MONODROMY FILTRATIONS AND THE TOPOLOGY OF TROPICAL VARIETIES

DAVID HELM AND ERIC KATZ

Abstract. We find restrictions on the topology of tropical varieties that arise from a certain natural class of varieties. We develop a theory of tropical degenerations that is a nonconstant coefficient analogue of Tevelev's theory of tropical compactifications, and use it to construct normal crossings degenerations of a subvariety X of a torus, under mild hypotheses on X. These degenerations allow us to construct a natural, "multiplicity-free" parameterization of $\mathrm{Trop}(X)$ by a topological space Γ_X. We give a geometric interpretation of the cohomology of Γ_X in terms of the action of a monodromy operator on the cohomology of X. This gives bounds on the Betti numbers of Γ_X in terms of the Betti numbers of X. When X is a sufficiently general complete intersection, this allows us to show that the cohomology of $\mathrm{Trop}(X)$ vanishes in degree less than $\dim X$.

1. Introduction

Tropicalization is a procedure which takes as input a sub-variety of a torus, $X \subset (K^*)^n$, and associates to it a balanced weighted rational polyhedral complex $\mathrm{Trop}(X) \subseteq \mathbb{R}^n$. Several estions naturally arise in this framework; for instance, one may ask which polyhedral complexes arise as tropicalizations. The work of Ardila and Klivans [AK] considers certain complexes associated to matroids; it follows implicitly from their work or from the work of Develin, Santos, and Sturmfels [DSS, Thm 7.3] that such a complex is the tropicalization of a linear subspace if and only if its associated matroid is realizable. This result was strengthened by Mihalkin, Sturmfels, and Ziegler [M] to prove that the weighted polyhedral complex is a tropicalization if and only if its matroid is realizable. In [Sp1], Speyer addresses the analogous question in the non-constant coefficient case. One may also ask what constraints being a tropicalization places on the topology of a polyhedral complex. In [H], Hacking proved that if X is a subvariety of $(\mathbb{C}^*)^n$ satisfying a certain genericity condition, then the link of the fan $\mathrm{Trop}(X)$ only has reduced rational homology in the top dimension. Hacking's result holds for a number of examples, including generic intersections of ample hypersurfaces in projective toric varieties. In [Sp3, Sec. 10], Speyer showed that if C is a genus g curve in $(K^*)^n$ satisfying a genericity condition then there exists a balanced metric graph Γ with $b_1(\Gamma) \leq g$ and a parameterization $i : \Gamma \to \mathrm{Trop}(C)$ that is affine-linear on edges. Our results can be seen as the analogue of Hacking's result for varieties defined over $K = \mathbb{C}((t))$ or as a higher-dimensional generalization of Speyer's result.

All of our results require that the variety X be *schön*, a natural condition introduced in [T]. This condition means that the ambient torus $(K^*)^n$ may be compactified to a toric degeneration $\mathbb{P}$ over $\mathrm{Spec}\, \mathbb{C}[[t]]$ such that $\mathcal{X} = \overline{X} \subset \mathbb{P}$ intersects the open torus orbits U_P in $\mathbb{P}$ transversely. This allows us to construct normal crossings degenerations of X (c.f. 3.11) over a finite extension of $\mathbb{C}[[t]]$. Our construction is essentially a "nonconstant coefficient" version of the tropical compactification construction of [T].

The existence of such normal crossings degenerations for a schön X allows us to construct a natural "parameterizing space" Γ_X. This generalizes a construction introduced by Speyer when X has dimension 1. This space comes equipped with a canonical map to $\mathrm{Trop}(X)$; moreover a choice of sufficiently fine triangulation of $\mathrm{Trop}(X)$ gives Γ_X the structure of a polyhedral complex. When Γ_X is viewed in such a way, the natural parameterization $\Gamma_X \to \mathrm{Trop}(X)$ is affine-linear on polyhedra. This parameterization has several nice properties. For instance, it is natural under monomial morphisms: if X and Y are schön subvarieties of tori T and T' and $\phi : T \to T'$ is a homomorphism taking X to Y, then there is an induced map of complexes $\Gamma_X \to \Gamma_Y$ that commutes with parameterizations. Moreover, Γ_X satisfies a balancing condition analogous to the one satisfied by all tropical varieties. Finally, it is "not far" from $\mathrm{Trop}(X)$: if the intersections of $\mathcal{X}$ with open torus orbits U_P in $\mathbb{P}$ satisfy certain connectedness hypotheses, we may equate the cohomology of Γ_X and $\mathrm{Trop}(X)$ in certain degrees. We hope that parameterizing complexes will be seen as a fundamental object in tropical geometry.

Our main results (principally Theorem 5.7 and Corollary 5.11) relate the cohomology of Γ_X to geometric invariants of X. We take the viewpoint that a variety defined over $\operatorname{Spec} K$ is analogous to a family of varieties defined over a punctured disc. The fundamental group of the punctured disc acts on the cohomology of a general fiber of this family by monodromy. The analogue of this monodromy action for varieties over $\operatorname{Spec} K$ is the action of $\mathrm{Gal}(\overline{K}/K) \cong \hat{\mathbb{Z}}(1)$ on the étale cohomology $H^*_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$. After a possible finite base-extension of $\mathbb{C}((t))$, this action is unipotent, and is given by the *monodromy operator*

$$N : H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l(-1))$$

an endomorphism of the étale cohomology that is essentially the (matrix) logarithm of the monodromy action. (We refer the reader to section 5 for precise definitions). The action of N induces an increasing filtration on the cohomology. For X schön, Theorem 5.7 relates the cohomology of Γ_X to a the smallest nontrivial piece of the monodromy filtration of the cohomology of $\overline{X}_K$, the closure of X_K in the generic fiber of $\mathbb{P}$:

$$H^r(\Gamma_X, \mathbb{Q}_l) \cong H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}.$$

As a consequence, Corollary 5.8 gives a generalization of Speyer's result, bounding the Betti numbers of Γ_X in terms of those of X:

$$b_r(\Gamma_X) \leq \frac{1}{r+1} b_r(X).$$

We show that if X is the generic intersection of ample hypersurfaces in a toric degeneration $\mathbb{P}$, then $H^r(\mathrm{Trop}(X), \mathbb{Q}_l)$ vanishes for $1 \leq r < \dim X$, a non-constant coefficient analog of Hacking's result.

It is interesting to compare this result to results of Berkovich [B] on rigid analytic spaces. In particular, Berkovich shows that the cohomology group $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$ arising in our result is isomorphic to the cohomology of the Berkovich space X^{an} attached to X. Our result thus suggests a strong link between Γ_X and X^{an}. Links between tropical geometry and rigid geometry have also appeared in [EKL] and Speyer's thesis [Sp2].

The main tool we use is the Rapoport-Zink weight spectral sequence [RZ]. Under the schön condition, after a finite base-extension of $\mathbb{C}((t))$, we may compactify the ambient torus to a toric degeneration $\mathbb{P}$ defined over $\mathcal{O} = \mathbb{C}[[t]]$ so that the central fiber of the closure $\mathcal{X}$ of X in $\mathbb{P}$ is a divisor with normal crossings. The divisor, a degeneration of X, has a stratification coming from intersections of its irreducible components. The Rapoport-Zink spectral sequence then gives a very explicit formula for the cohomology on X, and the monodromy action, in terms of these strata. The E_1-term of the weight spectral sequence is formed from the cohomology groups of the strata with boundary maps built from the data of restriction maps and Gysin maps. The spectral sequence converges to the cohomology of the general fiber. Moreover, there is an explicitly given endomorphism of the E_1-term that descends to N in the limit. By a recent result of Ito [I], the spectral sequence degenerates at E_2 and N is an isomorphism of certain graded pieces of the E_2-term. This gives us an explicit description of the smallest nontrivial piece of the filtration which is isomorphic to the cohomology groups of Γ_X.

Our arguments are very similar to those of Hacking and Speyer. Hacking uses a spectral sequence coming from a weight filtration on a complex of differential forms while we use the Rapoport-Zink spectral sequence. Speyer's results use a resolution of the structure sheaf of a degeneration of C coming from a stratification induced by a toric degeneration.

We should mention the related results of Gross and Siebert [GS]. There, the authors construct a scheme X_0 from an integer affine manifold and a toric polyhedral decomposition. If X_0 is embedded in a family $\mathcal{X}$ over $\mathbb{C}[[t]]$, they are able to determine the limiting mixed Hodge structure in terms of the combinatorial data.

We work over the formal Laurent series field $\mathbb{C}((t))$ instead of the field of Puiseux series, $\mathbb{C}\{\{t\}\}$. This is no limitation since any scheme of finite type defined over Puiseux series is defined over some field $\mathbb{C}((t^{\frac{1}{d}}))$ and our results apply. We restrict our attention to this ground field primarily for convenience. In fact, our results hold without significant change if we replace $\mathbb{C}[[t]]$ with any equal characteristic discrete valuation ring with algebraically closed residue field. Working over an equal characteristic discrete valuation ring whose residue field is not algebraically closed introduces only some minor technical issues; one must take *geometric* connected components in the construction of the space Γ_X in section 4. Over a discrete valuation ring of mixed characteristic our results become conditional on Deligne's conjecture on the purity of the monodromy filtration, which is only known in equal characteristic. They can be made into unconditional results by rephrasing all statements about monodromy filtrations in terms of weight filtrations.

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2. Toric Degenerations

We begin by reviewing a construction that attaches a degenerating family of toric varieties over $\mathbb{C}[t]$ to a rational polyhedral complex in $\mathbb{R}^n$. This has appeared several times in the literature [Sp2] [NS] [S]. We follow the approach of [NS] here.

