ON l-ADIC FAMILIES OF CUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract. We compute the universal deformations of cuspidal representations π of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_l$, where F is a local field of residue characteristic p and l is an odd prime different from p. When π is supercuspidal there is an irreducible, two dimensional representation ρ of G_F that corresponds to π via the mod l local Langlands correspondence of [Vi2]; we show that there is a natural isomorphism between the universal deformation rings of ρ and π that induces the usual (suitably normalized) local Langlands correspondence on characteristic zero points. Our work establishes certain cases of a conjecture of Emerton [Em].

1. Introduction

A standard technique in the theory of automorphic forms is to study their variation in algebraic families, and to compare such families of forms to families of Galois representations. On the other hand, many questions about automorphic forms are much more naturally phrased in the language of automorphic representations. It is thus natural to try to study the variation of automorphic representations in families. Unfortunately, at present there is no good notion of automorphic representations over rings other than fields of characteristic zero, and hence no good notion of a family of automorphic representations.

If one instead works in a local setting, say with admissible representations of reductive groups over a local field F, the situation is much better. Indeed, Vignéras [Vi1] has extensively studied the representations of such groups over general coefficient rings. Moreover, in the case of GL_n she obtains a mod l local Langlands correspondence for $GL_n(F)$ [Vi2], when the characteristic of F is different from l, that is compatible in a certain sense with "reduction mod l" from the classical (characteristic zero) local Langlands correspondence.

Vignéras' results provide a framework in which one can study the deformation theory of mod l admissible representations of $\mathrm{GL}_n(F)$. It is then natural to ask if one can extend the mod l local Langlands correspondence to deformations; that is: if π is the admissible representation associated to a representation ρ of W_F by mod l local Langlands, then is there a natural bijection between deformations of ρ and those of π? Such a bijection would amount to a "local Langlands correspondence in families."

We address this question in the case when π is a supercuspidal representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_l$ (so $\rho : W_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible.) In this case π admits a universal deformation, which we compute via the theory of types. Using the explicit description of local Langlands due to Bushnell-Henniart [BH], we show that there

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is a natural isomorphism between the universal deformation ring of π and the universal deformation ring of ρ (see Theorem 6.1 for the precise statement.) This gives a bijection between deformations of ρ over a complete Noetherean local $W(\overline{\mathbb{F}}_l)$-algebra A and deformations of π over A, extending the ordinary local Langlands correspondence.

The question of constructing a local Langlands correspondence of GL_2 in algebraic families has also been considered in [Em]. Emerton considers families of two-dimensional representations ρ of G_F over reduced complete l-torsion free Noetherean local $W(k)$-algebras A, and shows that for such a representation there is at most one admissible $A[\mathrm{GL}_2(F)]$-module π such that:

- π is A-torsion free,
- at generic points η of $\operatorname{Spec} A$, the smooth dual $\pi_\eta^\vee$ of π_η corresponds to ρ_η under a certain "modified local Langlands correspondence", and
- at the closed point x of $\operatorname{Spec} A$, π_x is a quotient of the $\pi(\rho_x)^\vee$, where $\pi(\rho_x)$ is the representation attached to ρ_x by this modified local Langlands correspondence.

(We refer the reader to Theorems 1.2 and 1.3 of [Em] for details.) Emerton conjectures further that such a π always exists; our result establishes this in the case when ρ_x is absolutely irreducible. (More precisely, in this setting ρ is a deformation of ρ_x; Theorem 6.1 gives a corresponding deformation of $\pi(\rho_x)$, and the representation π conjectured by Emerton is the smooth dual of this deformation.)

The structure of the paper is as follows: we begin by studying the deformation theory of a representation of a finite group over a finite field k of characteristic l, and show that a very naive approach to constructing the universal deformation (that proceeds essentially by concatenating the characteristic zero lifts of this representation) actually produces a deformation that "agrees with the universal deformation up to l-torsion." We give a criterion in terms of character theory for this naive deformation to be universal. The next section applies this theory for mod l representations of $\mathrm{GL}_2(\mathbb{F}_q)$, where l is odd and prime to q.

