle \widehat{f}^!_z} \\
\mathrm{End}(\mathcal{S}(\widehat{z}))\text{-mod} \simeq \langle \mathcal{S}(\widehat{z}) \rangle & \hookrightarrow & \mathrm{Ind}(\mathrm{KPerf}_{\Lambda_z}(\widehat{\mathcal{L}}_z(X/G))).
\end{array}$$

We state the weakly-equivariant variant as well; it has the additional feature that there is a functor from the category of D-modules to the category of coherent sheaves on the derived loop space, which we use to formulate a conjecture regarding irreducible objects. Recall that for smooth schemes X, $\mathrm{KPerf}(\mathbb{T}_X[-1]) = \mathrm{Coh}(\mathbb{T}_X[-1])$. Note that this result does not depend on Theorem 5.23 and follows from Corollary 5.2 of [BN12].

Proposition 5.29. *Let X/G be a smooth quasiprojective quotient stack by a reductive group, and $\Lambda \subset \mathrm{Sing}(\mathcal{L}(X/G))$ a singular support condition, with restriction $\Lambda_z = f_z^! \Lambda$. Let $\mathcal{S} \in \mathrm{Coh}_\Lambda(\mathcal{L}(X/G))^{S^1}$ be such that $\mathcal{S}(\widehat{z}) := \widehat{f}_z^! \mathcal{S}$ is in the z-trivial block and choose a graded lift*

[39] I.e. for a k-linear category $\mathbf{C}$ with a $\widetilde{B\mathbb{G}_a} = B\mathbb{G}_a \rtimes \mathbb{G}_m$-action, the equivariant category $\mathbf{C}^{\widetilde{B\mathbb{G}_a}}$ is linear over $k[u]\text{-mod}^{\mathbb{G}_m}$, and $\mathbf{C}^{\widetilde{\mathrm{Tate}}}$ is obtained by passing to the generic point.

[40] I.e. the unit of the adjunction $M \to \mathrm{Hom}(X, X \otimes_{\mathrm{End}(X)} M)$ is an equivalence when $\mathrm{Hom}(X, -)$ commutes with colimits.

$\widetilde{\mathcal{S}}(z)$. *Then, there is a commuting diagram*

$$\begin{array}{ccccccc} \langle \mathcal{S} \rangle & \rightleftarrows & \langle \mathcal{S}(z) \rangle & \longleftarrow & \langle \widetilde{\mathcal{S}}(z) \rangle & \longrightarrow & \langle \mathbf{S}(z) \rangle \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Coh}_\Lambda(\mathcal{L}(X/G))^{S^1} & \rightleftarrows & \mathrm{Coh}_{\Lambda_z}(\mathcal{L}'_z(X/G))^{S^1} & \leftarrow & \mathrm{Coh}_{\Lambda_z}(\mathbb{T}_{X^z_\circ}/G^z)_z^{\widetilde{B\mathbb{G}_a}} & \rightarrow & \mathrm{Coh}_{\Lambda_z}(\mathbb{T}_{X^z_\circ}/G^z)_z^{\widetilde{\mathrm{Tate}}} \\ & & & & \downarrow \simeq & & \downarrow \simeq \\ & & & & F\mathcal{D}^\omega_{\Lambda_z}(X^z_\circ/G^z)_z & \longrightarrow & \mathcal{D}^\omega_{\Lambda_z}(X^z_\circ/G^z)_z. \end{array}$$

We now consider a more specific context where the sheaf $\mathcal{S}$ is of geometric origin: let $\mu : \widetilde{X} \to X$ be a G-equivariant proper map of smooth G-schemes, and define

$$\mathcal{S} := \mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\widetilde{X}/G)} = \mathcal{L}\mu_* \omega_{\mathcal{L}(\widetilde{X}/G)}.$$

We first verify the z-triviality condition required in the above results.

Lemma 5.30. *Let $z \in G$ be semisimple. The sheaf $\mathcal{S}(\hat{z})$ is z-trivial for every semisimple $z \in \widetilde{G}$ (and likewise for $\mathcal{S}(z)$).*

Proof. Follows by equivariant locaization and base change, i.e. $\mathcal{S}(\hat{z})$ is the pushforward of $\omega_{\hat{\mathcal{L}}(\widetilde{X}^z/G^z)}$, which is z-trivial since z acts trivially on $\widetilde{X}^z$ and z is central in G^z. ☐

In adddition to z-triviality being automatic in this setting, there is a canonical choice for graded lifts when $\mathcal{S} = \mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\tilde{X}/G)}$.

Definition 5.31. Let $\mathcal{S} = \mathcal{L}\mu_* \mathcal{O}_{\mathcal{L}(\tilde{X}/G)}$. For any $z \in G$ semisimple, there is a *geometric (or Hodge) graded lift* of $\mathcal{S}(\hat{z})$ and $\mathcal{S}(z)$. Namely, by base change along the diagram

$$\begin{array}{ccc} \widehat{\mathbb{T}}_{\widetilde{X}^z/G^z}[-1] \simeq \hat{\mathcal{L}}_z(\widetilde{X}/G) & \longrightarrow & \mathcal{L}(\widetilde{X}/G) \\ \downarrow \mathbb{T}\mu^z & & \downarrow \\ \widehat{\mathbb{T}}_{X^z/G^z}[-1] \simeq \hat{\mathcal{L}}_z(X/G) & \longrightarrow & \mathcal{L}(X/G) \end{array}$$

we have that $\mathcal{S}(\hat{z}) \simeq \mathbb{T}\mu^z_* \omega_{\widehat{\mathbb{T}}_{X^z/G^z}[-1]}$. We give $\mathcal{S}(\hat{z})$ the graded lift arising from the $\mathbb{G}_m$-equivariant structure on $\omega_{\widehat{\mathbb{T}}_{X^z/G^z}[-1]}$ arising via the natural $\mathbb{G}_m$-action on the odd tangent bundle. A similar natural lift can be made for the specialized Springer sheaf $\mathcal{S}(z)$.

Remark 5.32. The geometric graded lift has favorable functoriality properties with respect to Koszul duality. Namely, the dualizing sheaf $\omega_{\widehat{\mathbb{T}}_{\widetilde{X}^z/G^z}}$ corresponds under Koszul duality to the canonical sheaf D-module $\omega_{\widetilde{X}^z/G^z}$. By functoriality, $\mathcal{S}(\hat{z})$ is Koszul dual to the pushforward $\mu^z_* \omega_{\widetilde{X}^z/G^z}$. By a deep theorem of Saito [Sa88], this pushforward is a strict filtered complex of D-modules (see also [Gi12]).

5.3.1. *Application to coherent Springer theory.* We now let $\mu : \widetilde{\mathcal{N}} \to \mathfrak{g}$ denote the Springer resolution, discuss applications to conjectures from Section 4.1. In Theorem 4.1.1 of [Gi12] it is shown that for trivial semisimple parameter (i.e. for the Springer resolution over the nilpotent cone), the D-module $\mu_* \mathcal{O}_{\widetilde{\mathcal{N}}}$ has vanishing higher cohomology. As an approach to Conjecture 4.15, we conjecture that the same is true over all semisimple parameters.

Conjecture 5.33. The D-module $\mu^z_* \omega_{\widetilde{\mathcal{N}}^z/\widetilde{G}^z}$ has vanishing higher cohomology.

The equivalence $\mathcal{H}$-mod $\simeq \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ of Theorem 2.29 is not t-exact, and naturally leads to the following question.

Question 5.34. The equivalence $\mathcal{H}$-mod $\simeq \langle \mathcal{S} \rangle \subset \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ is not t-exact. Describe the corresponding "exotic" t-structures on each side of the equivalence, their correponding abelian categories and classify the simple objects.

The question can be probed by completing or specializing at semisimple parameters. The results in [CG97, KL87] give a bijection between irreducible $\mathcal{H}$-representations with central character (s,q) with a certain set of parameters consisting of a G^s-equivariant intersection cohomology sheaf on $\mathcal{N}^{(s,q)} = \{n \in \mathcal{N} \mid gng^{-1} = qn\}$ subject to the additional condition that it "appears in the Springer sheaf." Thus Proposition 5.29 defines for us a functor:

$$P_{s,q} : \langle \mu_*^{(s,q)} \mathcal{O}_{\widetilde{\mathcal{N}}^{(s,q)}} \rangle \longrightarrow \mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})).$$

In this way, we obtain a class of simple "skyscraper" objects in $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$. Explicitly, for a filtered D-module $(\mathcal{M}, F)$ the object $P_{s,q}(\mathcal{M}, F)$ is obtained by taking the associated graded and applying a sheared graded Koszul duality (see Proposition 5.19), and then pushing the resulting object forward to $\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G})$.

These objects do not necessarily lie in the heart of $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ (equipped with the usual t-structure). We pose the following conjecture.

Conjecture 5.35. There is an "exotic" t-structure on $\mathrm{Coh}(\mathcal{L}(\widehat{\mathcal{N}}/\widetilde{G}))$ which, after restricting to $\langle \mathcal{S} \rangle$, is identified with the standard t-structure on $\mathcal{H}$-mod. The simple objects in the heart of this t-structure on $\langle \mathcal{S} \rangle$ are given by the application of $P_{s,q}$ to simple objects in the Koszul dual non-standard t-structure in [BGS96, Ri13].

6. Moduli of Langlands parameters for GL_n

We now turn to arithmetic applications of our results, in particular the study of moduli spaces of Langlands parameters for $G = \mathrm{GL}_n$. Let F be a p-adic field, with residue field $\mathbb{F}_q$, and let $G^\vee$ denote a connected, split, reductive group over F (i.e. on the automorphic side of Langlands).