Definition 2.1. A rational polyhedral complex in $\mathbb{R}^n$ is a collection Σ of finitely many convex rational polyhedra $P \subset \mathbb{R}^n$ with the following properties:

- If $P \in \Sigma$ and P' is a face of P, then P' is in Σ.

- If $P, P' \in \Sigma$ then $P \cap P'$ is a face of both P and P'.

Given a Σ as above, we can construct a fan $\tilde{\Sigma}$ in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ as follows: for each $P \in \Sigma$ let $\tilde{P}$ be the closure in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ of the set

$$\{(x, a) \subset \mathbb{R}^n \times \mathbb{R}_{>0} : \frac{x}{a} \in P\}.$$

Then $\tilde{P}$ is a rational polyhedral cone in $\mathbb{R}^n \times \mathbb{R}_{>0}$. Its facets come in two types:

- cones of the form $\tilde{P}'$, where P' is a facet of P, and
- the cone $P_0 = \tilde{P} \cap (\mathbb{R}^n \times \{0\})$, which is the limit as a goes to zero of the polyhedron aP in $\mathbb{R}^n$.

We let $\tilde{\Sigma}$ be the collection of cones of the form $\tilde{P}$ and P_0 for P in Σ. It is a rational polyhedral fan in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Note that $\Sigma = \tilde{\Sigma} \cap (\mathbb{R}^n \times \{1\})$. On the other hand the fan Σ_0 given by $\tilde{\Sigma} \cap (\mathbb{R}^n \times \{0\})$ is the limit as a approaches zero of the polyhedral complexes $a\Sigma$.

Remark 2.2. In fact, the association $\Sigma \mapsto \tilde{\Sigma}$ defines a bijection between the set of polyhedral complexes in $\mathbb{R}^n$ and the set of fans in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ for which every cone contained in $\mathbb{R}^n \times \{0\}$ is the boundary of a cone that meets $\mathbb{R}^N \times \mathbb{R}_{>0}$. Its inverse is $\tilde{\Sigma} \mapsto \tilde{\Sigma} \cap (\mathbb{R}^n \times \{1\})$.

Let $X(\tilde{\Sigma})$ be the toric variety over $\mathbb{C}$ associated to the fan $\tilde{\Sigma}$. Projection from $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ induces a map of fans from $\tilde{\Sigma}$ to the fan $\{0, \mathbb{R}_{\geq 0}\}$ associated to $\mathbb{A}^1_{\mathbb{C}}$. This gives rise to a map of toric varieties $\pi : X(\tilde{\Sigma}) \to \mathbb{A}^1_{\mathbb{C}}$. As remarked in [NS], this map is flat and torus equivariant.

One easily describes the general fiber of π. Indeed, $\pi^{-1}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ is a union of torus orbits; the orbits in $\pi^{-1}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ correspond to cones of the form P_0.

We thus have:

Lemma 2.3 ([NS], Lemma 3.4)**.** *There is a toric isomorphism:*

$$\pi^{-1}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) \cong X(\Sigma_0) \times_{\mathbb{C}} \mathbb{C}^{\times}.$$

The fiber over any particular t in $(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\})$ is isomorphic to $X(\Sigma_0)$.

If Σ is *integral*, i.e. the vertices of every polyhedron in Σ lie in $\mathbb{Z}^n$, then Nishinou and Seibert show ([NS], p. 10) that the special fiber $\pi^{-1}(0)$ is reduced. In this case $\pi^{-1}(0)$ is the union of the torus orbits corresponding to the cones in $\tilde{\Sigma}$ of the form $\tilde{P}$, for P in Σ. This defines an inclusion-reversing bijection between closed torus orbits in $\pi^{-1}(0)$ and polyhedra P in Σ; the irreducible components of $\pi^{-1}(0)$ correspond to vertices in Σ; the intersection of a collection of irreducible components corresponds to the smallest polyhedron in Σ containing all of their vertices.

Note that making a base change of the form $t \mapsto t^d$ on $\mathbb{A}^1_{\mathbb{C}}$ has the effect of rescaling Σ by d; that is, the base change of the family $X(\tilde{\Sigma}) \to \mathbb{A}^1_{\mathbb{C}}$ is the family $X(\widetilde{(d\Sigma)}) \to \mathbb{A}^1_{\mathbb{C}}$. In particular, given any toric degeneration coming from a polyhedral complex Σ, we can choose d such that $d\Sigma$ is integral; after the base change $t \mapsto t^d$ the special fiber of the family $X(\tilde{\Sigma}) \to \mathbb{A}^1_{\mathbb{C}}$ will be reduced.

This construction gives rise to a large class of toric degenerations. In fact, we have:

Proposition 2.4. *Let $\mathcal{O}$ be the ring $\mathbb{C}[[t]]$, and let K be its field of fractions. Let $\mathcal{T}$ be an n-dimensional split torus over $\mathcal{O}$, and let T be the torus $\mathcal{T} \times_{\mathcal{O}} \operatorname{Spec} K$ over*

K. Suppose $\mathcal{X}$ is a separated, finite type scheme over $\mathcal{O}$ with an action of $\mathcal{T}$ such that:

(1) *$\mathcal{X}$ is normal and flat over $\mathcal{O}$,*
(2) *$\mathcal{X} \times_{\mathcal{O}} \operatorname{Spec} K$ is a toric variety of $\mathcal{T} \times_{\mathcal{O}} \operatorname{Spec} K$,*
(3) *the open immersion $T \to \mathcal{X}$ taking T to the open torus orbit in $\mathcal{X} \times_{\mathcal{O}} \operatorname{Spec} K$ is dominant, and*
(4) *the closure of every torus orbit in the general fiber of $\mathcal{X}$ meets the special fiber.*

Then there is a polyhedral complex $\Sigma \subseteq \mathbb{R}^n$ such that $\mathcal{X}$ is the base change of $X(\tilde{\Sigma})$ to $\operatorname{Spec} \mathcal{O}$.

Proof. We adapt the proof of the correspondence between normal toric varieties and fans as given in [KKMS]. Every point $z \in \mathcal{X} \times_{\mathcal{O}} \operatorname{Spec} \mathbb{C}$ has an open affine invariant neighborhood $\operatorname{Spec} R$ in $\mathcal{X}$. R has a decomposition into $\mathcal{O}$-modules graded by M, the character lattice of $\mathcal{T}$. By property (3) R is a graded submodule of the ring $K[M]$ of rational functions on T. Therefore, R is generated as an $\mathcal{O}$-module by elements $t^{a_m} x^m$ for $m \in S$, S a semigroup in M and $a \in \mathbb{Z}$.

Consider the semigroup $\tilde{S} \subset M \times \mathbb{Z}$ given by (m, a) for all $t^a x^m \in R$. Then $R = \mathbb{C}[\tilde{S}] \otimes \mathcal{O}$. By normality, $\tilde{S}$ is a saturated semigroup, and is therefore dual to a cone $\tilde{\sigma}$ in $M^\vee \times \mathbb{Z}$. Since $(0,1) \in \tilde{S}$, $\tilde{\sigma} \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. By standard arguments, the $\tilde{\sigma}$'s glue together to give a fan $\tilde{\Sigma}$ in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. By property (4), $\mathcal{X} = X(\tilde{\Sigma}) \times_{\mathbb{C}[t]} \operatorname{Spec} \mathcal{O}$. Moreover, the closure of every torus orbit in the general fiber of $\mathcal{X}$ meets the special fiber of $\mathcal{X}$; in terms of the fan $\tilde{\Sigma}$ this means that every cone of $\tilde{\Sigma}$ contained in $\mathbb{R}^n \times \{0\}$ is the boundary of a cone that meets $\mathbb{R}^n \times \mathbb{R}_{>0}$. Thus $\tilde{\Sigma}$ arises from the polyhedral complex Σ given by $\tilde{\Sigma} \cap (\mathbb{R}^n \times \{1\})$ in the manner described above. □

We will be particularly interested in degenerations of toric varieties in which the special fiber is a divisor with normal crossings. These are easy to construct, because the boundary of a smooth toric variety is always a divisor with normal crossings:

Proposition 2.5. *Let Σ be a rational polyhedral complex in $\mathbb{R}^n$. There exists an integer d, and a subdivision Σ' of $d\Sigma$ such that the general fiber of the degeneration $X(\tilde{\Sigma}')$ is a smooth toric variety and the special fiber of $X(\tilde{\Sigma}')$ is a divisor with normal crossings.*

Proof. Choose d such that $d\Sigma$ is integral. Fulton [F], section 2.6, gives an algorithm for constructing a subdivision $\tilde{\Sigma}'$ of the fan $\widetilde{(d\Sigma)}$ such that all the cones of $\tilde{\Sigma}'$ are simplicial and unimodular. We will show that this algorithm can be performed in such a way that $\tilde{\Sigma}' \cap (\mathbb{R}^n \times \{1\})$ is *integral*.

Each step of Fulton's algorithm involves taking a cone C of $\tilde{\Sigma}$ that fails to be simplicial or unimodular and is of the smallest possible dimension, choosing a vector v in its relative interior, and replacing each cone of $\tilde{\Sigma}$ that contains v with the join of v and its faces. The only constraint v must satisfy for the algorithm to work is that if C is simplicial and $v_1, \ldots, v_r$ are the minimal lattice vectors along the one-dimensional faces of C, then v must be of the form

$$\sum_i t_i v_i; \quad 0 \leq t_i < 1.$$

Note that every cone of $\widetilde{(d\Sigma)}$ that meets $\mathbb{R}^n \times \{1\}$ is a cone over the lattice polyhedron dP for some $P \in \Sigma$. Therefore at each step of Fulton's algorithm that involves a cone meeting $\mathbb{R}^n \times \{1\}$ we can always take v to be a lattice point in the interior of dP. In this way we ensure that the resulting subdivision $\tilde{\Sigma}'$ has the property that the polyhedral complex $\tilde{\Sigma}' \cap (\mathbb{R}^n \times \{1\})$ is integral. Call this polyhedral complex Σ'. Observe that $\widetilde{\Sigma'} = \tilde{\Sigma}'$.