In section 4 we turn to the deformation theory of cuspidal representations π of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_l$, where F is a local field with residue field $\mathbb{F}_q$. This proceeds by the theory of types; we show that deforming such a representation is equivalent to deforming its type. We then use this to attach to a supercuspidal π a character χ of either $F^\times$ or $E^\times$, where E is a quadratic extension of F, in such a way that deformations of π are naturally in bijection with deformations of χ. (See Theorem 4.9 for details.) We also compute the universal deformations of representations of π that are cuspidal but not supercuspidal; although this is not necessary for our main results it is of a piece conceptually with the ideas described here, and has applications to Emerton's conjecture that we will address in a future paper.

The deformation theory of two-dimensional irreducible representations of G_F is well-understood, and we summarize the necessary facts in section 5. Combining this with the explicit local Langlands for GL_2 of [BH] establishes Theorem 6.1.

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2. Deformation Theory

We begin with some general preliminaries on the deformation theory of representations of a finite group G. Let k be a finite field of characteristic l and let π be an absolutely irreducible k-representation of G. Attached to π we have the universal deformation ring R_π^{univ} of π; it parameterizes lifts of π to representations over finite length local $W(k)$-algebras with residue field k. The ring R_π^{univ} is a complete Noetherian local $W(k)$-algebra, with residue field k, and comes equipped with is a universal representation π^{univ} of G over R_π^{univ}, and an isomorphism:

$$\iota : \pi^{\mathrm{univ}} \otimes_{R_\pi^{\mathrm{univ}}} k \cong \pi.$$

We refer the reader to [Ma] for more details.

Under certain hypotheses that we explain further below, it is possible to compute the ring R_π^{univ} directly from the character table of G. We begin by constructing a particular deformation of π:

Let K be an algebraic closure of the field of fractions of $W(k)$, and let $\mathcal{O}$ be the ring of integers in K. We say a K-reprsentation $\tilde{\pi}$ of G lifts π if there is a G-stable $\mathcal{O}$-lattice $\tilde{\pi}_{\mathcal{O}}$ in $\tilde{\pi}$ such that $\tilde{\pi}_{\mathcal{O}} \otimes_{\mathcal{O}} \overline{k}$ is isomorphic to $\pi \otimes_k \overline{k}$. One sees easily that the set S of isomorphism classes of such $\tilde{\pi}$ is in bijection with the set of K-points of $\operatorname{Spec} R_\pi^{\mathrm{univ}}$.

For $\tilde{\pi}$ in S, let $K_{\tilde{\pi}}$ be the minimal field of definition of $\tilde{\pi}$. Define:

$$R_K := \prod_{\tilde{\pi}\in S} K_{\tilde{\pi}}$$

$$\pi_K := \prod_{\tilde{\pi}\in S} \tilde{\pi}.$$

In the definition of π_K we consider each $\tilde{\pi}$ to be a representation over $K_{\tilde{\pi}}$, so that π_K is an R_K-representation of G.

Each $\tilde{\pi}$ arises by base change from π^{univ} via a canonical map $R_\pi^{\mathrm{univ}} \to K_{\tilde{\pi}}$; together these give a canonical map $R_\pi^{\mathrm{univ}} \to R_K$. Let R be its image, and let π_R be the representation $\pi^{\mathrm{univ}} \otimes_{R_\pi^{\mathrm{univ}}} R$. We have $\pi_K \cong \pi_R \otimes_R R_K$.

On the other hand, let R_0 be the $W(k)$-subalgebra of R generated by the traces of elements σ of G on π_R. By [Ca], Theorem 2, π_R is defined over R_0. Thus the surjection

$$R_\pi^{\mathrm{univ}} \to R$$

factors through R_0. We must therefore have $R_0 = R$.

One thus has an explicit description of R in terms of the character table of G: for each σ in G, let x_σ be the element of R_K defined by:

$$(x_\sigma)_{\tilde{\pi}} = \operatorname{tr} \sigma|_{\tilde{\pi}}.$$

Then x_σ is the trace of σ on π_R, and hence R is simply the $W(k)$-algebra generated by the x_σ.

Remark 2.1. The closed immersion $\operatorname{Spec} R \to \operatorname{Spec} R_\pi^{\mathrm{univ}}$ induced by the above surjection is a bijection on K-points. One can show that the nilpotents of R_π^{univ} are supported on the special fiber of $\operatorname{Spec} R_\pi^{\mathrm{univ}}$; it follows that after inverting l the map $R_\pi^{\mathrm{univ}} \to R$ is an isomorphism. In other words R is simply the quotient of R_π^{univ} by its l-power torsion. We will not make use of this, however.