The derived category $D(G^\vee)$ of smooth complex representations of $G^\vee(F)$ admits a decomposition into blocks. The so-called *principal block* of $D(G^\vee)$ (that is, the block containing the trivial representation) is naturally equivalent to the category of $\mathcal{H}_q$-modules, where $\mathcal{H}_q$ now denotes the affine Hecke algebra associated to $G^\vee$, with parameter q. Theorem 4.12 then gives a fully faithful embedding from this principal block into $\mathrm{QC}^!(\mathcal{L}_q(\widehat{\mathcal{N}}/G))$.

The space $\mathcal{L}_q(\widehat{\mathcal{N}}/G)$ has a natural interpretation in terms of Langlands (or Weil-Deligne) parameters for $G^\vee(F)$. Recall that a Langlands parameter for $G^\vee$ is a pair (ρ, N), where $\rho : W_F \to G(\mathbb{C})$ is a homomorphism with open kernel, and N is a nilpotent element of $\mathrm{Lie}\, G$ such that, for all σ in the inertia group I_F of W_F, one has $\mathrm{Ad}(\rho(\mathrm{Fr}^n \sigma))(N) = q^n N$, where Fr denotes a Frobenius element of W_F.

On the other hand, the underlying stack of $\mathcal{L}_q(\widehat{\mathcal{N}}/G)$ can be regarded as the moduli stack of pairs (s, N), where $s \in G(\mathbb{C})$, $N \in \mathrm{Lie}\, G$, and $\mathrm{Ad}(s)(N) = qN$, up to G-conjugacy (i.e. the map ρ above vanishes on inertia). To such a pair we can attach the Langlands parameter (ρ, N), where ρ is the unramified representation of W_F taking Fr to s. Such a Langlands parameter is called *unipotent*, and this construction identifies $\mathcal{L}_q(\widehat{\mathcal{N}}/G)$ with the moduli stack of unipotent Langlands parameters, modulo G-conjugacy.[41] We thus obtain a fully faithful embedding from the principal block of $D(G^\vee)$ into the category of ind-coherent sheaves on the moduli stack of unipotent Langlands parameters.

It is natural to ask if this extends to an embedding of *all* of $D(G^\vee)$ into a category of sheaves on the moduli stack of *all* Langlands parameters. We will show that, at least when $G = \mathrm{GL}_n$ over F, this is indeed the case. For the remainder of the section, we will take $G = G^\vee = \mathrm{GL}_n$.

6.1. **Blocks, semisimple types, and affine Hecke algebras.** Our argument proceeds by reducing to the principal block. On the representation theory side, this reduction is a consequence of the Bushnell-Kutzko theory of types and covers [BK97, BK99], which we now recall. *For this subsection only, we will reverse our conventions* to avoid cumbersome notation; that is, we let G be a connected reductive split group over F on the automorphic side of Langlands duality.

[41] Strictly speaking, a Langlands parameter is a pair (ρ, N) as above in which *ρ is semisimple*. When building a moduli space of Langlands parameters we must drop this condition, however, as the space of semisimple parameters is not a well-behaved geometric object. In particular the locus in $\mathcal{L}_q$ consisting of pairs (s, N) in which s is semisimple is neither closed nor open in $\mathcal{L}_q$.

6.1.1. *Supercuspidal support.* Let $P \subset G$ be a parabolic subgroup with Levi M and unipotent radical U, and let π be a smooth complex representation of M. Recall that the parabolic induction $i_P^G(\pi)$ is obtained by inflating π to a representation of P, twisting by the square root of the modulus character of P, and inducing to G. The parabolic induction functor i_P^G has a natural left adjoint, the parabolic restriction r_G^P (restriction to P, untwist, and U-coinvariants).

Definition 6.1. A complex representation π of G is *supercuspidal* if, for all proper parabolic subgroups P of G, the parabolic restriction $r_G^P(\pi)$ vanishes. Let π be an irreducible supercuspidal representation of M; an irreducible complex representation Π has *supercuspidal support* (M, π) if Π is isomorphic to a subquotient of $i_P^G(\pi)$ (this is well-defined up to conjugacy).

A character χ of M is *unramified* if it is trivial on every compact open subgroup of M, and the Levi-supercuspidal pairs (M, π) and (L, π') are *inertially equivalent* if there exists an unramified character χ of L such that (M, π) and $(L, \pi' \otimes \chi)$ are G-conjugate.

For such a pair (M, π) up to inertial equivalence, following Bernstein-Deligne [BeDe84], we define $D(G)_{[M,\pi]} \subset D(G)$ to be the full subcategory of objects such that every subquotient of Π has supercuspidal support inertially equivalent to (M, π). Then Bernstein-Deligne show:

Theorem 6.2. *The full subcategory $D(G)_{[M,\pi]}$ is a block of $D(G)$, i.e. summing over supercuspidals up to inertial equivalence,*

$$D(G) = \bigoplus D(G)_{[M,\pi]}.$$

6.1.2. *Types and Hecke algebras.* We recall the notion of a type.

Definition 6.3. A *type* for G is a pair (K, τ), where $K \subset G$ is a compact open subgroup and τ is an irreducible complex representation of K, such that[42] the full subcategory $\mathrm{Rep}^{sm}(G, K, \tau) \subset \mathrm{Rep}^{sm}(G)$ consisting of representations V which are generated by the image of the evaluation map $\mathrm{Hom}_K(\tau, V) \otimes \tau \to V$. Attached to a type we have its Hecke algebra

$$\mathcal{H}(G, K, \tau) := \mathrm{End}_G(\mathrm{cInd}_K^G(\tau))$$

and an equivalence of abelian categories $\mathrm{Rep}^{sm}(G, K, \tau)^{\heartsuit} \simeq \mathcal{H}(G, K, \tau)\text{-mod}^{\heartsuit}$.

The main result of [BK99] describes an arbitrary block of $D(G)$ as a category of modules for a certain tensor product of Hecke algebras, via the theory of G-covers, providing a connection between parabolic induction methods (which involve subgroups which are not compact open) and Hecke algebra methods (which only make sense for compact open subgroups).

We first consider the block $D(L)_{[L,\pi]}$ (i.e. where $L = G$). Let L be a Levi subgroup of G and π a supercuspidal representation of L. We denote by $L_0 \subset L$ the smallest subgroup containing every compact open; then L/L_0 is free abelian of rank equal to $\dim(Z(L))$. Furthermore, the unramified characters of L are in bijection with the characters of L/L_0. There is a bijection

$$X^\bullet(L/L_0)/H \longleftrightarrow \mathrm{Irr}(D(L)^{\heartsuit}_{[L,\pi]}), \qquad \chi \mapsto \pi \otimes \chi$$

where we denote $X^\bullet(L/L_0) = \mathrm{Hom}(L/L_0, \mathbb{C}^\times)$ and $H \subset X^\bullet(L/L_0)$ is the subgroup of unramified characters χ such that $\pi \otimes \chi \simeq \pi$. Moreover, there is an equivalence of categories:

$$D(L)_{[L,\pi]} \simeq \mathbb{C}[X^\bullet(L/L_0)]^H\text{-mod}, \qquad \pi \otimes \chi \mapsto \mathbb{C}_\chi.$$

We may rephrase this equivalence in terms of types and Hecke algebras as follows: first, we may (by Section 1.2 in [BK99]) choose a maximal simple cuspidal type (K_L, τ_L) occurring in π. One then has a natural support-preserving isomorphism of $\mathcal{H}(L, K_L, \tau_L) \simeq \mathbb{C}[X^\bullet(L/L_0)]^H$, and thus an (inverse) equivalence

$$\mathbb{C}[X^\bullet(L/L_0)]^H\text{-mod} \simeq D(G)_{[L,\pi]}, \qquad V \mapsto V \otimes_{\mathcal{H}(L,K_L,\tau_L)} \mathrm{cInd}_{K_L}^L \tau_L.$$

We are interested in understanding the induction of (L, π) to G. This is achieved by the following composite of results of [BK99]; we refer the reader to *op. cit.* for the definitions of simple type and G-cover.

[42] See pp. 594 of [BK97] for why this is necessary.

Theorem 6.4 ([BK99]). *Let $[L,\pi]$ and the cuspidal type (K_L,τ_L) be as above, and let $P \subset G$ be a parabolic subgroup with Levi factor L. There exists an intermediate*[43] *Levi subgroup $L \subset L^\dagger \subset G$, and types $(K^\dagger,\tau^\dagger)$ of $L^\dagger$ and (K,τ) of G with the following properties:*

(1) The type $(K^\dagger,\tau^\dagger)$ is a simple type of $L^\dagger$.

(2) (K,τ) is a G-cover of $(K^\dagger,\tau^\dagger)$, and $(K^\dagger,\tau^\dagger)$ is an $L^\dagger$-cover of (K_L,τ_L). In particular we have natural injections:

$$T_{P\cap L^\dagger}:\ \mathcal{H}(L,K_L,\tau_L) \hookrightarrow \mathcal{H}(L^\dagger,K^\dagger,\tau^\dagger)$$

$$T_{L^\dagger P}:\ \mathcal{H}(L^\dagger,K^\dagger,\tau^\dagger) \xhookrightarrow{\simeq} \mathcal{H}(G,K,\tau)$$

with $T_{(L^\dagger)P}$ an isomorphism.