The upshot is that $X(\tilde{\Sigma}')$ is a smooth toric variety with a birational morphism $X(\tilde{\Sigma}') \to X(\tilde{\Sigma})$. The induced map $X(\tilde{\Sigma}') \to \mathbb{A}^1_{\mathbb{C}}$ is the toric degeneration associated to the integral polyhedral complex Σ'.

The general fiber of $X(\tilde{\Sigma}')$ over $\mathbb{A}^1_{\mathbb{C}}$ corresponds to the fan Σ'_0, and is therefore smooth. The special fiber is a union of irreducible components of the boundary of $X(\tilde{\Sigma}')$, and is therefore a divisor of $X(\tilde{\Sigma}')$ with normal crossings. □

3. Tropical Degenerations

Let $\mathcal{O}$ be the ring $\mathbb{C}[[t]]$, and let K be its field of fractions. We now turn to the question of studying degenerations of varieties over K via tropical geometry. The approach we take is analogous to Tevelev's approach to compactifications [T] in the "constant coefficient case"; our construction is equivalent to that appearing in Speyer's thesis [Sp2]. We note that the extension of Tevelev's work to the DVR case was done by Zhenhua Qu in part of his Ph.D. thesis [Q].

Let $\overline{K}$ be the algebraic closure of K; it is the field of Puiseaux series $\mathbb{C}\{\{t\}\}$ in one variable. There is a unique valuation

$$\mathrm{ord} : \overline{K}^{\times} \to \mathbb{Q}$$

such that $\mathrm{ord}(t) = 1$.

Let $\mathcal{T} \cong \mathbb{G}_m^n$ be a split n-dimensional torus over $\mathcal{O}$, and let $T = \mathcal{T} \times_{\mathcal{O}} K$ be the corresponding torus over K. The valuation ord gives rise to a map

$$\mathrm{val} : \mathcal{T}(\overline{K}) \to \mathbb{Q}^n,$$

by fixing an isomorphism of $\mathcal{T}$ with $\mathbb{G}_m^n$ (and hence an isomorphism of $\mathcal{T}(\overline{K})$ with $(\overline{K}^{\times})^n$.) Let X be a closed subvariety of T, defined over K.

Definition 3.1 ([EKL], 1.2.1)**.** : The tropical variety $\mathrm{Trop}(X)$ associated to X is the closure of $\mathrm{val}(X(\overline{K}))$ in $\mathbb{R}^n$.

Given such an X, we wish to find a well-behaved compactification $\overline{X}$ of X, and a well-behaved degeneration of $\overline{X}$ over $\mathcal{O}$. This turns out to be intimately connected to the set $\mathrm{Trop}(X)$.

We proceed as follows: Let Σ be a rational polyhedral complex in $\mathbb{R}^n$, and let $\mathbb{P}$ be the corresponding toric degeneration over $\mathbb{C}[t]$. By base change from $\mathbb{C}[t]$ to $\mathcal{O}$, we view $\mathbb{P}$ as a toric degeneration over $\mathcal{O}$. We identify the group of cocharacters of $\mathcal{T}$ with $\mathbb{Z}^n$ in $\mathbb{R}^n$; this identifies $\mathbb{T}$ with the open torus orbit on $\mathbb{P}$.

We can thus take the closure of $\mathcal{X}$ of X in $\mathbb{P}$. We then have:

Proposition 3.2. *The scheme $\mathcal{X}$ is proper over $\mathcal{O}$ if, and only if,* $\mathrm{Supp}\,\Sigma$ *contains* $\mathrm{Trop}(X)$.

Proof. Over K this is [T], 2.3. The general case is proven in [Sp2], 2.4.1. □

We assume henceforth that $\mathrm{Supp}\,\Sigma$ contains $\mathrm{Trop}(X)$. Let $\overline{X}$ be the fiber of $\mathcal{X}$ over K, and $\mathcal{X}_{\mathbb{C}}$ be the special fiber of $\mathcal{X}$. The natural multiplication map

$$\mathcal{T} \times_{\mathcal{O}} \mathbb{P} \to \mathbb{P}$$

restricts to a multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}.$$

Definition 3.3. The pair $(X, \mathbb{P})$ is *tropical* if the map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}$$

is faithfully flat, and $\mathcal{X} \to \mathcal{O}$ is proper.

Note that this is slightly different from the definition of a tropical pair in [T], as Tevelev always works over $\mathbb{C}$ rather than over $\mathcal{O}$ (and hence with toric varieties over $\mathbb{C}$ rather than toric degenerations.) In particular being a tropical pair *over* $\mathcal{O}$ is a "relative version" of the tropical pair condition of [T].

Proposition 3.4 (c.f. [T], 2.5)**.** *Suppose $(X, \mathbb{P})$ is tropical and let $\mathbb{P}' \to \mathbb{P}$ be a morphism of toric degenerations corresponding to a refinement Σ' of Σ. Then $(X, \mathbb{P}')$ is also tropical.*

Proof. The proof of this is exactly the same as the proof of [T], 2.5, except that Tevelev works over $\mathbb{C}$ rather than $\mathcal{O}$. □

Proposition 3.5 (c.f. [T], 2.5)**.** *If $(X, \mathbb{P})$ is a tropical pair then* $\operatorname{Supp} \Sigma = \operatorname{Trop}(X)$.

Proof. This result is also proven by Tevelev in the constant coefficient case, as part of [T], 2.5. The only element of that proof that does not carry over directly is [T], Lemma 2.2; Speyer proves a nonconstant analogue of this lemma in [Sp2], Lemma 2.3.1. □

Following Speyer ([Sp2], 2.4) If $(X, \mathbb{P})$ is a tropical pair, we call $\overline{X}$ a *tropical compactification* of X, and $\mathcal{X}_{\mathbb{C}}$ a *tropical degeneration* of X.

In fact, the "visual contour" construction of Tevelev shows that every X admits a tropical pair $(X, \mathbb{P})$. In particular, let $\mathbb{P}$ be a projective toric variety of $\mathcal{T}$ over $\mathcal{O}$, and let $\overline{X}$ be the closure of X in $\mathbb{P}$. Let $\operatorname{Hilb}(\mathbb{P})$ be the Hilbert scheme over $\mathcal{O}$ parameterizing subschemes of $\mathbb{P}$ with the same Hilbert polynomial as $\overline{X}$. The torus $\mathcal{T}$ acts on $\operatorname{Hilb}(\mathbb{P})$ via $t[Y] = [t^{-1}Y]$, where Y is a subscheme of $\mathbb{P}$ and $[Y]$ is the corresponding Hilbert point. The map $T \to \mathbb{P}$ defined by

$$t \mapsto t[X]$$

identifies T/T_X with a locally closed subscheme of $\operatorname{Hilb}(\mathbb{P})$, where T_X is the normalizer of X in T.

Let $\mathbb{P}_{\mathrm{Gr}}$ be the normalization of the closure of T/T_X in $\operatorname{Hilb}(\mathbb{P})$; by Proposition 2.4, it is a toric degeneration over $\mathcal{O}$ associated to a polyhedral complex Σ. If $\mathcal{T}_X$ is the closure of T_X in $\mathcal{T}$, then $\mathcal{T}/\mathcal{T}_X$ acts simply transitively on an open dense subset of $\mathbb{P}_{\mathrm{Gr}}$.

Note that T_X acts freely on T, and hence on X. We have $X/T_X \subset T/T_X \subset \mathcal{T}/\mathcal{T}_X \subset \mathbb{P}_{\mathrm{Gr}}$. Let $\mathcal{X}_{\mathrm{vc}}$ be the closure of X/T_X in $\mathbb{P}_{\mathrm{Gr}}$, and let $\mathbb{P}_{\mathrm{vc}}$ be the subset of $\mathbb{P}_{\mathrm{Gr}}$ consisting of torus orbits that meet $\mathcal{X}_{\mathrm{vc}}$.

Theorem 3.6 ([T], 1.7)**.** *The pair $(X/T_X, \mathbb{P}_{Gr})$ is tropical, and $\mathcal{X}_{\mathrm{vc}}$ is the corresponding tropical degeneration.*

Proof. The proof of [T], 1.7 carries through verbatim to this situation. □

The combinatorics of the special fiber of a tropical degeneration of X is closely related to the combinatorics of $\mathrm{Trop}(X)$. In particular if $(X, \mathbb{P})$ is a tropical pair, and $\mathcal{X}$ is the corresponding tropical degeneration, then a polyhedron P of Σ corresponds to the closure of a torus orbit in the special fiber of $\mathbb{P}$. Call this torus orbit closure $\mathbb{P}_P$. Then the intersection $\mathcal{X}_P$ of $\mathcal{X}$ with $\mathbb{P}_P$ is a closed subscheme of $\mathcal{X}_{\mathbb{C}}$. Moreover, if P and P' are polyhedra of Σ, and Q is the smallest polyhedron in Σ containing both P and P', then the intersection of $\mathcal{X}_P$ and $\mathcal{X}_{P'}$ is $\mathcal{X}_Q$.