We will give criteria for the natural map $R_\pi^{\mathrm{univ}} \to R$ to be an isomorphism. For each $\tilde{\pi}$, let $e_{\tilde{\pi}}$ be the idempotent in $K[G]$ corresponding to $\tilde{\pi}$. Also note that for any finite length local $W(k)$-algebra A with residue field k, and any deformation π_A of π, the center $Z(W(k)[G])$ of $W(k)[G]$ acts on π_A by scalars because of Schur's lemma. We thus obtain a natural map $Z(W(k)[G]) \to A$.

Theorem 2.2. *Suppose that the following conditions both hold:*

(1) *The sum* $e_\pi := \sum_{\tilde{\pi}\in S} e_{\tilde{\pi}}$ *lies in* $\mathcal{O}[G]$*, and*
(2) *For any finite length* $W(k)$*-algebra* A *with residue field* k*, and any deformation* π_A *of* π*, the* $W(k)$*-subalgebra* A_0 *of* A *generated by the traces* $\operatorname{tr}\sigma|_{\pi_A}$ *for* $\sigma \in G$ *is contained in the image of the natural map* $Z(W(k)[G]) \to A$*.*

Then the natural map $R_\pi^{\mathrm{univ}} \to R$ *is an isomorphism.*

Proof. The second condition implies that the natural map $Z(W(k)[G]) \to R$ is surjective; i.e. R is generated by the endomorphisms of π_R arising from class functions in $W(k)[G]$. Let $a_1, \dots, a_r$ be such class functions, and let $P \in W(k)[x_1, \dots, x_r]$ be a polynomial such that $P(a_1, \dots, a_r)$ annihilates π_R. We will show that $P(a_1, \dots, a_r)$ annihilates any A-deformation π_A of π, over any A.

It suffices to show this after making a base change from $W(k)$ to $\mathcal{O}$. Note that e_π is the identity on $\tilde{\pi}$ for any $\tilde{\pi}$, and hence e_π is the identity on π. By Nakayama's lemma it follows that $e_\pi \pi_A = \pi_A$. It thus suffices to show that $e_\pi P(a_1, \dots, a_r)$ is zero in $\mathcal{O}[G]$, or even in $K[G]$. As a $K[G]$ module, $e_\pi K[G]$ is a direct sum of copies of various $\tilde{\pi} \in S$. Each such $\tilde{\pi}$ arises by base change from π_R and hence is annihilated by $P(a_1, \dots, a_r)$, as required.

It follows that any such $P(a_1, \dots, a_r)$ annihilates π^{univ}. We thus obtain a well defined map

$$R \to R_\pi^{\mathrm{univ}}$$

that, given an element of R, lifts it to an element of $Z(W(k)[G])$ and sends it to the corresponding element of R_π^{univ}. It suffices to show this map is surjective. But by [Ca], Theorem 2, R_π^{univ} is generated by the traces of elements of G on π^{univ}, and hence (by condition 2) by the image of $Z(W(k)[G])$ in R_π^{univ}. Since this is in the image of R the result follows. $\square$

3. Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

We now apply the results of the previous section to the group $G = \mathrm{GL}_2(\mathbb{F}_q)$, for $q = p^r$. We first recall some basic facts about the representation theory of G (see for instance [BH], chapter 6). Let B be the standard Borel subgroup of G, N its unipotent radical, and T the standard torus in B. Finally, let Z denote the center of G.

Let K be an algebraic closure of $\mathbb{Q}_l$, and consider two characters $\tilde{\phi}_1, \tilde{\phi}_2 : \mathbb{F}_q^\times \to K^\times$. We can view the pair $\tilde{\phi}_1, \tilde{\phi}_2$ as a character of T in the obvious way, and extend it to a character $\tilde{\phi}$ of B trivial on N. Then the representation $\mathrm{Ind}_B^G \tilde{\phi}$ is irreducible of dimension $q+1$ if $\tilde{\phi}_1 \neq \tilde{\phi}_2$. If $\tilde{\phi}_1 = \tilde{\phi}_2$, then $\mathrm{Ind}_B^G \tilde{\phi}$ decomposes as a direct sum of the character $\tilde{\phi}_1 \circ \det$ with an irreducible "Steinberg representation" of dimension q.