(3) The functors

$$\mathrm{Hom}_K(\tau,-):\ D(G)_{[L,\pi]} \xrightarrow{\simeq} \mathcal{H}(G,K,\tau)\text{-mod}$$

$$\mathrm{Hom}_{K^\dagger}(\tau^\dagger,-):\ D(L^\dagger)_{[L,\pi]} \xrightarrow{\simeq} \mathcal{H}(L^\dagger,K^\dagger,\tau^\dagger)\text{-mod}$$

$$\mathrm{Hom}_{K_L}(\tau_L,-):\ D(L)_{[L,\pi]} \xrightarrow{\simeq} \mathcal{H}(L,K_L,\tau_L)\text{-mod}$$

are equivalences of categories. Moreover, for any representation V in $D(L)$, one has an isomorphism of $\mathcal{H}(G,K,\tau)$-modules:

$$\mathrm{Hom}_K(\tau, i^G_{P'}V) \cong \mathrm{Hom}_{K_L}(\tau_L,V)\otimes_{\mathcal{H}(L,K_L,\tau_L)}\mathcal{H}(G,K,\tau),$$

where P' denotes the opposite parabolic to P, and where $\mathcal{H}(G,K,\tau)$ is regarded as an $\mathcal{H}(L,K_L,\tau_L)$-module via the map $T_P := T_{L^\dagger P}\circ T_{P\cap L^\dagger}$.

(4) Suppose $L^\dagger \simeq \prod_i L_i^\dagger$, with each $L_i^\dagger \simeq \mathrm{GL}_{n_i}$ for some n_i. Let L_i be the projection of L to $L_i^\dagger$, and let π_i be the projection of π to L_i. Let H_i denote the group of unramified characters χ of $L_i^\dagger$ such that $\pi\otimes\chi \simeq \pi$, and let r_i denote the order of H_i. Then $n_i = r_i m_i$ for some positive integer m_i, and there is a natural isomorphism (depending on π):

$$\mathcal{H}(L^\dagger,K^\dagger,\tau^\dagger) \cong \bigotimes_i \mathcal{H}_{q^{r_i}}(m_i),$$

where $\mathcal{H}_{q^{r_i}}(m_i)$ denotes the affine Hecke algebra associated to GL_{m_i} with parameter q^{r_i}.

These constructions are naturally compatible with parabolic induction, in the following sense: let M be a Levi with $L \subset M \subset G$, and with parabolic $Q = MP$. Then Theorem 6.4 gives us an M-cover (K_M,τ_M) of (K_L,τ_L) and a G-cover (K,τ) of (K_L,τ_L), as well as maps:

$$T_{P\cap M}: \mathcal{H}(L,K_L,\tau_L)\to\mathcal{H}(M,K_M,\tau_M), \qquad T_P: \mathcal{H}(L,K_L,\tau_L)\to\mathcal{H}(G,K,\tau).$$

We then have:

Theorem 6.5 ([BK99]). *There exists a unique map:*

$$T_Q: \mathcal{H}(M,K_M,\tau_M)\to\mathcal{H}(G,K,\tau)$$

such that $T_P = T_Q\circ T_{P\cap M}$. Moreover, for any $V \in D(M)$, we have an isomorphism of $\mathcal{H}(G,K,\tau)$-modules:

$$\mathrm{Hom}_K(\tau, i^G_{Q'}V) \cong \mathrm{Hom}_{K_M}(\tau_M,V)\otimes_{\mathcal{H}(M,K_M,\tau_M)}\mathcal{H}(G,K,\tau).$$

Example 6.6. The fundamental (and motivating) example for this is when $L = T$ is the standard maximal torus with parabolic $P = B$ the standard Borel, and $\tau = 1$ is the trivial character of T. In this setting K_L is the maximal compact subgroup $T_0 = T(O) \subset T$, and τ_L is the trivial character. Moreover $L^\dagger = G$, the subgroup $K = I \subset G$ is the Iwahori subgroup, and τ is the trivial representation of I. We then have natural identifications of the Hecke algebra:

$$\mathcal{H}(L,K_L,1) \simeq \mathbb{C}[T/T_0] \simeq \mathbb{C}X_\bullet(T).$$

[43] Defined to be the smallest Levi containing the G-normalizer of the type (K_L,τ_L).

and a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[X_\bullet] & \xrightarrow{\simeq} & \mathcal{H}(T, T(O), 1) \\ \big\downarrow & & \big\downarrow{\scriptstyle T_P} \\ \mathcal{H}_q & \xrightarrow{\simeq} & \mathcal{H}(G, I, 1). \end{array}$$

More generally, if $M \subset G$ is a Levi subgroup and Q is its standard parabolic, then K_M is the Iwahori subgroup $I \cap M$ of M, and the map

$$T_Q : \mathcal{H}(M, I \cap M, 1) \to \mathcal{H}(G, I, 1)$$

is uniquely determined by the following properties:

- $T_Q \circ T_{B \cap M} = T_B$,
- If $w \in W(M)$ is an element of the Iwahori-Weyl group of M, then $T_Q(I_M w I_M) = IwI$.

This picture is compatible with the general situation in the following sense. Suppose for simplicity that $L^\dagger = G$. Then L is a product of m copies of $\mathrm{GL}_{\frac{n}{m}}$ for some divisor m of n, and (after an unramified twist) we may assume that π has the form $\pi_0^{\otimes m}$. There is an extension E/F of degree $\frac{n}{m}$ and ramification index r, and an embedding $\mathrm{GL}_m(E) \subset G = \mathrm{GL}_n(F)$, such that the intersection $L \cap GL_m(E)$ is the standard maximal torus of $GL_m(E)$.

We denote the subgroup $GL_m(E)$ by G_E, its standard maximal torus by T_E and its standard Iwahori by I_E. Let M be a Levi such that $L \subset M \subset G$, define $M_E = M \cap G_E$ and take (K_M, τ_M) to be a cover of (K_L, τ_L) via Theorem 6.4. The choice of π then gives rise to an isomorphism $\mathbb{C}X_\bullet(T) \simeq \mathcal{H}(L, K_L, \tau_L)$, such that for each coharacter $\lambda \in X_\bullet(T)$ the image of λ is supported on the double coset $K_L \lambda(\varpi_E) K_L$, and such that the induced action of $X_\bullet(T)$ on the Hecke module attached to π is trivial. We then have:

Theorem 6.7 (Theorem 6.4 [BK93])**.** *Assume that $L^\dagger = G$. There is an isomorphism $\mathcal{H}_{q^r}(m) \simeq \mathcal{H}(G, K, \tau)$ fitting into a commutative diagram:*

$$\begin{array}{ccccc} \mathcal{H}(T_E, (T_E)_0, 1) & \cong & \mathbb{C}X_\bullet(T) & \cong & \mathcal{H}(L, K_L, \tau_L) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}(M_E, I_E \cap M_E, 1) & \cong & \bigotimes_{m_i} \mathcal{H}_{q^r}(m_i) & \cong & \mathcal{H}(M, K_M, \tau_M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}(G_E, I_E, 1) & \cong & \mathcal{H}_{q^r}(m) & \cong & \mathcal{H}(G, K, \tau). \end{array}$$

Thus when $[L, \pi]$ is "simple" (that is, when $L^\dagger = G$), we have a natural reduction of $D(G)_{[L,\pi]}$ to the principal block of $D(G_E)$, in a manner compatible with parabolic induction. In general we obtain a reduction of $D(G)_{[L,\pi]}$ to a tensor product of such principal blocks.

6.2. **The moduli spaces $X^\nu_{F,G}$.** We now turn to our study of moduli stacks of Langlands parameters for $G = \mathrm{GL}_n$. Henceforth we revert to our default notation, where G denotes a group on the spectral side of Langlands duality.

Moduli stacks of Langlands parameters for GL_n have been studied extensively in mixed characteristic, for instance in [H20] in the case of GL_n, or more recently in [BG19, BP19], and [DHKM20] for more general groups. Since in our present context we work over $\mathbb{C}$, the results we need are in general simpler than the results of the above papers, and have not appeared explicitly in the literature in the form we need.

We first consider these moduli spaces as underived stacks; it will follow by Proposition 4.3 that they have trivial derived structure. As in the previous section, we take $G = \mathrm{GL}_n$, considered as the Langlands dual of $G^\vee = \mathrm{GL}_n(F)$. We use $X_{F,G}$ to denote the moduli scheme whose quotient stack is the moduli stack $\mathbb{L}_{F,G}$ in the introduction.

Definition 6.8. Let I be an open normal subgroup of the inertia subgroup $I_F \subset W_F$. Then there is a scheme $X^I_{F,G}$ parameterizing pairs (ρ, N), where $\rho : W_F/I \to \mathrm{GL}_n$ is a homomorphism, and N is a nilpotent n by n matrix such that for all $\sigma \in I_F$, $\mathrm{Ad}\,\rho(\mathrm{Fr}^n\,\sigma)(N) = q^n N$. For any $\nu : I_F/I \to \mathrm{GL}_n(\mathbb{C})$, we may consider the subscheme $X^\nu_{F,G} \subset X^I_{F,G}$ corresponding to pairs (ρ, N)

such that the restriction of ρ to I_F is conjugate to ν; it is easy to see that $X^{\nu}_{F,G}$ is both open and closed in $X^{I}_{F,G}$. We will say that a Langlands parameter is of "type ν" if it lies in $X^{\nu}_{F,G}$.

Example 6.9. When $\nu = 1$ is the trivial representation, the quotient stack $X^{1}_{F,G}/G$ is isomorphic to the underlying underived stack of $\mathcal{L}_q(\widehat{\mathcal{N}}/G)$, as we remarked in the previous section.

We will show that in fact, for ν arbitrary, the stack $X^{\nu}_{F,G}/G$ is isomorphic to a product of stacks of the form $\mathcal{L}_{q^{r_i}}(\widehat{\mathcal{N}}_i/G_i)$, in a manner that exactly parallels the type-theoretic reductions of the previous section. This will allow us to transfer the structures we have built up on $\mathcal{L}_{q^{r_i}}(\widehat{\mathcal{N}}_i/G_i)$ to stacks of the form $X^{\nu}_{F,G}/G$ for arbitrary ν. Our approach very closely parallels the construction of Sections 7 and 8 of [H20] with the exception that we are able to work with the full inertia group I_F, whereas the integral ℓ-adic setting of [H20] requires one to work with the prime-to-ℓ inertia instead.