Let U_P be the open torus orbit corresponding to P. Fix a point p in U_P, and consider the fiber over p of the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}.$$

On the one hand, $m^{-1}(p)$ is nonempty of dimension equal to the dimension of X, since m is flat and surjective. On the other hand, by projection onto $\mathcal{X}$, $m^{-1}(p)$ is isomorphic to the product $\mathcal{T}_P \times (\mathcal{X} \cap U_P)$, where $\mathcal{T}_P$ is the subgroup of $\mathcal{T}_0$ that acts trivially on U_P. Since $(\mathcal{X} \cap U_P)$ is dense in $\mathcal{X}_P$ we find that $\mathcal{X}_P$ is nonempty of dimension equal to the dimension of X minus the dimension of P.

On the other hand, let w be a point in the relative interior of P. Then w corresponds to a cocharacter of t, and $\lim_{t\to 0} w(t)$ is a point p in U_P. Projection onto $\mathcal{T}$ identifies $m^{-1}(p)$ with the limit $\mathrm{in}_w X$ of $w(t)X$ as t approaches zero. (More formally, $\mathrm{in}_w X$ can be defined as the special fiber of the closure in $\mathcal{T}$ of the subscheme $w(t)X$ of T.) In particular we have

$$\mathrm{in}_w X \cong \mathcal{T}_P \times (\mathcal{X} \cap U_P)$$

for any w in the relative interior of P. We have thus shown:

Lemma 3.7. *The irreducible components of $\mathcal{X} \cap U_P$ are in bijection with the irreducible components of $\mathrm{in}_w X$.*

We will be particularly interested in tropical pairs $(X, \mathbb{P})$ where the multiplication map $m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}$ is *smooth*. This is a generalization of the schön condition of [T] to the nonconstant coefficient case.

Definition 3.8. (c.f. [T], definition 1.3) A subvariety X of $\mathcal{T}$ is *schön* if there exists a tropical pair $(X, \mathbb{P})$ such that the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}$$

is smooth.

Proposition 3.9 ([T], 1.4)**.** *If X is schön, then for* any *tropical pair $(X, \mathbb{P})$, the multiplication map*

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}$$

is smooth.

Proof. As usual, the proof of the constant coefficient version of this result [T], 1.4, carries over to the nonconstant case without change. □

Note that if X is schön then it is smooth (consider the preimage of the identity in T under the multiplication map.) In fact, we have:

Proposition 3.10. *The following are equivalent:*

(1) *X is schön.*
(2) *$\mathrm{in}_w X$ is smooth for all $w \in \mathrm{Trop}(X)$.*

(3) *For any tropical pair $(X, \mathbb{P})$, and any polyhedron P in Σ, $\mathcal{X}$ meets the torus orbit U_P transversely.*

Proof. Statements 2) and 3) are clearly equivalent since we have seen that $\mathrm{in}_w X$ is the product of a torus with $\mathcal{X} \cap U_P$, where P is the polyhedron in Σ that contains w in its relative interior.

As for the equivalence of 1) and 2), fix a tropical pair $(X, \mathbb{P})$. We have seen that the fibers of the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \to \mathbb{P}$$

are isomorphic to $\mathrm{in}_w X$ as w ranges over $\mathrm{Trop}(X)$. So 1) implies 2) is clear. For the converse, note that since m is faithfully flat, to show it is smooth it suffices to show that it has smooth fibers. □

It is easy to construct tropical degenerations of schön varieties X in which the special fiber is a divisor with normal crossings. In particular we have:

Proposition 3.11 (c.f. [H], proof of 2.4)**.** *Let X be schön. There exists an integer d and a tropical pair $(X, \mathbb{P})$ over $\mathcal{O}[t^{\frac{1}{d}}]$ such that $\overline{X}$ is smooth over $K[t^{\frac{1}{d}}]$, and $\mathcal{X}_{\mathbb{C}}$ is a divisor in $\mathcal{X}$ with normal crossings.*

Proof. Let $(X, \mathbb{P})$ be any tropical pair over $\mathcal{O}$, and let Σ be the rational polyhedral complex corresponding to $\mathbb{P}$. By Proposition 2.5, we can find a refinement Σ' of Σ and an integer d such that the corresponding toric degeneration $\mathbb{P}'$ (viewed over $\mathcal{O}[t^{\frac{1}{d}}]$) has smooth general fiber, and special fiber a divisor with normal crossings.

Then $(X, \mathbb{P}')$ is also tropical, and the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X}' \to \mathbb{P}'$$

is smooth by the previous proposition. Since the special fiber of $\mathbb{P}'$ is a divisor with normal crossings, so is the special fiber of $\mathcal{T} \times_{\mathcal{O}} \mathcal{X}'$. Hence the special fiber of $\mathcal{X}'$ is a divisor with normal crossings as well. Similarly, the general fiber of $\mathcal{X}'$ is smooth because the general fiber of $\mathbb{P}'$ is smooth. □

Definition 3.12. We call a pair $(X, \mathbb{P})$ of the sort produced by Proposition 3.11 a *normal crossings pair*. If $(X, \mathbb{P})$ is a normal crossings pair, and Σ is the polyhedral decomposition of $\mathrm{Trop}(X)$ corresponding to $\mathbb{P}$, we say that Σ is a *normal crossings decomposition* of $\mathrm{Trop}(X)$.

Remark 3.13. In much of what follows, we will often need to attach a normal crossings pair to a schön variety X over $\mathcal{O}$. To do this we may need to replace $\mathcal{O}$ with a finite extension $\mathcal{O}[t^{\frac{1}{d}}]$; this is harmless and we often do so without comment.

4. Parameterized Tropical Varieties

In this section, given a schön subvariety X of $\mathcal{T}$, we construct a natural parameterization of $\mathrm{Trop}(X)$ by a topological space Γ_X. This parameterization is functorial in a sense we make precise below. Moreover, we will see in the next section that the space Γ_X encodes more precise information about the cohomology of X than $\mathrm{Trop}(X)$ does. Our approach generalizes a construction of Speyer ([Sp3], proof of Theorem 10.8) when X has dimension 1.

Suppose we have a normal crossings pair $(X, \mathbb{P})$ so $\mathrm{Supp}(\Sigma) = \mathrm{Trop}(X)$. We associate to $(X, \mathbb{P})$ a polyhedral complex $\Gamma_{(X,\mathbb{P})}$ as follows: its k-cells are pairs (P, Y), where P is a polyhedron in Σ and Y is an irreducible component of $\mathcal{X}_P$.

The cells on the boundary of (P, Y) are the cells of the form (P_i, Y_i), where P_i is a facet of P and Y_i is the irreducible component of $\mathcal{X}_{P_i}$ containing Y (there is exactly one such irreducible component because $\mathcal{X}_{P_i}$ is smooth, so its irreducible components do not meet). The complex $\Gamma_{(X,\mathbb{P})}$ maps naturally to Σ by sending (P, Y) to P.

Proposition 4.1. *The underlying topological space of* $\Gamma_{(X,\mathbb{P})}$ *depends only on* X*.*

Proof. Any two polyhedral decompositions of $\mathrm{Trop}(X)$ have a common refinement; we can further refine this to be a normal crossings decomposition of $\mathrm{Trop}(X)$. It thus suffices to show that if Σ and Σ' are normal crossings decompositions of $\mathrm{Trop}(X)$, with associated normal crossings pairs $(X, \mathbb{P})$ and $(X, \mathbb{P}')$, and Σ' refines Σ, then the underlying topological spaces of $\Gamma_{(X,\mathbb{P})}$ and $\Gamma_{(X,\mathbb{P}')}$ are isomorphic.

Since Σ' is a refinement of Σ, we have a map $\mathbb{P}' \to \mathbb{P}$. Let $\mathcal{X}'$ be the degeneration corresponding to the pair $(X, \mathbb{P}')$. If P is a polyhedron of Σ, and P' is a polyhedron of Σ' contained in P, then this map induces a map of $\mathcal{X}'_{P'}$ to $\mathcal{X}_P$. In particular, for every pair (P', Y') of $\Gamma_{X,\mathbb{P}'}$, the image of Y' in $\mathcal{X}$ is contained in an unique irreducible component Y of $\mathcal{X}_P$. The map taking (P', Y') to (P, Y) is then a map of polyhedral complexes

$$\Gamma_{(X,\mathbb{P}')} \to \Gamma_{(X,\mathbb{P})}.$$

The natural map

$$\Gamma_{(X,\mathbb{P}')} \to \mathrm{Trop}(X)$$

is the composition of the map

$$\Gamma_{(X,\mathbb{P}')} \to \Gamma_{(X,\mathbb{P})}$$

with the natural map

$$\Gamma_{(X,\mathbb{P})} \to \mathrm{Trop}(X).$$

The fiber of $\Gamma_{(X,\mathbb{P})} \to \mathrm{Trop}(X)$ over a point w is in canonical bijection with the set of connected components of $\mathcal{X}_P$, and is therefore also in bijection with the set of connected components of $\mathcal{X} \cap U_P$, as $\mathcal{X} \cap U_P$ is dense in $\mathcal{X}_P$, and $\mathcal{X}_P$ is smooth. By Lemma 3.7 this latter set is in bijection with the set of connected components of $\mathrm{in}_w X$, which is *independent of* $\mathbb{P}$. Thus the map

$$\Gamma_{(X,\mathbb{P}')} \to \Gamma_{(X,\mathbb{P})}$$

is bijective, and is therefore a homeomorphism on the underlying topological spaces of $\Gamma_{(X,\mathbb{P}')}$ and $\Gamma_{(X,\mathbb{P})}$. □

In light of this proposition, we denote by Γ_X the underlying topological space of $\Gamma_{(X,\mathbb{P})}$ for *any* normal crossings pair $(X, \mathbb{P})$. We think of Γ_X, together with its natural map to $\mathrm{Trop}(X)$, as a "parameterized tropical variety". Note that Γ_X inherits an integral affine structure by pullback from $\mathrm{Trop}(X)$; more precisely, for any normal crossings pair $(X, \mathbb{P})$, the map $\Gamma_X \to \mathrm{Trop}(X)$ is linear on any polyhedron in $\Gamma_{(X,\mathbb{P})}$.