Those irreducible K-representations of G that do not occur as subquotients of some $\mathrm{Ind}_B^G \tilde{\phi}$ are called supercuspidal, and can be constructed as follows: Let $\tilde{\psi} : N \to K^\times$ be a nontrivial character. Fix a subgroup E of G isomorphic to

$\mathbb{F}_{q^2}^\times$, and a character $\tilde{\theta} : E \to K^\times$, such that $\tilde{\theta}^q \neq \tilde{\theta}$. Consider the character $\tilde{\theta}_{\tilde{\psi}}$ of ZN defined by $\tilde{\theta}_{\tilde{\psi}}(zu) = \tilde{\theta}(z)\tilde{\psi}(u)$ for $z \in Z$, $u \in N$. Then one can compute the character of the virtual representation

$$\pi_{\tilde{\theta}} = \mathrm{Ind}_{ZN}^G \tilde{\theta}_{\tilde{\psi}} - \mathrm{Ind}_E^G \tilde{\theta}.$$

One has:

$$\begin{array}{rcll} \mathrm{tr}\,\pi_{\tilde{\theta}}(z) & = & (q-1)\tilde{\theta}(z) & \text{for } z \in Z \\ \mathrm{tr}\,\pi_{\tilde{\theta}}(zu) & = & -\tilde{\theta}(z) & \text{for } z \in Z,\ u \in N \setminus \{\mathrm{Id}\} \\ \mathrm{tr}\,\pi_{\tilde{\theta}}(t) & = & 0 & \text{for } t \in T \setminus Z \\ \mathrm{tr}\,\pi_{\tilde{\theta}}(y) & = & -\tilde{\theta}(y) - \tilde{\theta}^q(y) & \text{for } y \in E \setminus Z. \end{array}$$

This exhausts the conjugacy classes of G, and one easily verifies that $\pi_{\tilde{\theta}}$ is an irreducible representation of G of dimension $q-1$. Moreover, $\pi_{\tilde{\theta}}$ is independent of $\tilde{\psi}$. We have $\pi_{\tilde{\theta}} = \pi_{\tilde{\theta}^q}$, and $\pi_{\tilde{\theta}} \neq \pi_{\tilde{\theta}'}$ unless $\tilde{\theta}' \in \{\tilde{\theta}, \tilde{\theta}^q\}$. A simple count then shows that every supercuspidal K-representation of G is of this form.

We also observe that the representation $\mathrm{Ind}_N^G \tilde{\psi}$ contains every irreducible representation of G except for those of the form $(\tilde{\phi}_1 \circ \det)$, with multiplicity one (c.f. [BH], p. 48.)

From this information it is straightforward to compute the irreducible representations of G over $\overline{\mathbb{F}}_l$, for l odd and prime to p. (All that is necessary is modular character theory at the level of [Se], chapter 18.) This is well-known, but we summarize the results below for ease of reference:

Since the order of G is $(q^2-1)(q^2-q)$, if q is not congruent to ± 1 mod l the representation theory over $\overline{\mathbb{F}}_l$ is "the same" as in characteristic zero; reduction mod l takes the character of a representation in characteristic zero to the character of the corresponding representation mod l, and this correspondence is a bijection. There are thus two cases of interest to us: l an odd divisor of either $q-1$ or $q+1$.

First assume $q \equiv 1$ mod l. Given $\phi_1, \phi_2 : \mathbb{F}_q^\times \to \overline{\mathbb{F}}_l^\times$ we obtain a character ϕ of B as above; then $\mathrm{Ind}_B^G \phi$ is irreducible if $\phi_1 \neq \phi_2$, and splits as a direct sum of a character and a Steinberg representation otherwise.

The irreducible representations that do not arise in the above way can be described as follows: for each character $\theta : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{F}}_l^\times$, satisfying $\theta^q \neq \theta$, we can lift θ to a character $\tilde{\theta}$ with values in $K^\times$. It is easy to see by computing the modular character that the mod l reduction of $\pi_{\tilde{\theta}}$ is an irreducible $\overline{\mathbb{F}}_l$-representation π_θ of G, that depends only on θ and not on $\tilde{\theta}$. As in characteristic zero, we have $\pi_\theta = \pi_{\theta'}$ if, and only if, $\theta' \in \{\theta, \theta^q\}$. A counting argument shows that every irreducible $\overline{\mathbb{F}}_l^\times$-representation of G that does not arise from parabolic induction is of the form π_θ.