Our strategy will be to rigidify the moduli space $X^{\nu}_{F.G}$. For any $\mathbb{C}$-algebra R, let us fix a representative $\rho : W_F/I \to \mathrm{GL}_n(R)$ of type ν, i.e. of the conjugacy class.

For any irreducible complex representation η of I_F, let W_η be the finite index subgroup of W_F consisting of all $w \in W_F$ such that η^w is isomorphic to η. Then η extends to a representation of W_η, although not uniquely; let $\tilde{\eta}$ be a choice of such an extension. This choice defines a natural W_η/I_F-action on the space $\mathrm{Hom}_{I_F}(\eta, \rho)$, and an injection of W_η-representations

$$\tilde{\eta} \otimes \mathrm{Hom}_{I_F}(\eta, \rho) \hookrightarrow \rho.$$

Frobenius reciprocity then gives an injection:

$$\mathrm{Ind}^{W_F}_{W_\eta}\left(\tilde{\eta} \otimes \mathrm{Hom}_{I_F}(\eta, \rho)\right) \hookrightarrow \rho.$$

The image of this injection is the sum of the I_F-subrepresentations of ρ isomorphic to a W_F-conjugate of η. We thus have a direct sum decomposition of W_F-representations:

$$\rho \cong \bigoplus_{\eta} \mathrm{Ind}^{W_F}_{W_\eta}\left(\tilde{\eta} \otimes \mathrm{Hom}_{I_F}(\eta, \rho)\right),$$

where η runs over a set of representatives for the W_F-orbits of irreducible representations of I_F/I. Moreover, the map[44] N is I_F-equivariant, and thus induces, for each η, a nilpotent endomorphism N_η of $\mathrm{Hom}_{I_F}(\eta, \rho)$. If Fr_η is a Frobenius element of W_η, we have $\mathrm{Fr}_\eta N_\eta \mathrm{Fr}_\eta^{-1} = q^{r_\eta} N_\eta$.

Let $n_\eta(\rho)$ be the dimension of the space $\mathrm{Hom}_{I_F}(\eta, \rho)$; since $n_\eta(\rho)$ only depends on the type ν of ρ, we may also write this as $n_\eta(\nu)$. A choice of R-basis for $\mathrm{Hom}_{I_F}(\eta, \rho)$ then gives a homomorphism:

$$\rho_\eta : W_F/I_F \to \mathrm{GL}_{n_i}(R)$$

and realizes N_η as a nilpotent element of $M_{n_i}(R)$ such that (ρ_η, N_η) is an R-point of $X^{1}_{E_\eta, \mathrm{GL}_{n_\eta(\rho)}}$. We thus define:

Definition 6.10. A *pseudo-framing* of a Langlands parameter (ρ, N) over R is a choice, for all η such that $n_\eta(\rho)$ is nonzero, of an R-basis for $\mathrm{Hom}_{I_F}(\eta, \rho)$. Let $\widetilde{X}^{\nu}_{F,G}$ be the moduli scheme parameterizing parameters (ρ, N) of type ν together with a pseudo-framing, and define

$$G_\nu := \prod_{\{\eta | n_\eta(\nu) \neq 0\}} \mathrm{GL}_{n_\eta}.$$

The scheme $\widetilde{X}^{\nu}_{F,G}$ is equipped with a $G \times G_\nu$-action.

We denote by E_η the fixed field of W_η, by r_η the degree of E_η over F, and by d_η the dimension of η. We see that G_ν acts on $\widetilde{X}^{\nu}_{F,G}$ via "change of pseudo-framing", and this action makes $\widetilde{X}^{\nu}_{F,G}$ into a G_ν-torsor over $X^{\nu}_{F,G}$. On the other hand, given an R-point (ρ, N) of $\widetilde{X}^{\nu}_{F,G}$,

[44] I.e. viewed as a map $N : I_F \twoheadrightarrow I_F/P_F \simeq \prod_{\ell'} \overline{\mathbb{Q}}_{\ell'} \twoheadrightarrow \overline{\mathbb{Q}}_\ell \simeq \mathbb{C} \to \mathrm{GL}_n(R)$.

the pseudo-framing gives, for each η, an R-point (ρ_η, N_η) of $X^1_{E_\eta,\mathrm{GL}_{n_\eta(\nu)}}$. We thus obtain a natural map:

$$\widetilde{X}^\nu_{F,G} \to \prod_\eta X^1_{E_\eta,\mathrm{GL}_{n_\eta(\nu)}}$$

which is a torsor for the conjugation action of G on $\widetilde{X}^\nu_{F,G}$. We thus obtain natural isomorphisms of quotient stacks:

$$X^\nu_{F,G}/G \cong \widetilde{X}^\nu_{F,G}/(G \times G^\nu) \cong \left(\prod_\eta X^1_{E_\eta,\mathrm{GL}_{n_\eta(\nu)}}\right)/G_\nu \simeq \prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{n_\eta(\nu)}/\,\mathrm{GL}_{n_\eta(\nu)}).$$

Note that the composite isomorphism depends on the choice, for each η, of an extension $\tilde{\eta}$ of η to W_F.

6.3. **The ν-Springer sheaves.** We define a Springer sheaf by transporting across the above isomorphism.

Definition 6.11. We define the *ν-Springer sheaf* $S_\nu \in \mathrm{Coh}(X^\nu_{F,G}/G)$ to be the product, over η, of the sheaves $\mathcal{S}_{q^{r_\eta}}$ on the moduli stack $X^1_{E_\eta,\mathrm{GL}_{n_\eta(\nu)}}/\,\mathrm{GL}_{n_\eta(\nu)}$.

By Theorem 4.12, the endomorphisms of the ν-Springer sheaf are a tensor product of affine Hecke algebras, and we introduce the notation

$$\mathcal{H}_\nu := \bigotimes_\eta \mathcal{H}_{q^{r_\eta}}(n_\eta(\nu)).$$

We thus obtain a fully faithful embedding $\mathcal{H}_\nu$-mod $\hookrightarrow \mathrm{QC}^!(X^\nu_{F,G}/G)$. However, since our identifications depend, ultimately, on our choices of $\tilde{\eta}$, this embedding will also depend on these choices. (By contrast, the sheaf $\mathcal{S}_\nu$ itself is, at least up to isomorphism, independent of the choices of $\tilde{\eta}$.) We can remove this dependence by rephrasing this embedding in terms of smooth representations of $G^\vee$, via the type theory of the previous section.

Proposition 6.12. *There is a G-type (K_ν, τ_ν) such that $\mathcal{H}(G^\vee, K_\nu, \tau_\nu) \simeq \mathcal{H}_\nu$ (depending on choices), and an identification of dg algebras*

$$\mathrm{End}^\bullet(\mathcal{S}_\nu) \simeq \mathcal{H}(G^\vee, K_\nu, \tau_\nu)$$

which is is independent of the choices of $\tilde{\eta}$.

Proof. Let $L^\vee_\nu$ be the standard Levi of $G^\vee$ corresponding to block diagonal matrices whose blocks consist, for each η, of $n_\eta(\nu)$ blocks of size $r_\eta d_\eta$. Let π^0_η be the cuspidal representation of $\mathrm{GL}_{r_\eta d_\eta}$ corresponding to $\mathrm{Ind}^{W_F}_{W_\eta} \tilde{\eta}$ under the local Langlands correspondence, and let π_ν be the cuspidal representation:

$$\pi_\nu := \bigotimes_\eta (\pi^0_\eta)^{\otimes n_\eta(\nu)}$$

of $L^\vee_\nu$. Then representations in the block $D(G^\vee)_{[L^\vee_\nu, \pi_\nu]}$ correspond, via local Langlands, to Langlands parameters for G of type ν.

For each η, we can find a cuspidal type (K_η, τ_η) in $\mathrm{GL}_{r_\eta d_\eta}$ for π^0_η. From this we can form the type $(K_{L_\nu}, \tau_{L_\nu})$ in $L^\vee_\nu$, by setting $K_{L_\nu} = \prod_\eta K_\eta^{n_\eta(\nu)}$ and $\tau_{L_\nu} = \bigotimes_\eta \tau_\eta^{\otimes n_\eta(\nu)}$. This type is associated to the block $[L^\vee_\nu, \pi_\nu]$ in $D(L^\vee_\nu)$. Let $P^\vee$ be the standard parabolic of $G^\vee$ with Levi $L^\vee$, and let $(P')^\vee$ denote the opposite parabolic. The theory of section 6.1 then gives us a Levi subgroup $(L^\dagger)^\vee$ of $G^\vee$ containing $L^\vee_\nu$, an $(L^\dagger)^\vee$-cover $(K^\dagger_\nu, \tau^\dagger_\nu)$ of $(K_{L_\nu}, \tau_{L_\nu})$, and a $G^\vee$-cover (K_ν, τ_ν) of $(K^\dagger_\nu, \tau^\dagger_\nu)$. These covers depend on a choice of parabolic with Levi $L^\vee$; we choose our covers to be the ones associated to *the opposite parabolic* $(P')^\vee$. In particular we obtain a map

$$T_{(P')^\vee} : \mathcal{H}(L^\vee_\nu, K_{L^\vee_\nu}, \tau_{L^\vee_\nu}) \to \mathcal{H}(G^\vee, K_\nu, \tau_\nu)$$

that is compatible with the parabolic induction functor $i^{G^\vee}_{P^\vee}$ on $D(L^\vee_\nu)$ in the sense of Theorem 6.4.