Note that for any $w \in \mathrm{Trop}(X)$, the number of preimages of w in Γ_X is equal to the number of connected components of $\mathrm{in}_w X$. Therefore, if Σ is a normal crossings decomposition of $\mathrm{Trop}(X)$, and w is in the relative interior of a top dimensional cell P of Σ, then the number of preimages of w is equal to the *multiplicity* of P in $\mathrm{Trop}(X)$. This suggests that we should give Γ_X the structure of a weighted polyhedral complex by giving every polyhedron on Γ_X weight one.

If we do this, then Γ_X satisfies a "balancing condition" analogous to the well-known balancing condition on $\text{Trop}(X)$. Fix a normal crossings decomposition Σ of $\text{Trop}(X)$, with corresponding normal crossings pair $(X, \mathbb{P})$. Consider a polyhedron (P, Y) of $\Gamma_{(X,\mathbb{P})}$ of dimension $\dim X - 1$, and let $\{(P_i, Y_i)\}$ be the top dimensional polyhedra of $\Gamma_{(X,\mathbb{P})}$ containing (P, Y).

Fix a point w with rational coordinates in the relative interior of P, and let V_P be the linear span, $\text{Span}(P - w)$. Similarly, for each P_i, let V_i be the positive span of $\text{Span}^+(P_i - w)$. Then V_i/V_P is a ray in $\mathbb{R}^n/V_P$; this collection of rays is the fan attached to the toric variety $\mathbb{P}_P$. Let v_i be the smallest integer vector along the ray V_i/V_P.

Proposition 4.2. *The v_i's satisfy the "balancing property":*

$$\sum_{(P_i,Y_i)} v_i = 0.$$

Proof. Torus-equivariant rational functions on $\mathbb{P}_P$ correspond to lattice vectors u in the space $(\mathbb{R}^n/V_P)^*$ dual to $\mathbb{R}^n/V_P$. The valuation of u along the divisor of $\mathbb{P}_P$ corresponding to v_i is simply $u(v_i)$.

Now restrict u to the curve $\mathcal{X}_P$. For any polyhedron P' of Σ containing P, $\mathcal{X}_P$ intersects the boundary divisior $\mathbb{P}_{P'}$ in one point for each cell (P_i, Y_i) of $\Gamma_{(X,\mathbb{P})}$ with $P_i = P'$. The divisor of u is therefore equal to

$$\sum_{(P_i,Y_i)} u(v_i)Y_i,$$

as $\mathcal{X}_P$ intersects each boundary divisor $\mathbb{P}_{P_i}$ transversely. This divisor is a principal divisor and thus has degree zero. □

We have thus attached to any schön subvariety X of $\mathcal{T}$, a canonical, multiplicity free parameterization by the topological space Γ_X. Moreover, this construction is functorial: let $\mathcal{T}$ and $\mathcal{T}'$ be tori over $\mathcal{O}$, and let T and T' be their general fibers. Suppose we have schön subvarieties X and Y of T and T', respectively, and a homomorphism of tori $T \to T'$ that takes X to Y. We then have a natural map $f : \text{Trop}(X) \to \text{Trop}(Y)$.

Proposition 4.3. *There is a natural map $\Gamma_X \to \Gamma_Y$ that makes the diagram*

$$\begin{array}{ccc} \Gamma_X & \to & \Gamma_Y \\ \downarrow & & \downarrow \\ \text{Trop}(X) & \to & \text{Trop}(Y) \end{array}$$

commute.

Proof. Let Σ' be a normal crossings decomposisition of $\text{Trop}(Y)$. By proposition 3.11 we can find a normal crossings decomposition Σ of $\text{Trop}(X)$ such that the image of any cell of Σ under the map f is contained in a cell of Σ'. Let $(X, \mathbb{P})$ and $(Y, \mathbb{P}')$ be the tropical pairs corresponding to Σ and Σ', and let $\mathcal{X}$ and $\mathcal{Y}$ denote the associated tropical degenerations. Since each cell of Σ maps into a cell of Σ', we obtain a map from $\mathcal{X}$ to $\mathcal{Y}$ extending the map $X \to Y$.

Now let P be a polyhedron in Σ, and P' be the polyhedron of Σ' containing the image of P. Then our map $\mathcal{X} \to \mathcal{Y}$ induces a map $\mathcal{X}_P \to \mathcal{Y}_{P'}$.

If (P, X_i) is a polyhedron of $\Gamma_{(X,\mathbb{P})}$, then by definition X_i is a connected component of $\mathcal{X}_P$. The image of X_i in $\mathcal{Y}_{P'}$ is contained in a connected component Y_i of

$\mathcal{Y}_{P'}$. We can thus construct a map of polyhedral complexes

$$\Gamma_{(X,\mathbb{P})} \to \Gamma_{(Y,\mathbb{P}')}$$

that maps (P, X_i) to (P', Y_i) by the map $P \to P'$. The induced map $\Gamma_X \to \Gamma_Y$ on underlying topological spaces is clearly continuous and makes the diagram commute.

To see that it is independent of choices, note that for $w \in \mathrm{Trop}(X)$, the preimage of w in Γ_X is in canonical bijection with the set of connected components of $\mathrm{in}_w X$. Similarly the preimage of $f(w)$ in Γ_Y is in bijection with the connected components of $\mathrm{in}_{f(w)} Y$. The map $X \to Y$ induces a natural map $\mathrm{in}_w X \to \mathrm{in}_{f(w)} Y$; in terms of this map the map $\Gamma_X \to \Gamma_Y$ takes a point over w (corresponding to a connected component Z of $\mathrm{in}_w X$) to the point over $f(w)$ that corresponds to the connected component of $\mathrm{in}_{f(w)} Y$ containing the image of Z. □

Remark 4.4. Although Proposition 4.3 is stated for maps $X \to Y$ that are *monomial morphisms* (i.e., that arise from morphisms of the ambient tori), we can avoid this issue if X and Y are intrinsically embedded. Recall that X is *very affine* if it can be embedded as a closed subscheme of a torus T. In this case (c.f. [T], section 3) there is an intrinsic torus T_X associated to X a *canonical* embedding of X in T_X. Moreover, if X and Y are very affine and $f : X \to Y$ is a morphism, there is a morphism of tori $T_X \to T_Y$ that induces f.

We also record, for later use, the following result relating the cohomology of Γ_X to that of $\mathrm{Trop}(X)$:

Lemma 4.5. *Let X be schön, and let Σ be a normal crossings decomposition of* $\mathrm{Trop}(X)$. *Suppose that for each polyhedron P in Σ, $\mathcal{X}_P$ is either connected or has dimension zero. Then the natural map*

$$H^r(\mathrm{Trop}(X), \mathbb{Z}) \to H^r(\Gamma_X, \mathbb{Z})$$

is an isomorphism for $0 \leq r < \dim X$, and an injection for $r = \dim X$.

Proof. Let $(X, \mathbb{P})$ be the normal crossings pair attached to Σ, so that $\Gamma_{(X,\mathbb{P})}$ is a triangulation of Γ. The polyhedra P in $\Gamma_{(X,\mathbb{P})}$ with $\dim P < \dim X$ are, by construction, in bijection with the polyhedra in Σ with $\dim P < \dim X$. Thus $\Gamma_{(X,\mathbb{P})}$ is obtained from Σ by adding additional top-dimensional cells; the result follows immediately. □

5. The Weight Spectral Sequence

The control that tropical geometry gives over the degenerations of schön subvarieties X of T has significant consequences on the level of cohomology. In particular the theory of vanishing cycles allows one to relate the étale cohomology of a nice tropical compactification of X to that of its tropical degeneration. When the degeneration is a divisor with normal crossings, this relationship is given by the Rapoport-Zink weight spectral sequence.

Let X be a variety over K, and consider the base change $X_{\overline{K}}$ of X to $\overline{K}$. The group $\mathrm{Gal}(\overline{K}/K)$ is canonically isomorphic to $\hat{\mathbb{Z}}(1)$, where $\hat{\mathbb{Z}}(1)$ is the inverse limit of the groups μ_n of nth roots of unity under the system of maps $\mu_{mn} \to \mu_n$ given by $\zeta \mapsto \zeta^m$. For an arbitrary profinite abelian group A, let $A(1)$ be the "Tate twisted" group $A \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}(1)$. Then $\hat{\mathbb{Z}}(1) \cong \mathrm{Gal}(\overline{K}/K)$ acts on the étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ for any prime l. This action is quasi-unipotent, i.e. a subgroup of H

of $\hat{\mathbb{Z}}(1)$ of finite index acts unipotently on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$. In particular there is a nilpotent map

$$N : H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \to H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l(-1))$$

called the *monodromy operator* such that for all $\sigma \in H$, σ acts on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ by $\exp(t_l(\sigma)N)$, where t_l is the canonical projection $\hat{\mathbb{Z}}(1) \to \mathbb{Z}_l(1)$.

Remark 5.1. A choice of a compatible system of primitive nth roots of unity gives an isomorphism of $\hat{\mathbb{Z}}(1)$ with $\hat{\mathbb{Z}}$. In our setting this isomorphism is $\mathrm{Gal}(\overline{K}/K)$-equivariant as K contains all roots of unity. Thus if one is willing the fix such a choice, one can can safely ignore all of the Tate twists in this discussion.