If $q \equiv -1$ mod l, the situation is slightly different. As in the previous case, given ϕ_1, ϕ_2 we have $\mathrm{Ind}_B^G \phi$ irreducible if $\phi_1 \neq \phi_2$. On the other hand, if $\phi_1 = \phi_2$, the Jordan-Holder constituents of $\mathrm{Ind}_B^G \phi$ consist of two copies of the character $(\phi_1 \circ \det)$ and an irreducible representation π_{ϕ_1} of G of dimension $q-1$. The representation π_{ϕ_1} can be described as follows: let $\theta : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{F}}_l^\times$ be defined by $\theta(x) = \phi_1(x^{q+1})$. Then $\theta^q = \theta$, but we can lift θ to a character $\tilde{\theta}$ with values in a field of characteristic zero. Moreover, since $q \equiv -1$ mod l, we can choose $\tilde{\theta}$ such that $\tilde{\theta}^q \neq \tilde{\theta}$. (This is easily seen to be possible only when q is congruent to -1 mod l.) Then the characteristic zero representation $\pi_{\tilde{\theta}}$ reduces mod l to π_{ϕ_1}. (One verifies that the mod l reduction of $\pi_{\tilde{\theta}}$ is irreducible and that its modular character agrees with that of π_{ϕ_1}.) Conversely, if $\theta : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{F}}_l^\times$ is a character satisfying $\theta^q = \theta$, then we have $\theta(x) = \phi_1(x^{q+1})$ for some character ϕ_1 of $\mathbb{F}_q^\times$; we define π_θ to be the representation π_{ϕ_1} constructed above. The representations π_θ are not supercuspidal, as they arise as Jordan-Holder constituents of a representation of the form $\mathrm{Ind}_B^G \phi$, but they are *cuspidal*; i.e., they do not arise as subrepresentations of a representation of the form $\mathrm{Ind}_B^G \phi$.

As in the $q \equiv 1 \bmod l$ case, the remaining representations of G over $\overline{\mathbb{F}}_l$ are parametrized by the characters $\theta : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{F}}_l^\times$ such that $\theta^q \neq \theta$. Given such a character we define π_θ to be the mod l reduction of $\pi_{\tilde{\theta}}$, where $\tilde{\theta}$ is any lift of θ to characteristic zero. As before, π_θ does not depend on the particular $\tilde{\theta}$ chosen, and is an irreducible supercuspidal representation of G over $\overline{\mathbb{F}}_l^\times$.

Now fix a particular q, l, and $\theta : \mathbb{F}_{q^2}^\times \to \overline{\mathbb{F}}_l^\times$. We require that θ admits a lift $\tilde{\theta}$ to characteristic zero such that $\tilde{\theta}^q \neq \tilde{\theta}$; in this case the representation π_θ is well-defined. Let k be a finite field of characteristic l over whice π_θ is defined. Our main goal is to apply the theory of the previous section to the deformation theory of π_θ.

Let S_θ denote the set of equivalence classes of characters $\tilde{\theta} : \mathbb{F}_{q^2}^\times \to K^\times$ such that $\tilde{\theta}^q \neq \tilde{\theta}$ and the mod l reduction of $\tilde{\theta}$ is θ. We consider two such characters $\tilde{\theta}, \tilde{\theta}'$ equivalent if $\tilde{\theta}^q = \tilde{\theta}'$, and use the notation $[\tilde{\theta}]$ for the equivalence class of $\tilde{\theta}$. (Note that this equivalence relation is trivial unless $\theta^q = \theta$ and l divides $q + 1$.) Then the correspondence $\tilde{\theta} \mapsto \pi_{\tilde{\theta}}$ defines a bijection between the equivalence classes in S_θ and the lifts of π_θ to representations over K.

Proposition 3.1. *The conditions of Theorem 2.2 apply to* π_θ.

Proof. For each $\tilde{\theta} \in S_\theta$, let $e_{\tilde{\theta}}$ be the idempotent in $K[G]$ corresponding to $\pi_{\tilde{\theta}}$. The idempotent e_{π_θ} appearing in condition 1) of Theorem 2.2 is given by:

$$e_{\pi_\theta} = \sum_{\tilde{\theta} \in S_\theta} e_{\tilde{\theta}}.$$

It follows directly from our calculation of the characters of the representations $\pi_{\tilde{\theta}}$ that e_{π_θ} lies in $\mathcal{O}[G]$, so condition 1) is satisfied.