One verifies, by compatibility of local Langlands with unramified twists, that for each η the group of unramified characters χ of $\mathrm{GL}_{r_\eta d_\eta}$ such that $\pi^0_\eta \otimes \chi$ is isomorphic to π^0_η is r_η. Thus there is an isomorphism of Hecke algebras $\mathcal{H}(G^\vee, K_\nu, \tau_\nu) \simeq \mathcal{H}_\nu$. Moreover, the composition:

$$\mathcal{H}(G^\vee, K_\nu, \tau_\nu) \cong \mathcal{H}_\nu \cong \mathrm{End}(\mathcal{S}_\nu)$$

is independent of the choices of $\tilde{\eta}$. This essentially boils down to the compatibility of the local Langlands correspondence with unramified twists and parabolic induction. □

Since $D(G^\vee)_{[L^\vee_\nu, \pi_\nu]}$ is canonically equivalent to the category of $\mathcal{H}(G^\vee, K_\nu, \tau_\nu)$-modules, and this equivalence associates the representations $\mathrm{cInd}_{K_\nu}^{G^\vee} \tau_\nu$ to the free $\mathcal{H}(G^\vee, K_\nu, \tau_\nu)$-module of rank one, we have shown:

Theorem 6.13. *For each ν there is a natural fully faithful functor:*

$$\mathrm{LL}_{G,\nu} : D(G^\vee)_{[L^\vee_\nu, \pi_\nu]} \hookrightarrow \mathrm{QC}^!(X^\nu_{F,G})$$

that takes the generator $\mathrm{cInd}_{K_\nu}^{G^\vee} \tau_\nu$ to $\mathcal{S}_\nu$.

Remark 6.14. We will say that an inertial type ν is *cuspidal* if the representations of W_F corresponding to points of $X^\nu_{F,G}$ are irreducible. For $G = \mathrm{GL}_n$ this happens precisely when $n_\eta = 1$ for a single η and is zero for all other η. In such cases $X^\nu_{F,G}$ is simply a copy of $\mathbb{G}_m$, the sheaf $\mathcal{S}_\nu$ is the structure sheaf, and the corresponding affine Hecke algebra is simply $\mathbb{C}[T, T^{-1}]$, which our choices above identify with the global functions on $X^\nu_{F,G} \cong \mathbb{G}_m$. In particular for such ν the functor $\mathrm{LL}_{G,\nu}$ is an abelian equivalence, that takes an irreducible $\mathbb{C}[T, T^{-1}]$-module to a skyscraper sheaf on the corresponding point of $X^\nu_{F,G}$.

By taking products of the above picture we see that a similar statement holds for Levi subgroups M of G (with a suitable torus in place of $\mathbb{G}_m$.)

6.3.1. *A direct construction of $\mathcal{S}_\nu$.* In this section we give a more intrinsic construction of $\mathcal{S}_\nu$. Fix a particular ν, and let L_ν denote the Langlands dual of $L^\vee_\nu$; we identify L_ν with the standard block diagonal Levi of G containing $n_\eta(\nu)$ blocks of size $r_\eta d_\eta$. Let $\nu' : I_F \to L_\nu$ be the representation of I_F on L whose projection to each block of L_ν of type η is the sum of the W_F-conjugates of η. We then have a moduli space $X^{\nu'}_{F,L_\nu}$ parameterizing Langlands parameters for L_ν that are of type ν'.

Let P be the standard (block upper triangular) parabolic of G containing L_ν. We then also have a moduli space $X^{\nu'}_{F,P}$ parameterizing Langlands parameters for G that factor through P, and whose projection to L_ν is of type ν'. The inclusion of $P \hookrightarrow G$, and the projection of $P \twoheadrightarrow L$ induce parabolic induction maps

$$X^{\nu'}_{F,L_\nu} \xleftarrow{\pi_P} X^{\nu'}_{F,P} \xrightarrow{\iota_P} X^{\nu'}_{F,G}$$

We then have:

Theorem 6.15. *There are natural isomorphisms:*

$$\mathcal{S}_\nu \cong (\iota_P)_* \mathcal{O}^{\nu'}_{F,P} \cong (\iota_P)_* \pi_P^* \mathcal{O}^{\nu'}_{F,L_\nu},$$

where $\mathcal{O}^{\nu'}_{F,P}$ and $\mathcal{O}^{\nu'}_{F,L_\nu}$ denote the structure sheaves on $X^{\nu'}_{F,P}/P$ and $X^{\nu'}_{F,L_\nu}/L_\nu$, respectively.

Proof. Let $L^\dagger$ be the standard Levi of G that is block diagonal of block sizes $n_\eta(\nu) r_\eta d_\eta$. Let Q be the standard block upper triangular parabolic of G with Levi $L^\dagger$, and let ν'' be the composition of ν' with the inclusion of L_ν in $L^\dagger$. We then have spaces $X^{\nu''}_{F,L^\dagger}$ and $X^{\nu''}_{F,Q}$, where the former parameterizes pairs (ρ, N) for $L^\dagger$ that are of type ν'', and the latter parameterizes pairs (ρ, N) for G that factor through Q and whose projection to $L^\dagger$ is of type ν''. We may also consider the space $X^{\nu'}_{F,P \cap L^\dagger}$, which parameterizes pairs (ρ, N) for $L^\dagger$ that factor through $P \cap L^\dagger$ and whose projection to L is of type ν'. We then have a natural Cartesian diagram:

$$\begin{array}{ccc}
X^{\nu'}_{F,P}/P & \longrightarrow & X^{\nu'}_{F,P\cap L^\dagger}/P\cap L^\dagger \\
\big\downarrow & & \big\downarrow{\scriptstyle \iota_{P\cap L^\dagger}} \\
X^{\nu''}_{F,Q}/Q & \xrightarrow{\ \iota_Q\ } & X^{\nu''}_{F,L^\dagger}/L^\dagger
\end{array}$$

from which we conclude that $(\iota_P)_*\pi_P^*\mathcal{O}^{\nu'}_{F,L_\nu}$ is isomorphic to $(\pi_Q)_*\iota_Q^*(\iota_{P\cap L^\dagger})_*\pi^*_{P\cap L^\dagger}\mathcal{O}^{\nu'}_{F,L_\nu}$, where $\pi_Q : X^{\nu''}_{F,Q}/Q \to X^{\nu}_{F,G}/G$, and $\pi_{P\cap L^\dagger} : X^{\nu'}_{F,P\cap L^\dagger}/(P\cap L^\dagger) \to X^{\nu'}_{F,L_\nu}/L_\nu$.

On the other hand, let B_η and T_η denote the standard Borel subgroup and maximal torus of $\mathrm{GL}_{n_\eta(\nu)}$, for each η. We then have a commutative diagram (note that we transport derived structures across the isomorphisms by definition):

$$\begin{array}{ccc}
\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{T_\eta}/T) & \cong & X^{\nu'}_{F,L}/L_\nu \\
\uparrow & & \uparrow \\
\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{B_\eta}/B) & \cong & X^{\nu'}_{F,P\cap L^\dagger}/(P\cap L^\dagger) \\
\downarrow & & \downarrow \\
\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{n_\eta(\nu)}/G_{n_\eta(\nu)}) & \cong & X^{\nu''}_{F,L^\dagger}/L^\dagger \\
\uparrow & & \uparrow \\
\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{n_\eta(\nu)}/G_{n_\eta(\nu)}) & \cong & X^{\nu''}_{F,Q}/Q \\
\downarrow & & \downarrow \\
\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{n_\eta(\nu)}/G_{n_\eta(\nu)}) & \cong & X^{\nu}_{F,G}/G
\end{array}$$

where the bottom two vertical maps on the left are the identity. It follows that the iterated pull-push $(\iota_Q)_*\pi_Q^*(\iota_{P\cap L^\dagger})_*\pi^*_{P\cap L^\dagger}\mathcal{O}^{\nu'}_{F,L_\nu}$ corresponds, under the bottom isomorphism, to $\mathcal{S}_\nu$, as the latter is simply the pushforward to $\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{n_\eta(\nu)}/G_{n_\eta(\nu)})$ of the structure sheaf on $\prod_\eta \mathcal{L}_{q^{r_\eta}}(\widehat{\mathcal{N}}_{B_\eta}/B)$. $\square$

6.3.2. *Compatibility with parabolic induction.* As in the previous subsection, we fix a particular ν and let $L_\nu^\vee$, L_ν and P be as above. Let Q be a standard Levi subgroup of G whose standard Levi subgroup M contains L_ν, and let $M^\vee$ and $Q^\vee$ be the corresponding dual subgroups of $G^\vee$. Let ν' be the inertial type $I_F \to L_\nu$ constructed in the previous subsection, and let ν'' be the composition of ν' with the inclusion of L_ν in M. We have a diagram with the square Cartesian:

$$\begin{array}{ccccc}
X^{\nu'}_{F,L_\nu}/L_\nu & \xleftarrow{\ \pi_{P\cap M}\ } & X^{\nu'}_{F,P\cap M}/P\cap M & \xleftarrow{\ \pi_{P,P\cap M}\ } & X^{\nu'}_{F,P}/P \\
 & & \big\downarrow{\scriptstyle \iota_{P\cap M}} & & \big\downarrow{\scriptstyle \iota_{P,Q}} \\
 & & X^{\nu''}_{F,M}/M & \xleftarrow{\ \pi_Q\ } & X^{\nu''}_{F,Q}/Q \\
 & & & & \big\downarrow{\scriptstyle \iota_Q} \\
 & & & & X^{\nu}_{F,G}/G.
\end{array}$$

Theorem 6.15 shows that $\mathcal{S}_\nu$ is isomorphic to the pushforward to $X^\nu_{F,G}/G$ of the structure sheaf on $X^{\nu'}_{F,P}/P$, and the corresponding sheaf $\mathcal{S}_{\nu,M}$ on $X^{\nu''}_{F,M}$ is the pushforward to $X^{\nu''}_{F,M}/M$ of the structure sheaf on $X^{\nu'}_{F,P\cap M}/(P\cap M)$. The above diagram then gives us a natural isomorphism:

$$\mathcal{S}_\nu \cong (\iota_Q)_*\pi_Q^*\mathcal{S}_{\nu,M}.$$

Via functoriality and this isomorphism one obtains an embedding of $\mathrm{End}(\mathcal{S}_{\nu,M})$ in $\mathrm{End}(\mathcal{S}_\nu)$.