Now, if V is any finite dimensional vector space, with a nilpotent endomorphism N such that $N^r = 0$, then there is a unique increasing filtration $\{V_i\}$ on V such that:

- $V_r = V$,
- $V_{-r} = 0$,
- N maps V_i to V_{i-2}, and
- N^i induces an isomorphism $V_i/V_{i-1} \to V_{-i}/V_{-i-1}$.

(see [D] I, 1.7.2 for details.) We thus obtain a natural filtration, called the monodromy filtration, on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$.

Remark 5.2. If V consists of a single Jordan block of dimension r, one sees easily that V_i/V_{i-1} is one-dimensional for $i \in \{r-1, r-3, \dots, -r+1\}$, and zero otherwise. Moreover, V_{r-1-2k} is the image of N^k for $0 \leq k \leq r-1$. It is thus straightforward to read off the filtration coming from an arbitrary V and N from a Jordan normal form for N. The filtration is independent of choices, even though the Jordan normal form of N is not.

If X degenerates over $\mathcal{O}$ to a divisor with normal crossings, the etale cohomology of X and the monodromy filtration have a nice description in terms of the special fiber. Let $\mathcal{X}$ be a proper scheme over $\mathcal{O}$, of relative dimension n, whose fiber X_K over $\operatorname{Spec} K$ is smooth and whose fiber $\mathcal{X}_{\mathbb{C}}$ over $\operatorname{Spec} \mathbb{C}$ is a divisor with normal crossings. Then the Rapoport-Zink weight spectral sequence relates the étale cohomology of $X_{\overline{K}}$ to the geometry of the special fiber $\mathcal{X}_{\mathbb{C}}$. More precisely, let $\mathcal{X}_{\mathbb{C}}^{(r)}$ denote the disjoint union of $(r+1)$-fold intersections of irreducible components of $\mathcal{X}_{\mathbb{C}}$; it is smooth of dimension $n-r$ over $\mathbb{C}$. We then have:

Theorem 5.3 ([RZ], Satz 2.10; see also [I])**.** *There is a spectral sequence:*

$$E_1^{-r,w+r} = \bigoplus_{s \geq \max(0,-r)} H^{w-r-2s}_{\text{ét}}(\mathcal{X}_{\mathbb{C}}^{(2s+r)}, \mathbb{Q}_l(-r-s)) \Rightarrow H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l).$$

The boundary maps of this spectral sequence are completely explicit, and can be described as follows: up to sign, they are direct sums of restriction maps

$$H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m))) \to H^i_{\text{ét}}(Y', \mathbb{Q}_l(-m))$$

where Y is an irreducible component of $\mathcal{X}_{\mathbb{C}}^{(j)}$ and Y' is an irreducible component of $\mathcal{X}_{\mathbb{C}}^{(j+1)}$ contained in Y, or Gysin maps

$$H^i_{\text{ét}}(Y', \mathbb{Q}_l(-m)) \to H^{i+2}_{\text{ét}}(Y, \mathbb{Q}_l(-m+1))$$

where Y and Y' are as above.

More precisely, each term $E_1^{p,q}$ is a direct sum of terms of the form $H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m))$ for some irreducible component Y of $\mathcal{X}_{\mathbb{C}}^{(j)}$. If Y' is an irreducible component of $\mathcal{X}_{\mathbb{C}}^{(j+1)}$, then we have:

- Whenever $H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m))$ is a direct summand of $E_1^{p,q}$, and $H^i_{\text{ét}}(Y', \mathbb{Q}_l(-m))$ is a direct summand of $E_1^{p+1,q}$, then the corresponding direct summand of the boundary map $E_1^{p,q} \to E_1^{p+1,q}$ is (up to sign) the restriction
$$H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m))) \to H^i_{\text{ét}}(Y', \mathbb{Q}_l(-m)).$$
- Whenever $H^i_{\text{ét}}(Y', \mathbb{Q}_l(-m))$ is a direct summand of $E_1^{p,q}$, and $H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m+1))$ is a direct summand of $E_1^{p+1,q}$, then the corresponding direct summand of the boundary map $E_1^{p,q} \to E_1^{p+1,q}$ is (up to sign) the Gysin map
$$H^i_{\text{ét}}(Y', \mathbb{Q}_l(-m))) \to H^{i+2}_{\text{ét}}(Y, \mathbb{Q}_l(-m+1)).$$

The spectral sequence E is equipped with a monodromy operator $N : E_1^{p,q} \to E_1^{p+2,q-2}(-1)$, which gives the action of the monodromy operator N on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$. It is easily described: if $H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m))$ occurs as a direct summand of $E_1^{p,q}$, and $H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m+1))$ occurs as a direct summand of $E_1^{p+2,q-2}$, then the corresponding direct summand of N is the identity
$$H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m)) \to H^i_{\text{ét}}(Y, \mathbb{Q}_l(-m+1))(-1).$$
All other direct summands of N are the zero map.

The following difficult theorem (the "weight-monodromy conjecture") is due to Ito. It remains an open problem in mixed characteristic.

Theorem 5.4 ([I], Theorem 1.1)**.** *The spectral sequence E degenerates at E_2. Moreover, the map*
$$N^r : E_2^{-r,w+r} \to E_2^{r,w-r}$$
is an isomorphism for all r, w. In particular the filtration on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ induced by E coincides (up to a shift in degree) with the monodromy filtration; that is,
$$H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)_{-r} / H^w_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)_{-r-1} \cong E_2^{r,w-r}.$$

Remark 5.5. The filtration on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ induced by the spectral sequence E is called the *weight filtration*; we discuss it further in the appendix.

Example 5.6. Suppose that X_K is a curve of genus g. Then $\mathcal{X}_{\mathbb{C}}^{(0)}$ is the normalization of $\mathcal{X}_{\mathbb{C}}$; it is a disjoint union of smooth curves C_i of genus g_i. On the other hand, $\mathcal{X}_{\mathbb{C}}^{(1)}$ is the set of singular points of $\mathcal{X}_{\mathbb{C}}$; each such point lies on exactly two of the C_i. The corresponding weight spectral sequence is nonzero only for $-1 \leq r \leq 1$ and $0 \leq w + r \leq 2$; it looks like:
$$\begin{array}{ccccc} \bigoplus_{p \in \mathcal{X}_{\mathbb{C}}^{(1)}} \mathbb{Q}_l(-1) & \to & \bigoplus_i H^2_{\text{ét}}(C_i, \mathbb{Q}_l) & & 0 \\ 0 & & \bigoplus_i H^1_{\text{ét}}(C_i, \mathbb{Q}_l) & & 0 \\ 0 & & \bigoplus_i H^0_{\text{ét}}(C_i, \mathbb{Q}_l) & \to & \bigoplus_{p \in \mathcal{X}_{\mathbb{C}}^{(1)}} \mathbb{Q}_l \end{array}$$
The sequence clearly degenerates at E_2. The monodromy operator N is nonzero only from $E_1^{-1,2}$ to $E_1^{1,0}(-1)$; it is simply the identity map on
$$\bigoplus_{p \in \mathcal{X}_{\mathbb{C}}^{(1)}} \mathbb{Q}_l(-1).$$

We thus find that the middle quotient of the monodromy filtration on $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is isomorphic to the direct sum of $H^1_{\text{ét}}(C_i, \mathbb{Q}_l)$, whereas the top and bottom quotients are isomorphic to $H_1(\Gamma, \mathbb{Q}_l)$, (resp. $H^1(\Gamma, \mathbb{Q}_l)$) where Γ is the dual graph of $\mathcal{X}_{\mathbb{C}}$. We thus recover the usual picture of the stable reduction of a family of curves.

We now return to the setting of tropical geometry. Let X be schön. By Proposition 3.11 there is a polyhedral complex Σ, with support equal to $\mathrm{Trop}(X)$ and corresponding toric degeneration $\mathbb{P}$, such that the pair $(X, \mathbb{P})$ is tropical, the corresponding compactification $\overline{X}$ of X is smooth with normal crossings boundary, and the special fiber of the corresponding tropical degeneration $\mathcal{X}$ is a divisor with normal crossings.

The polyhedral complex $\Gamma_{(X,\mathbb{P})}$ encodes the combinatorics of the special fiber $\mathcal{X}_{\mathbb{C}}$. In particular $\mathcal{X}_{\mathbb{C}}$ is a union of smooth irreducible varieties connected varieties $\mathcal{X}_v$, where v runs over the vertices of $\Gamma_{(X,\mathbb{P})}$. The varieties $\mathcal{X}_{v_1}, \dots, \mathcal{X}_{v_r}$ meet if and only if $v_1, \dots, v_r$ are the vertices of a polyhedron in $\Gamma_{(X,\mathbb{P})}$. [Note that since $\mathcal{X}_{\mathbb{C}}$ is a normal crossings divisor, if $Y_0, \dots, Y_r$ intersect in codimension r then they are the only irreducible components of $\mathcal{X}_{\mathbb{C}}$ containing their intersection.]

We have a natural map $\Gamma_{(X,\mathbb{P})} \to \Sigma$. Since Σ is a triangulation of $\mathrm{Trop}(X)$, and $\Gamma_{(X,\mathbb{P})}$ is a triangulation of Γ_X, this induces a natural map

$$H^r(\mathrm{Trop}\, X, \mathbb{Q}_l) \to H^r(\Gamma_X, \mathbb{Q}_l).$$

By the proof of Lemma 4.5, this map is an isomorphism if $\mathcal{X}_P$ is connected for every polyhedron P in Σ, or (equivalently) if $\mathrm{in}_w X$ is connected for every w in $\mathrm{Trop}(X)$.