We now turn to condition 2) of Theorem 2.2. Note that it suffices to verify this condition after base change from $W(k)$ to $W(k')$, for k' a finite extension of k. We may therefore assume we have a nontrivial character Ψ of N with values in $k^\times$; Ψ lifts uniquely to a character $\tilde{\Psi}$ with values in $W(k)^\times$. We consider Ψ and $\tilde{\Psi}$ as characters of ZN that are trivial on Z. We may also assume that all nth roots of unity are in k for $n|q^2 - 1$, and n prime to l. (In other words, that all characters $E \to \overline{\mathbb{F}_l}^\times$ take values in $k^\times$.) Let χ be the restriction of θ to the center Z of G, and extend it to a character of ZN trivial on N. Note that $\mathrm{Res}_B^G \pi_\theta = \mathrm{Ind}_{ZN}^B \chi\Psi$.

Let A be a finite length $W(k)$-algebra with residue field k, and let π_A be an A-deformation of π_θ. Let χ_A be the central character of π_A. By definition, Z acts on π_A by χ_A; since N has order prime to l and π_θ contains a subspace on which N acts by χ, π_A contains a N-stable direct summand on which N acts by $\tilde{\Psi}$. Thus π_A contains a ZN-subrepresentation isomorphic to $\chi_A\tilde{\Psi}$, so we obtain a nonzero map $\mathrm{Ind}_{ZN}^B \chi_A\tilde{\Psi} \to \mathrm{Res}_B^G \pi_A$. This map reduces modulo the maximal ideal

m_A of A to the corresponding map $\mathrm{Ind}_{ZN}^B \chi\Psi \to \pi_\theta$, which is an isomorphism of B-representations. Thus $\mathrm{Res}_B^G \pi_A$ is isomorphic to $\mathrm{Ind}_{ZN}^B \chi_A \tilde{\Psi}$.

One sees easily from this that for any $\sigma \in B$, the trace of σ on π_A is in the subalgebra of A generated by $\chi_A(Z)$, and hence in the image of the map $Z(W(k)[G]) \to A$. Since any σ in G is conjugate either to an element of B or an element of $E \setminus Z$, it remains to verify condition 2) for elements of $E \setminus Z$.

Suppose first that l does not divide $q-1$. Let σ be an element of $E \setminus Z$, and note that there are $q(q-1)$ elements conjugate to σ. The element: $\sum_{\sigma' \sim \sigma} \sigma'$ of $Z(W(k)[G])$ acts on π_A with trace $q(q-1)\,\mathrm{tr}(\sigma)$ on π_A. On the other hand, it acts on π_A by a scalar c so its trace is $(q-1)c$. Thus $\mathrm{tr}(\sigma)$ is $\frac{c}{q}$ which indeed lies in the image of $Z(W(k)[G])$ in A.

Now suppose that l does not divide $q+1$. In this case $\mathrm{Res}_E^G \pi_\theta$ is a sum of characters $\theta' : E \to k^\times$, where θ' agrees with χ on Z and $\theta' \notin \{\theta, \theta^q\}$. There are exactly $q-1$ such θ' and each occurs with multiplicity one. Moreover, each such θ' admits a unique lift to an $A^\times$-valued character θ'_A such that $\theta'_A(\sigma) = \chi_A(\sigma)$ for $\sigma \in Z$. Then $\mathrm{Res}_E^G \pi_A$ is the direct sum of these θ'_A.

It follows that for $\sigma \in E$, the trace of σ on χ_A is given by $-\theta_A(\sigma) - \theta_A^q(\sigma)$, where θ_A is the unique lift of θ that agrees with χ_A on Z. Since l does not divide $q+1$, we can write $\sigma = \tau\tau'$ where τ is in Z and τ' has order prime to l. Then $\theta_A(\tau')$ is just the Teichmuller lift of $\theta(\tau')$, and therefore lies in the image of $W(k)$ in A. On the other hand, $\theta_A(\tau)$ is equal to $\chi_A(\tau)$, and hence lies in the image of $Z(W(k)[G])$ in A. □

This gives us an explicit description of the universal deformation ring $R_{\pi_\theta}^{\mathrm{univ}}$ of π_θ. In particular, the maps

$$f_{\tilde{\theta}} : R_{\pi_\theta}^{\mathrm{univ}} \to K$$

arising from the representations $\pi_{\tilde{\theta}}$ for $\tilde{\theta} \in S_\theta$ identify $R_{\pi_\theta}^{\mathrm{univ}}$ with the W(k)-subalgebra of $\prod_{[\tilde{\theta}] \in S_\theta} K$ generated by the elements x_σ for $\sigma \in G$, where

$$(x_\sigma)_{[\tilde{\theta}]} = \mathrm{tr}\, \sigma|_{\pi_{\tilde{\theta}}}.$$