Recall that we have identified these endomorphism rings with certain Hecke algebras via type theory. In particular, we have the type $(K_{L_\nu}, \tau_{L_\nu})$ of $L_\nu^\vee$, an $M^\vee$-cover $(K_{M^\vee}, \tau_{M^\vee})$ coming from the parabolic $(P')^\vee \cap M^\vee$ opposite $P^\vee \cap M^\vee$, and a $G^\vee$-cover (K,τ) coming from the parabolic $(P')^\vee$ opposite $P^\vee$. Theorem 6.5 then gives us a map:

$$T_{(Q')^\vee} : \mathcal{H}(M^\vee, K_{M^\vee}, \tau_{M^\vee}) \to \mathcal{H}(G^\vee, K, \tau).$$

Lemma 6.16. *We have a commutative diagram:*

$$\begin{array}{ccc} \mathcal{H}(M^\vee, K_{M^\vee}, \tau_{M^\vee}) & \xrightarrow{\simeq} & \mathrm{End}(\mathcal{S}_{\nu,M}) \\ {\scriptstyle T_{(Q')^\vee}}\downarrow & & \downarrow \\ \mathcal{H}(G^\vee, K, \tau) & \xrightarrow{\simeq} & \mathrm{End}(\mathcal{S}_\nu) \end{array}$$

where the right hand map is induced by the isomorphism of $\mathcal{S}_\nu \simeq (\iota_Q)_* \pi_Q^* \mathcal{S}_{\nu,M}$.

Proof. The machinery of the previous subsection, together with the compatibility of the general case with the Iwahori case in section 6.1 allow us to reduce to the case where $\nu = 1$. In this case the claim reduces to the compatibility of the Ginsburg-Kazhdan-Lusztig interpretation of the affine Hecke algebra as K_0 of the Steinberg variety with parabolic induction, checked in the proof of Theorem 4.12. □

As a consequence, we deduce:

Theorem 6.17. *We have a commutative diagram of functors:*

$$\begin{array}{ccc} D(M^\vee)_{[L_\nu, \tau_\nu]} & \xhookrightarrow{\mathrm{LL}_{M,\nu}} & \mathrm{QC}^!(X_{F,M}^\nu) \\ {\scriptstyle i_{Q^\vee}^{G^\vee}}\downarrow & & \downarrow{\scriptstyle (\iota_Q)_* \pi_Q^*} \\ D(G^\vee)_{[L_\nu, \tau_\nu]} & \xhookrightarrow{\mathrm{LL}_{G,\nu}} & \mathrm{QC}^!(X_{F,G}^\nu). \end{array}$$

Proof. We have isomorphisms:

$$\begin{aligned} \mathrm{LL}_{G,\nu}(i_{Q^\vee}^{G^\vee} V) &\cong \mathrm{Hom}(\mathrm{cInd}_K^{G^\vee} \tau, i_{Q^\vee}^{G^\vee} V) \otimes_{\mathcal{H}(G^\vee, K, \tau)} \mathcal{S}_\nu \\ &\cong \mathrm{Hom}_{M^\vee}(\mathrm{cInd}_{K_{M^\vee}}^{M^\vee} \tau_{M^\vee}, V) \otimes_{\mathcal{H}(M^\vee, K_{M^\vee}, \tau_{M^\vee})} (\iota_Q)_* \pi_Q^* \mathcal{S}_{M^\vee,\nu} \\ &\cong (\iota_Q)_* \pi_Q^* (\mathrm{LL}_{M,\nu} V) \end{aligned}$$

from which the result follows. □

Appendix A. Proofs

This appendix contains proofs of technical results used in the body of the paper.

A.1. Functoriality of Hochschild homology in geometric settings.

Proof of Proposition 3.14. The first and second statements are Theorem 2.21 (or Proposition 5.5) in [BN19]. We give a direct argument for the third statement (which can also be adapted toward the second). We let $Z := X \times_Y X$, and denote the diagonals by $\Delta_X : X \hookrightarrow X \times X$ (and likewise for Y), the relative diagonal by $\Delta : X \hookrightarrow Z = X \times_Y X$, and its inclusion by $i : Z = X \times_Y X \hookrightarrow X \times X$.

Note that we use !-integral transforms in our convention; thus to describe the integral transforms it is convenient to pass between $*$-pullbacks and !-pullbacks. For any quasi-smooth map $g : E \to B$ we denote by $\beta_g^* : f^*(-) \simeq f^!(-) \otimes_{\mathcal{O}_X} \omega_{E/B}^{-1}$ and $\beta_g^! : f^!(-) \simeq f^*(-) \otimes_{\mathcal{O}_X} \omega_{E/B}$ the canonical equivalences.

The integral transform corresponding to $f_* f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(Y)$ is given by the kernel

$$\mathcal{K}_{f_* f^*} := \Delta_{Y*} f_* (\omega_X \underset{\mathcal{O}_X}{\otimes} \omega_{X/Y}^{-1}).$$

Letting η_f denote the unit for the adjunction (f^*, f_*), the unit $\eta \in \mathrm{Hom}_{Y \times Y}(\Delta_{Y*}\omega_Y, \mathcal{K}_{f_* f^*})$ is defined:

$$\eta := \Delta_{Y*}(\beta_f^* \circ \eta_f) : \Delta_{Y*}\omega_Y \longrightarrow \Delta_{Y*}(f_* f^* \omega_Y) \simeq \Delta_{Y*}(f_*(f^! \omega_Y \underset{\mathcal{O}_X}{\otimes} \omega_{X/Y}^{-1})).$$

The integral transform corresponding to $f^* f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ is given by the kernel:

$$\mathcal{K}_{f^* f_*} := i_* (\omega_Z \underset{\mathcal{O}_Z}{\otimes} \omega_{Z/X}^{-1}).$$

Letting η_Δ denote the unit for the adjunction (Δ^*, Δ_*), the counit $\epsilon \in \mathrm{Hom}_{X\times X}(\mathcal{K}_{f^*f_*}, \Delta_{X*}\omega_X)$ is defined:

$$\epsilon := i_*(\beta_\Delta^{!-1} \circ \eta_\Delta) : i_*(\omega_Z \underset{\mathcal{O}_Z}{\otimes} \omega_{Z/X}^{-1}) \to i_*\Delta_*(\Delta^*\omega_Z \underset{\mathcal{O}_Z}{\otimes} \omega_{X/Z}) \simeq i_*\Delta_*\omega_X$$

where we implicitly use the canonical identification $\Delta^*\omega_{Z/X}^{-1} \simeq \omega_{X/Z}$ (i.e. since $\omega_{X/X}$ is canonically trivial). We leave verification of the adjunction identites to the reader.

The functoriality $\omega(\mathcal{L}_\phi Y) \to \omega(\mathcal{L}_\phi X)$ is given by composing the unit and counit after applying $\Gamma \circ \Gamma_{\phi_Y}^!$ and $\Gamma \circ \Gamma_{\phi_X}^!$ (where, somewhat confusingly, Γ denotes the global sections functor, and Γ_ϕ denotes the graph). Recall the factorization and notation of Lemma 3.10, let $p_X : \mathcal{L}_\phi Y_X \to X$ and $p_Y : \mathcal{L}_\phi Y_X \to Y$ denote the natural maps, and $\mathrm{ev}_X : \mathcal{L}_\phi X \to X$ the evaluation (and likewise for Y). For the unit map η, we have

$$\Gamma_{\phi_Y}^! \eta : \Gamma_{\phi_Y}^! \Delta_{Y*}\omega_Y \longrightarrow \Gamma_{\phi_Y}^! \Delta_{Y*} f_*(\omega_X \otimes_{\mathcal{O}_X} \omega_{X/Y}^{-1}).$$

We perform a base change along the diagram:

$$\begin{array}{ccccc} \mathcal{L}_\phi Y_X & \xrightarrow{\pi} & \mathcal{L}_\phi Y & \xrightarrow{\mathrm{ev}_Y} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle \mathrm{ev}_Y} & & \downarrow{\scriptstyle \Gamma_{\phi_Y}} \\ X & \xrightarrow{f} & Y & \xrightarrow{\Delta_Y} & Y \times Y. \end{array}$$

to find

$$\Gamma_{\phi_Y}^! \Delta_{Y*} f_*(\omega_X \otimes_{\mathcal{O}_X} \omega_{X/Y}^{-1}) \simeq p_{Y*}p_X^!(\omega_X \otimes_{\mathcal{O}_X} \omega_{X/Y}^{-1}) \simeq p_{Y*}(\omega_{\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi Y_X}} p_X^*\omega_{X/Y}^{-1})$$
$$\simeq p_{Y*}(\omega_{\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi Y_X}} \omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y}^{-1}) \simeq \mathrm{ev}_{Y*}\pi_*\pi^*\omega_{\mathcal{L}_\phi Y}$$

and an identification of η with the unit η_π for the adjunction (π^*, π_*):

$$\eta \simeq \mathrm{ev}_{Y*}(\eta_\pi(\omega_{\mathcal{L}_\phi Y})) : \mathrm{ev}_{Y*}\omega_{\mathcal{L}_\phi Y} \longrightarrow \mathrm{ev}_{Y*}\pi_*\pi^*\omega_{\mathcal{L}_\phi Y}.$$