Theorem 5.7. *There is a natural isomorphism*

$$H^r(\Gamma_X, \mathbb{Q}_l) \cong H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r},$$

and hence a natural map

$$H^r(\mathrm{Trop}(X), \mathbb{Q}_l) \to H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}.$$

This map is an isomorphism if $\mathcal{X}_P$ is connected for every polyhedron P in Σ.

Proof. The bottom nonzero row of the E_1 term of the Rapoport-Zink spectral sequence (i.e., the $w = -r$ row) is the complex:

$$H^0_{\text{ét}}(\mathcal{X}^{(0)}_{\mathbb{C}}, \mathbb{Q}_l) \to H^0_{\text{ét}}(\mathcal{X}^{(1)}_{\mathbb{C}}, \mathbb{Q}_l) \to H^0_{\text{ét}}(\mathcal{X}^{(2)}_{\mathbb{C}}, \mathbb{Q}_l) \to \dots$$

in which the horizontal maps are restriction maps. This is simply the coboundary complex of $\Gamma_{(X,\mathbb{P})}$. We thus have a natural isomorphism

$$E_2^{r,0} \cong H^r(\Gamma_X, \mathbb{Q}_l).$$

By the weight-monodromy theorem, this yields an isomorphism

$$H^r(\Gamma_X, \mathbb{Q}_l) \cong H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}.$$

□

The following result, due to Speyer ([Sp3], Theorem 10.8) for curves, follows immediately:

Corollary 5.8. *Let $b_r(\Gamma_X)$ and $b_r(X)$ denote the rth Betti numbers of Γ_X and X, respectively. Then we have:*

$$b_r(\Gamma_X) \leq \frac{1}{r+1} b_r(X).$$

Proof. Theorem 5.7 shows that $b_r(\Gamma_X)$ is the dimension of $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$. The dimension of this piece of the monodromy filtration counts the number of Jordan blocks of size $r+1$ in a Jordan normal form for N acting on $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)$. In particular the dimension of the latter is at least $r+1$ times the dimension of the former. □

Remark 5.9. Theorem 5.7 shows in particular that the space $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$, which *a priori* depends on $\overline{X}$ and thus a choice of Σ, in fact depends only on X and is *independent* of Σ. One can see this directly on the level of cohomology. In particular, we will show the space $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$ can be interpreted as the "weight zero" part of $H^r_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$, and hence depends only on X. We refer the reader to the appendix for details. In particular, we prove the following result:

Proposition 5.10. *Let X be a smooth variety over K, and let $\overline{X}$ be a smooth compactification of X such that $\overline{X} \setminus X$ is a divisor with normal crossings. Then $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$ depends only on X and not on $\overline{X}$.*

The above results allow us to translate results about the cohomology of complete intersections in toric varieties into results about their tropicalizations. For instance:

Corollary 5.11. *Let X be a schön subvariety of T, and $\mathbb{P}_K$ a toric variety of T such that:*

1. *the closure Z of X in $\mathbb{P}_K$ is a smooth complete intersection, and*
2. *the boundary $Z \setminus X$ is a divisor with normal crossings.*

Then $H^r(\Gamma_X, \mathbb{Q}_l) = 0$ for $1 \le r < \dim X$.

Proof. By Proposition 3.11 and Theorem 5.7 there is a tropical pair $(X, \mathbb{P}')$, with corresponding compactification $\overline{X}$ of X, such that $H^r(\Gamma_X, \mathbb{Q}_l)$ is isomorphic to $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$. By Proposition 5.10 the latter is isomorphic to $H^r_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_l)_{-r}$.

Since Z is a complete intersection in $\mathbb{P}_K$, the Lefschetz hyperplane theorem shows that for $r < \dim X$, the restriction map

$$H^r_{\text{ét}}(\mathbb{P}_{\overline{K}}, \mathbb{Q}_l) \to H^r_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_l)$$

is an isomorphism. But $\mathbb{P}_K$ is a smooth toric variety, so the Galois action on its cohomology is trivial, and hence there is no nontrivial monodromy. □

Remark 5.12. We could avoid the use of Proposition 5.10 in the above result if we know that under the hypotheses 1) and 2) on $\mathbb{P}_K$, we could extend $\mathbb{P}_K$ to a toric degeneration $\mathbb{P}$ over $\mathcal{O}$ such that $(X, \mathbb{P})$ was a normal crossings pair. It is not immediately clear that this can always be done, however.

Under more restrictive hypotheses on X, we can turn the above result into a result about the cohomology of $\text{Trop}(X)$. This will be the main goal of the next section.

6. Complete Intersections

In the constant coefficient case, a (Zariski) general hyperplane section of a schön variety is schön. Unfortunately this is no longer true in the nonconstant coefficient case. To get analogous results in this case, it turns out one needs to work with the *rigid analytic* topology rather than the Zariski topology. (For the basics of the theory of rigid analytic spaces we refer the reader to [EKL] or [Sch]; we use very little here.)

To see the connection to rigid geometry, we first observe:

Lemma 6.1. *Let $\mathbb{P}$ be a toric degeneration, proper over $\mathcal{O}$, and let X be a subvariety of the open torus T in $\mathbb{P} \times_{\mathcal{O}} \operatorname{Spec} K$. Suppose that for all polyhedra P in the polyhedral complex Σ corresponding to P, the closure $\mathcal{X}$ of X in $\mathbb{P}$ intersects $\mathbb{P}_P$ transversely. Then X is schön, and $(X, \mathbb{P}')$ is a normal crossings pair, where $\mathbb{P}'$ is the open subset of $\mathbb{P}$ obtained by deleting all torus orbits that do not meet $\mathcal{X}$.*

Conversely, if X is schön and there exists a toric open subset $\mathbb{P}'$ of $\mathbb{P}$ such that $(X, \mathbb{P}')$ is a normal crossings pair, then the closure of X intersects $\mathbb{P}_P$ transversely for all polyhedra P in Σ.

Proof. Consider the multiplication map

$$m : \mathcal{T} \times \mathcal{X} \to \mathbb{P}'.$$

If y is a point in $\mathbb{P}'$ in the torus orbit corresponding to a polyhedron P in the subcomplex Σ' of Σ corresponding to $\mathbb{P}'$, then the fiber over y is isomorphic to the product $\mathcal{X} \cap \mathbb{P}'_P$ with a torus. By assumption, this is smooth, so m has smooth fibers. The argument of [H], Lemma 2.6 then shows that m is smooth. It follows that X is schön and $(X, \mathbb{P}')$ is a normal crossings pair. The converse is clear. □

Note that the lemma implies that $\operatorname{Trop}(X)$ will be equal to the support of Σ' for all such X. One can therefore use this result to study the space of schön subvarieties of a toric variety over K with a given tropicalization. We will not pursue this here, beyond a few straightforward observations.

Suppose $\mathbb{P}$ is projective. Fix an X as in the lemma, and let $\operatorname{Hilb}(\mathbb{P})$ be the Hilbert scheme over $\mathcal{O}$ parameterizing subschemes of $\mathbb{P}$ with the same Hilbert polynomial as the closure of X. Complex points of $\operatorname{Hilb}(\mathbb{P})$ correspond to subschemes of the special fiber of $\mathbb{P}$; those that meet each $\mathbb{P}_P$ transversely form an open subset U_0 of $\operatorname{Hilb}(\mathbb{P}) \times_{\mathcal{O}} \operatorname{Spec} \mathbb{C}$.

Now if y is a point of $\operatorname{Hilb}(\mathbb{P})(\overline{K})$, then y corresponds to a subscheme X_y of the general fiber of $\mathbb{P}$ over a finite extension of K. Then $X_y \cap T$ will satisfy the hypotheses of Lemma 6.1 if, and only if, y specializes to a point y_0 on the special fiber of $\operatorname{Hilb}(\mathbb{P})$ that lies in U_0. The set of points y that specialize to U_0 forms a "neighborhood of U_0" in the rigid analytic topology on $\operatorname{Hilb}(\mathbb{P})$. More precisely, let $\operatorname{Hilb}(\mathbb{P})^{\text{rig}}$ denote the rigid analytic space associated to the general fiber of $\operatorname{Hilb}(\mathbb{P})$; then $\operatorname{Hilb}(\mathbb{P})^{\text{rig}}$ is equipped with a "reduction mod t" map

$$\operatorname{Hilb}(\mathbb{P})^{\text{rig}} \to \operatorname{Hilb}(\mathbb{P}) \times_{\mathcal{O}} \operatorname{Spec} \mathbb{C}.$$

The preimage of U_0 under this map is an admissible open subset U^{rig} of $\operatorname{Hilb}(\mathbb{P})^{\text{rig}}$, and those $y \in \operatorname{Hilb}(\mathbb{P})(\overline{K})$ such that $X_y \cap T$ satisfies the hypotheses of Lemma 6.1 are precisely the $\overline{K}$-points of U^{rig}.

If we restrict our attention to complete intersections, we can say more than this. In particular fix a projective toric degeneration $\mathbb{P}$ over $\mathcal{O}$, and ample line bundles $L_1, \dots, L_s$ on $\mathbb{P}$. The space $\mathcal{H}$ parameterizing tuples $(D_1, \dots, D_s)$ such that for each i, D_i is an effective divisor in the linear system corresponding to L_i, and all the D_i's intersect transversely, is an open subset of a product of projective spaces over $\mathcal{O}$.