Explicitly, we have:

$$\begin{array}{rcll} (x_\sigma)_{[\tilde{\theta}]} & = & (q-1)\tilde{\theta}(\sigma) & \text{for } \sigma \in Z \\ (x_{\sigma\tau})_{[\tilde{\theta}]} & = & -\tilde{\theta}(\sigma) & \text{for } \sigma \in Z,\, \tau \in N \setminus \{\mathrm{Id}\} \\ (x_\sigma)_{[\tilde{\theta}]} & = & 0 & \text{for } \sigma \in T \setminus Z \\ (x_\sigma)_{[\tilde{\theta}]} & = & -\tilde{\theta}(\sigma) - \tilde{\theta}^q(\sigma) & \text{for } \sigma \in E \setminus Z. \end{array}$$

Suppose $\theta \neq \theta^q$. Then the equivalence relation on S_θ is trivial, so we can simply write $\tilde{\theta}$ in place of $[\tilde{\theta}]$ when discussing elements of S_θ. Define elements y_σ of $\prod_{\tilde{\theta} \in S_\theta} K$ by $(y_\sigma)_{\tilde{\theta}} = \tilde{\theta}(\sigma)$ for $\sigma \in E$. It is clear that $R_{\pi_\theta}^{\mathrm{univ}}$ is contained in the W(k)-subalgebra of $\prod_{\tilde{\theta} \in S_\theta} K$ generated by the y_σ. On the other hand, this subalgebra is simply R_θ^{univ}, the universal deformation ring of θ. We thus obtain a map

$$R_{\pi_\theta}^{\mathrm{univ}} \to R_\theta^{\mathrm{univ}}.$$

Theorem 3.2. *The map*

$$R_{\pi_\theta}^{\mathrm{univ}} \to R_\theta^{\mathrm{univ}}$$

is an isomorphism. In particular there is a natural bijection between A-deformations of π_θ and A-deformations of θ that induces the correspondence $\tilde{\theta} \mapsto \pi_{\tilde{\theta}}$ on lifts of θ to $K^\times$.

Proof. It suffices to show that each y_σ is in the $W(k)$-algebra generated by the x_σ. This is clear for $\sigma \in Z$; each such y_σ is $-x_{\sigma\tau}$ for a nontrivial τ in N.

As θ takes values in a field of characteristic l, it has order prime to l. Thus θ has a unique lift $\tilde{\theta}_l$ to with order prime to l; $\tilde{\theta}_l$ takes values in $W(k)^\times$. If we let E^l be the subgroup of E of order prime to l, then for any $\tilde{\theta}$ in S, $\tilde{\theta}_l^{-1}\tilde{\theta}$ is trivial on E^l. Note that since θ^{q-1} is a nontrivial character, so is $\tilde{\theta}_l^{q-1}$.

Then for $\sigma \in E \setminus Z$ we have:

$$\sum_{\tau \in E^l} \tilde{\theta}_l^{-1}(\tau)[\tilde{\theta}(\sigma\tau) + \tilde{\theta}^q(\sigma\tau)] = \sum_{\tau \in E^l} \tilde{\theta}(\sigma) + \tilde{\theta}_l^{q-1}(\tau)\tilde{\theta}^q(\sigma) = (\# E_l^\times)\tilde{\theta}(\sigma).$$

In terms of the y_σ and x_σ this becomes:

$$y_\sigma = \frac{1}{\# E^l} \sum_{\tau \in E_l^\times} \tilde{\theta}_l^{-1}(\tau) x_\sigma$$

for all $\sigma \in E \setminus Z$. As $\tilde{\theta}_l$ takes values in $W(k)^\times$ we are done. □

We now turn to the case where $\theta = \theta^q$. In this case q is congruent to -1 mod l, and the equivalence relation on the $\tilde{\theta}$ in S is nontrivial. For $\sigma \in E$, define an element y_σ of $\prod_{[\tilde{\theta}] \in S_\theta} K$ by

$$(y_\sigma)_{[\tilde{\theta}]} = \tilde{\theta}(\sigma) + \tilde{\theta}(\sigma)^q.$$