For the counit map ϵ, we have

$$\Gamma_{\phi_X}^! \epsilon : \Gamma_{\phi_X}^! i_*(\omega_Z \otimes_{\mathcal{O}_Z} \omega_{Z/X}^{-1}) \longrightarrow \Gamma_{\phi_X}^! \Delta_{X*}\omega_X.$$

We perform a base change along the diagram:

$$\begin{array}{ccccc} \mathcal{L}_\phi X & \xrightarrow{\delta} & \mathcal{L}_\phi Y_X & \xrightarrow{p_X} & X \\ \downarrow{\scriptstyle \mathrm{ev}_X} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \Gamma_{\phi_X}} \\ X & \xrightarrow{\Delta} & Z & \xrightarrow{i} & X \times X. \end{array}$$

to find that

$$\Gamma_{\phi_X}^! i_*(\omega_Z \otimes_{\mathcal{O}_Z} \omega_{Z/X}^{-1}) \simeq p_{X*}s^!(\omega_Z \otimes_{\mathcal{O}_Z} \omega_{Z/X}^{-1}) \simeq p_{X*}(\omega_{\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi Y_X}} s^*\omega_{Z/X}^{-1})$$
$$\simeq p_{X*}(\omega_{\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi Y_X}} \omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y}^{-1}) \simeq p_{X*}\delta^*\omega_{\mathcal{L}_\phi Y}.$$

Since the Calabi-Yau equivalence of Proposition 3.12 provides a canonical equivalence $\omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y} \simeq \mathcal{O}_{\mathcal{L}_\phi X}$, we have a canonical equivalence $\omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y_X} \simeq \delta^*\omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y}^{-1}$. Passing through this equivalence, we have

$$\Gamma_{\phi_X}^! i_*\Delta_*\omega_X \simeq p_{X*}s^!\Delta_*\omega_X \simeq p_{X*}\delta_*\delta^!\omega_{\mathcal{L}_\phi Y_X} \simeq p_{X*}\delta_*(\delta^*\omega_{\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi X}} \omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y_X})$$
$$\simeq p_{X*}\delta_*\delta^*(\omega_{\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi Y_X}} \omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y}^{-1}).$$

Thus, ϵ is identified with the unit η_δ for the adjunction (δ^*, δ_*):

$$\epsilon \simeq p_{X*}(\eta_\delta(\omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y_X} \otimes_{\mathcal{O}_{\mathcal{L}_\phi Y_X}} \omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y}^{-1})) : p_{X*}\pi^*\omega_{\mathcal{L}_\phi Y} \to \mathrm{ev}_{X*}\omega_{\mathcal{L}_\phi X}.$$

Taking global sections and composing, we see that the map

$$\omega(\mathcal{L}_\phi Y) \to \Gamma(\mathcal{L}_\phi Y_X, \omega_{\mathcal{L}_\phi Y_X} \otimes \omega_{\mathcal{L}_\phi Y_X/\mathcal{L}_\phi Y}^{-1}) \simeq \Gamma(\mathcal{L}_\phi Y_X, \omega_{\mathcal{L}_\phi Y_X} \otimes \delta_*\omega_{\mathcal{L}_\phi X/\mathcal{L}_\phi Y_X}) \to \omega(\mathcal{L}_\phi X)$$

is induced by the unit of the adjunction $(\mathcal{L}_\phi f^*, \mathcal{L}_\phi f_*)$, twisted by the Calabi-Yau equivalence.

□

The following is a generalization of Proposition 3.18. While Proposition 3.18 is stated in the setting of derived loop spaces, the arguments hold in the following more general setting.

Proposition A.1. *Let $f : X \to Y$ be a proper map of derived stacks, and let $Z = X \times_Y X$ with projections $p_1, p_2 : Z \to X$ and $p : Z \to Y$. There is a canonical equivalence:*

$$\zeta_f : p_* \mathcal{H}om_Z(\mathcal{O}_Z, \omega_Z) \simeq \mathcal{H}om_Y(f_*\mathcal{O}_X, f_*\omega_X).$$

In particular, if X is Calabi-Yau, then we have a natural equivalence $\omega(Z) \simeq \mathrm{End}_Y(f_*\omega_X)$. This equivalence is functorial in the following sense. Let $f' : X' \to Y'$ (and $p' : Z' \to Y'$) be as above.

- *Suppose that $\alpha_Y : Y \to Y'$ is proper, and that $X = X'$. We let $f : X \to Y$ be as above, $f' = \alpha_Y \circ f : X \to Y \to Y'$. We have commuting squares*

$$\begin{array}{ccc} \alpha_{Y*} p_* \mathcal{H}om_Z(\mathcal{O}_Z, \omega_Z) & \xrightarrow[\simeq]{\alpha_{Y*}(\zeta_f)} & \alpha_{Y*} \mathcal{H}om_Y(f_*\mathcal{O}_X, f_*\omega_X) \\ \downarrow{\scriptstyle \text{Def.3.13}} & & \downarrow{\scriptstyle \text{Def.3.17}} \\ p'_* \mathcal{H}om_{Z'}(\mathcal{O}_{Z'}, \omega_{Z'}) & \xrightarrow[\simeq]{\zeta_{f'}} & \mathcal{H}om_{Y'}(f_*\mathcal{O}_{X'}, f_*\omega_{X'}). \end{array}$$

- *Suppose that $\alpha_Y : Y \to Y'$ is Calabi-Yau, and that $X = X' \times_{Y'} Y$ (so α_X is also Calabi-Yau). Then we have commuting squares*

$$\begin{array}{ccc} p'_* \mathcal{H}om_{Z'}(\mathcal{O}_{Z'}, \omega_{Z'}) & \xrightarrow[\simeq]{\zeta_{f'}} & \mathcal{H}om_{Y'}(f_*\mathcal{O}_{X'}, f_*\omega_{X'}) \\ \downarrow{\scriptstyle \text{Def.3.13}} & & \downarrow{\scriptstyle \text{Def.3.17}} \\ \alpha_{Y*} p_* \mathcal{H}om_Z(\mathcal{O}_Z, \omega_Z) & \xrightarrow[\simeq]{\alpha_{Y*}(\zeta_f)} & \alpha_{Y*} \mathcal{H}om_Y(f_*\mathcal{O}_X, f_*\omega_X). \end{array}$$

Proof. The first statement is a formal consequence of adjunctions and base change:

$$p_* \mathcal{H}om_Z(\mathcal{O}_Z, \omega_Z) \simeq f_* \mathcal{H}om_X(\mathcal{O}_X, p_{1*}\omega_Z) \simeq f_* \mathcal{H}om_X(\mathcal{O}_X, f^! f_* \omega_X) \simeq \mathcal{H}om_Y(f_*\mathcal{O}_X, f_*\omega_X).$$

Functoriality for proper morphisms follows by a diagram chase on:

$$\begin{array}{ccc} \alpha_{Y*} p_* \mathcal{H}om_Z(\mathcal{O}_Z, \omega_Z) & \longrightarrow & p'_* \mathcal{H}om_{Z'}(\mathcal{O}_{Z'}, \omega_{Z'}) \\ \downarrow \simeq & & \downarrow \simeq \\ f'_* \mathcal{H}om_X(\mathcal{O}_X, p_{1*}\omega_Z) & \longrightarrow & f'_* \mathcal{H}om_X(\mathcal{O}_X, p'_{1*}\omega_{Z'}) \\ \downarrow \simeq & & \downarrow \simeq \\ f'_* \mathcal{H}om_X(\mathcal{O}_X, f^! f_* \omega_X) & \longrightarrow & f'_* \mathcal{H}om_X(\mathcal{O}_X, f'^! f'_* \omega_X) \\ \downarrow \simeq & & \downarrow \simeq \\ \alpha_{Y*} \mathcal{H}om_Y(f_*\mathcal{O}_X, f_*\omega_X) & \longrightarrow & \alpha_{Y*} \mathcal{H}om_{Y'}(f'_*\mathcal{O}_X, f'_*\omega_X) \end{array}$$

where we use the identification in the middle left terms $\alpha_{Y*} f_* \simeq f'_* \alpha_{X*} \simeq f'_*$ (i.e. since $X = X'$ and $\alpha_X = \mathrm{id}_X$), and the middle horizontal maps are given by functoriality of pushforwards of dualizing sheaves. In the Calabi-Yau case, we pass to left adjoints, apply the base change $\alpha^* f'_* \simeq f_* \alpha^*$ and chase the diagram:

$$\begin{array}{ccc} p_* \alpha_Z^* \mathcal{H}om_{Z'}(\mathcal{O}_{Z'}, \omega_{Z'}) & \longrightarrow & p_* \mathcal{H}om_Z(\mathcal{O}_Z, \omega_Z) \\ \downarrow \simeq & & \downarrow \simeq \\ f_* \alpha_X^* \mathcal{H}om_{X'}(\mathcal{O}_{X'}, p'_{1*}\omega_{Z'}) & \longrightarrow & f_* \mathcal{H}om_X(\mathcal{O}_X, p_{1*}\omega_Z) \\ \downarrow \simeq & & \downarrow \simeq \\ f_* \alpha_X^* \mathcal{H}om_{X'}(\mathcal{O}_{X'}, f'^! f'_* \omega_{X'}) & \longrightarrow & f_* \mathcal{H}om_{X'}(\mathcal{O}_X, f^! f_* \omega_X) \\ \downarrow \simeq & & \downarrow \simeq \\ \alpha_Y^* \mathcal{H}om_{Y'}(f'_*\mathcal{O}_{X'}, f'_*\omega_{X'}) & \longrightarrow & \mathcal{H}om_Y(f_*\mathcal{O}_X, f_*\omega_X) \end{array}$$

where the middle arrows arise by functoriality of Calabi-Yau pullback (as in Definition 3.13) after passing to left adjoints. □