By Bertini's theorem, the set of points in $\mathcal{H}(\mathbb{C})$ that correspond to divisors $(D_1, \dots, D_s)$ in $\mathbb{P} \times_{\mathcal{O}} \operatorname{Spec} \mathbb{C}$ such that $D_1 \cap \dots \cap D_s$ intersects each stratum $\mathbb{P}_P$ of

$\mathbb{P}$ transversely is an open dense subset U_0 of the special fiber of $\mathcal{H}$. The preimage of U_0 under the reduction map

$$\mathcal{H}^{\mathrm{rig}} \to \mathcal{H} \times_{\mathcal{O}} \operatorname{Spec} \mathbb{C}$$

is a (necessarily nonempty) admissible open subset U^{rig} of $\mathcal{H}^{\mathrm{rig}}$; the points of U^{rig} correspond precisely to those complete intersections $D_1 \cap \cdots \cap D_s$ whose intersection with T satisfies the conditions of Lemma 6.1.

Moreover, if $(D_1, \ldots, D_s)$ is a K-point of U^{rig}, and X is the corresponding complete intersection $D_1 \cap \cdots \cap D_s \cap T$ in T, then for each polyhedron P in Σ, $\mathcal{X}_P = D_1 \cap \cdots \cap D_s \cap \mathbb{P}_P$ is the intersection of ample divisors in the smooth toric variety $\mathbb{P}_P$, and is therefore either zero-dimensional or connected.

Lemma 4.5 and Theorem 5.7 now have immediate implications for the cohomology of $\operatorname{Trop}(X)$:

Theorem 6.2. *Let $(D_1, \ldots, D_s)$ be a K-point of U^{rig}, and set*

$$X = D_1 \cap \cdots \cap D_s \cap T.$$

Then $H^r(\operatorname{Trop}(X), \mathbb{Q}_l)$ vanishes for $1 \leq r < \dim X$, and the natural map:

$$H^r(\operatorname{Trop}(X), \mathbb{Q}_l) \to H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$$

is injective for $r = \dim X$.

Proof. The above discussion shows that X is schön and $(X, \mathbb{P})$ is a normal crossings pair. We thus apply Theorem 5.7 and Lemma 4.5 to see that the map

$$H^r(\operatorname{Trop}(X), \mathbb{Q}_l) \to H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$$

is an isomorphism for $0 \leq r < \dim X$ and injective for $r = \dim X$. On the other hand, $\overline{X}$ is a complete intersection in the general fiber of the smooth toric variety $\mathbb{P} \times_{\mathcal{O}} \operatorname{Spec} K$. The result thus follows from Corollary 5.11. $\square$

Remark 6.3. In the proof of the above result we know that $(X, \mathbb{P})$ is a normal crossings pair. Thus (c.f. Remark 5.12) the above result does not depend on Proposition 5.10 even though it invokes Corollary 5.11.

7. Appendix: weight filtrations over K

In this section we prove Proposition 5.10, using Deligne's theory of weights. Recall (c.f. [D], 1.2) that if F is a finite field of order q, a continuous l-adic representation ρ of $\operatorname{Gal}(F^{\mathrm{sep}}/F)$ has weight r if all the eigenvalues of the geometric Frobenius of F are algebraic integers α, all of whose Galois conjugates have complex absolute value equal to $q^{r/2}$. If A is a finitely generated $\mathbb{Z}$-algebra, an étale sheaf $\mathcal{F}$ on $\operatorname{Spec} A$ has weight r if for each closed point s of $\operatorname{Spec} A$, the stalk $\mathcal{F}_s$ has weight r when considered as a $\operatorname{Gal}(k(s)^{\mathrm{sep}}/k(s))$-module.

Following Ito ([I], 2.2), we extend this definition to the case where F is a purely inseparable extension of a finitely generated extension of $\mathbb{F}_p$ or $\mathbb{Q}$. For such F, one can find a finitely generated $\mathbb{Z}$-subalgebra A of F such that F is a purely inseparable extension of the field of fractions of A.

In this setting, a representation ρ of $\operatorname{Gal}(F^{\mathrm{sep}}/F)$ has weight r if there is an open subset U of $\operatorname{Spec} A$, and a smooth $\mathcal{F}$ on U, such that ρ arises from $\mathcal{F}$ by pullback to $\operatorname{Spec} F$. The Weil conjectures imply that for any proper smooth variety X over F, $H^r_{\text{ét}}(X_{F^{\mathrm{sep}}}, \mathbb{Q}_l)$ has weight r.

Let G be the absolute Galois group of the field $F((t))$. Then G admits a surjection $G \to \mathrm{Gal}(F^{\mathrm{sep}}/F)$, whose kernel is the inertia group I of $F((t))$. If M is a G-module on which I acts through a finite quotient, there is a finite index subgroup H in G such that $H \cap I$ acts trivially on M. Thus $\mathrm{Gal}(F^{\mathrm{sep}}/F')$ acts on M for some finite extension F' of F. We say M has weight r if it has weight r as a $\mathrm{Gal}(F^{\mathrm{sep}}/F')$-module. Note that this is independent of F'.

We say a G-module M is *mixed* if M admits an increasing G-stable filtration

$$\cdots \subset W_r M \subset W_{r+1} M \subset \dots$$

such that $W_r M / W_{r-1} M$ has weight r for all r. (Such a filtration, if it exists, will be unique.) We say M is mixed of weights between r and r' if M is mixed and the quotients $W_i M / W_{i-1} M$ are nonzero only when $r \leq i \leq r'$. The main result here is:

Theorem 7.1. *(c.f. [I], 2.3) Let X be a smooth, proper n-dimensional variety over $F((t))$, and let $L = F((t))^{\mathrm{sep}}$ be the separable closure of $F((t))$. Then $H^r_{\text{ét}}(X_L, \mathbb{Q}_l)$ is mixed of weights between* $\max(0, 2r-2n)$ *and* $\min(2n, 2r)$.

Proof. If X has strictly semistable reduction, i.e., X is isomorphic to the general fiber of a scheme $\mathcal{X}$ that is proper over $F[[t]]$, and whose special fiber is a divisor with normal crossings, then this follows from the weight spectral sequence. More precisely, the weight filtration *is* the filtration induced on $H^r_{\text{ét}}(X_L, \mathbb{Q}_l)$ by this spectral sequence, as every term in the jth row of the weight spectral sequence is a $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-module of weight j. The general case follows by de Jong's theory of alterations [dJ]. □

Proposition 7.2. *Let X be a smooth n-dimensional variety over $F((t))$, and $\overline{X}$ a compactification of X such that $\overline{X} - X$ is a divisor with normal crossings. Then for $r \leq n$, $H^r_{\text{ét}}(X_L, \mathbb{Q}_l)$ is mixed of weights between* 0 *and* $2r$*, and the natural map*

$$W_0 H^r_{\text{ét}}(\overline{X}_L, \mathbb{Q}_l) \to W_0 H^r_{\text{ét}}(X_L, \mathbb{Q}_l)$$

is an isomorphism.

Proof. Let D be the divisor $\overline{X} \setminus X$, and let $\overline{D}_1, \dots, \overline{D}_r$ be its irreducible components. Let X_i be the open subset $X \setminus \{\overline{D}_1 \cup \dots \cup \overline{D}_i\}$. We proceed by induction on i; the case $i = 0$ is clear.

Suppose the proposition is true for i. Define

$$D_i = \overline{D}_{i+1} \setminus \{\overline{D}_1 \cup \dots \cup \overline{D}_i\},$$

so that $X_i \setminus X_{i+1} = D_i$. By the inductive hypothesis the spaces $H^r_{\text{ét}}((X_i)_L, \mathbb{Q}_l)$ and $H^r_{\text{ét}}((D_i)_L, \mathbb{Q}_l)$ are mixed of weights between 0 and $2r$ for $r \leq n$. We have a Gysin sequence:

$$\begin{array}{llllll} H^{r-2}_{\text{ét}}((D_i)_L, \mathbb{Q}_l(-1)) & \to & H^r_{\text{ét}}((X_i)_L, \mathbb{Q}_l) & \to & H^r_{\text{ét}}((X_{i+1})_L, \mathbb{Q}_l) & \to \\ H^{r-1}_{\text{ét}}((D_i)_L, \mathbb{Q}_l(-1)) & \to & \dots, & & & \end{array}$$

and the first and last terms are mixed of weights between 2 and $2r$. It follows that $H^r_{\text{ét}}((X_{i+1})_L, \mathbb{Q}_l)$ is mixed of weights between 0 and $2r$ as required. We also obtain an isomorphism

$$W_0 H^r_{\text{ét}}((X_i)_L, \mathbb{Q}_l) \cong W_0 H^r_{\text{ét}}((X_{i+1})_L, \mathbb{Q}_l)$$

and hence by induction the desired isomorphism

$$W_0 H^r_{\text{ét}}(\overline{X}_L, \mathbb{Q}_l) \cong W_0 H^r_{\text{ét}}(X_L, \mathbb{Q}_l).$$

□

Proposition 5.10 now follows easily. If we have X and $\overline{X}$ over K satisfying the hypotheses of this proposition, then (as both X and $\overline{X}$ are of finite type over K), we can find a subfield F of $\mathbb{C}$, finitely generated over $\mathbb{Q}$, such that X, $\overline{X}$, and each irreducible component of $\overline{X} \setminus X$ are all defined over $F((t))$. By the weight-monodromy conjecture (Theorem 5.4), we find that $H^r_{\text{ét}}(\overline{X}_{\overline{K}}, \mathbb{Q}_l)_{-r}$ coincides with $W_0 H^r_{\text{ét}}(\overline{X}_L, \mathbb{Q}_l)$. By the previous proposition, this depends only on X and not on $\overline{X}$, as required.

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Department of Mathematics, University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257