A.2. **Horizontal trace of convolution categories.**

Proof of Theorem 3.23. We will employ the notation in Theorem 3.3.1 of [BNP17b] to point out how its argument can be modified to acommodate this more general setting. First, note that the surjectivity condition is not needed nor used in the proof of the theorem; it is subsumed by the singular support condition, so we omit it from the statement. The quasi-smoothness of q_n follows by quasi-smoothness of the graph Γ_ϕ. We replace, in the definition of $\mathcal{C}_\bullet$, the diagonal module $\mathrm{Perf}(X)$ with the module defined by the graph Γ_ϕ. In the definition of $Z_\bullet$, this amounts to replacing $\mathcal{L}Y$ with Y^ϕ (informally, introducing a twist by ϕ as we "come around the circle," i.e. in Lemma 3.3.2 the automorphism ℓ lives in $\mathrm{Map}_{Y(k)}(y, \phi(y))$). In the definition of $W_\bullet$, this amounts to replacing the last factor of $X \times_Y X = X \times_{f,Y,f} X$ representing the "segment containing the twist by ϕ" with $X \times_{f,Y,\phi_X \circ f} X$ (i.e. in Lemma 3.3.3, the final point x_n should lie in the fiber $f^{-1}(\phi(y))$ rather than $f^{-1}(y)$). The rest of the proof goes through without modification as the formulas still hold with the ϕ-twist. □

Proof of Proposition 3.30. The argument in Theorem 3.3.1 of [BNP17b] may be adapted in the following way. Let $\mathbf{M} = \mathrm{QC}^!(Z_{12})$ and $\mathbf{N} = \mathrm{QC}^!(Z_{23})$, and following the notation of *loc. cit.* we let $\mathbf{A} = \mathrm{QC}^!(Z_{22})$ and $\mathbf{B} = \mathrm{QC}^!(X_2)$. Then, writing $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N} = \mathbf{M} \otimes_{\mathbf{A}} \mathbf{A} \otimes_{\mathbf{A}} \mathbf{N}$, and (following the argument of *loc. cit.*) resolving $\mathbf{A}$ as a $\mathbf{A} \otimes_{\mathbf{B}} \mathbf{A}^{rv}$-module via the relative bar complex for $\mathbf{A}$ over $\mathbf{B}$, we find that $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$ can be realized as the geometric realization of the cosimplicial object:

$$\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N} = \mathrm{colim}(\mathrm{QC}^!_{\Lambda_n}(Z_n))$$

where we define

$$q_n : Z_n := X_1 \underset{Y}{\times} \overbrace{X_2 \underset{Y}{\times} \cdots \underset{Y}{\times} X_2}^{n+1} \underset{Y}{\times} X_3 \longrightarrow W_n := Z_{12} \times Z_{22}^n \times Z_{23},$$

$$\Lambda_n = q_n^!(\Lambda_{12} \boxtimes \overbrace{\mathbb{T}^{*[-1]}_{Z_{22}} \boxtimes \cdots \boxtimes \mathbb{T}^{*[-1]}_{Z_{22}}}^{n} \boxtimes \Lambda_{23}).$$

Explicitly, for $\eta = (x_1, x_2^{(0)}, \ldots, x_2^{(n)}, x_3) \in Z_n(k)$ with each coordinate living in the fiber over $y \in Y(k)$, we have

$$\mathbb{T}^{*[-1]}_{Z_n} = \{(\omega_{12}, \omega_{22}^{(01)}, \ldots, \omega_{22}^{(n-1,n)}, \omega_{23}) \in \mathbb{T}^*_{Y,y} \mid df_1^* \omega_{12} = 0, df_3^* \omega_{23} = 0, df_2^* \omega_{22}^{(i-1,i)} = df_2^* \omega_{22}^{(i,i+1)}\},$$

$$\Lambda_n = \{(\omega_{12}, \omega_{22}^{(01)}, \ldots, \omega_{22}^{(n-1,n)}, \omega_{23}) \in \mathbb{T}^*_{Y,y} \mid \omega_{12} \in \Lambda_{12,\eta}, \omega_{23} \in \Lambda_{23,\eta}, df_2^* \omega_{22}^{(i,i+1)} = 0\}.$$

Here, we note that the fiber of the singular support condition Λ_{ij} at the point $(x_i, x_j) \in Z_{ij}(k)$ in the fiber over y is naturally a subset $\Lambda_{ij,(x_i,x_j)} \subset \mathbb{T}^*_{Y,y}$. The singular support stability condition implies that the face maps $(Z_m, \Lambda_m) \to (Z_n, \Lambda_n)$ are maps of pairs. Pullback along the augmentation is conservative by definition of Λ_{13}. Analogous formulas in Lemma 3.3.9 of *op. cit.* hold in this situation (without the need to "loop around"), and the strictness condition follows by an argument analogous to Proposition 3.3.8 of *op. cit.* Thus, we have an equivalence

$$\mathrm{QC}^!_{\Lambda_{13}}(Z_{13}) \simeq \mathrm{Tot}(\mathrm{QC}^!_{\Lambda_n}(Z_n)).$$

For functoriality, we note that the resulting maps $(Z_n, \Lambda_n) \to (Z_n, \Lambda'_n)$ are maps of pairs by our description above for $n \geqslant 0$, and the case $n = -1$ is a straightforward verification. The claim then follows by functoriality of the descent with support discussed in Section 2.4 of [BNP17b]. We adopt the notation of *loc. cit.*: let $(X_\bullet, \Lambda_\bullet) \to (X_{-1}, \Lambda_{-1})$ and $(Y_\bullet, \Theta_\bullet) \to (Y_{-1}, \Theta_{-1})$ be augmented simplicial diagrams of maps of pairs satisfying the descent conditions of Theorem 2.4.1 and Corollary 2.4.2 of [BNP17b], and let $g_\bullet : (X_\bullet, \Lambda_\bullet) \to (Y_\bullet, \Theta_\bullet)$ be a level-wise proper map of augmented simplicial diagrams of pairs. We claim that we have a limit $\mathrm{Tot}(\mathfrak{g}^!_\bullet) \simeq \mathfrak{g}^!_{-1}$ and a colimit $\mathrm{Real}(\mathfrak{g}_{\bullet *}) \simeq \mathfrak{g}_{-1*}$, which proves the functoriality claims (i.e. since the maps $\mathfrak{g}_\bullet$ are the identity, the functors $\mathfrak{g}_{\bullet *}$ are the inclusion functors and $\mathfrak{g}^!_\bullet$ are the local cohomology functors). The first statement follows by commutativity of !-pullbacks with supports (see Remark 2.3.3 of

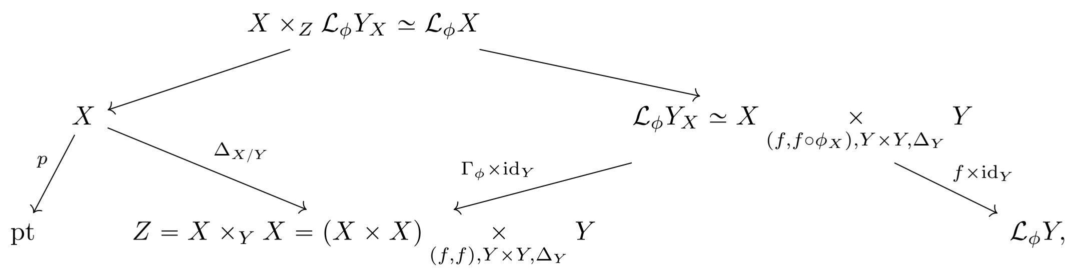
[BNP17b]) and by universal property of the limit. The second statement follows by passing to left adjoints (as in Corollary 2.4.2 of *op. cit.*). □

Proof of Proposition 3.36. Consider the functors

$$T(-) := - \otimes_{\mathrm{QC}(k)} \mathrm{QC}(X) : \mathbf{dgCat}_k \to \mathrm{QC}(Y)\text{-}\mathbf{mod},$$

$$T^R(-) := - \otimes_{\mathrm{QC}(Y)} \mathrm{QC}(X) : \mathrm{QC}(Y)\text{-}\mathbf{mod} \to \mathbf{dgCat}_k.$$

We claim that (T, T^R) are adjoint. Let $\Delta_X : X \to X \times X$ denote the diagonal, $p : X \to \mathrm{pt}$ denote the structure map, and $\Delta_{X/Y} : X \to X \times_Y X$ the relative diagonal. We define the unit $\eta : \mathrm{id}_{\mathbf{dgCat}_k} \to T^R \circ T$ via the functor $\Delta_{X/Y*}p^* : \mathrm{QC}(\mathrm{pt}) \to \mathrm{QC}(X \times_Y X)$ and the counit $\epsilon : T \circ T^R \to \mathrm{id}_{\mathrm{QC}(Y)\text{-}\mathbf{mod}}$ by the functor $f_* \Delta_X^* : \mathrm{QC}(X \times X) \to \mathrm{QC}(Y)$. Verification of the adjunction axioms is a straightforward application of base change and Theorem 4.7 of [BFN10]. To compute the trace, we apply base change and find that $[\mathrm{QC}(X), \phi_{X*}]$ is the pull-push of $k \in \mathrm{QC}(\mathrm{pt})$ along the diagram (where $\Delta_Y : Y \to Y \times Y$ is the diagonal):



i.e. $[\mathrm{QC}(X), \phi_{X*}] \simeq \mathcal{L}_\phi f_* \mathcal{O}_{\mathcal{L}_\phi X}$. □

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