(This is clearly independent of the choice of $\tilde{\theta}$ representing $[\tilde{\theta}]$.) For $\sigma \in \mathbb{F}_q^\times$, y_σ is simply an element of $W(k)^\times$. In particular the elements y_σ lie in the $W(k)$-subalgebra of $\prod_{[\tilde{\theta}] \in S_\theta} K$ generated by the x_σ; that is, we can consider the y_σ as elements of $R_{\pi_\theta}^{\mathrm{univ}}$. Moreover, each x_σ has a simple expression in terms of the y_σ. Thus $R_{\pi_\theta}^{\mathrm{univ}}$ is the subalgebra of $\prod_{[\tilde{\theta}] \in S_\theta} K$ generated by the y_σ.

As in the $\theta \neq \theta^q$ case, let $\tilde{\theta}_l$ be the unique lift of θ of order prime to l. (Note that since $\theta = \theta^q$, $\tilde{\theta}_l = \tilde{\theta}_l^q$ and so is *not* in S_θ.) Then $\tilde{\theta}_l$ takes values in $W(k)^\times$, and any lift $\tilde{\theta}$ of θ agrees with $\tilde{\theta}_l$ on $E_l^\times$. Thus for any $\tau \in E_l^\times$, $y_{\sigma\tau}$ differs from y_σ by a scalar factor in $W(k)^\times$. In particular $R_{\pi_\theta}^{\mathrm{univ}}$ is generated by y_σ for σ of l-power order. In fact, one verifies easily that if σ generates the pro-l part of E, then the single element y_σ generates all of $R_{\pi_\theta}^{\mathrm{univ}}$. We will find it slightly more convenient to use the generator $y_\sigma - 2$ instead; the minimal polynomial of $y_\sigma - 2$ is the polynomial Q defined by

$$Q(t)^2 = \prod_{\zeta^{l^n}=1; \zeta \neq 1} (t - \zeta - \zeta^{-1} + 2),$$

where $n = \mathrm{ord}_l(q^2 - 1)$. We thus have:

Theorem 3.3. *Suppose q is congruent to -1 mod l, and $\theta^q = \theta$. Then for any choice of generator σ of the pro-l part of E there is an isomorphism*

$$W(k)[[t]]/Q(t) \to R_{\pi_\theta}^{\mathrm{univ}}$$

sending t to $y_\sigma - 2$. Under this isomorphism, the trace of σ on $\pi_\theta^{\mathrm{univ}}$ is $-t - 2$.

Thus, for any complete Noetherean local $W(k)$-algebra A, and any α in A with $Q(\alpha) = 0$, there is a unique A-deformation $\pi_{\theta,\alpha}$ of π_θ for which $\mathrm{tr}\, \pi_{\theta,\alpha}(\sigma) = -\alpha - 2$.

4. Cuspidal representations of $\mathrm{GL}_2(F)$

We now turn to the deformation theory of cuspidal representations. Fix a p-adic field F, with residue field $\mathbb{F}_q$, and a prime l different from p. Our main goal will be to compute universal deformation rings of irreducible cuspidal representations π of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_l$.

Definition 4.1. Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_l$, and let A be a finite length local $W(\mathbb{F}_l)$-algebra, with maximal ideal m_A. An A-deformation of π is a free A-module π_A with an action of $\mathrm{GL}_2(F)$, together with an isomorphism $\pi_A/m_A\pi_A \cong \pi$, such that π_A is smooth; i.e every element of π_A is fixed by a compact open subgroup of $\mathrm{GL}_2(F)$.

Note that if U is a compact open subgroup of $\mathrm{GL}_2(F)$, of order prime to l, then taking U-invariants is exact; it follows that π_A^U will be finitely generated for all U of this sort. Since such U are cofinal in the set of all compact open subgroups of $\mathrm{GL}_2(F)$, π_A will be admissible as a $\mathrm{GL}_2(F)$-module.

The key tool we will use to compute the deformation theory of representations π is the theory of types. In characteristic l this theory is due to Vignéras [Vi1]. What we need will follow most easily from Bushnell-Henniart's very explicit description of the types of cuspidal $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_2(F)$.

Fix a character $\tilde{\psi}$ of F of level one, with values in $\overline{QQ}_l^{\times}$. Then for any simple stratum $(\mathfrak{A}, n, \alpha)$ in $M_2(F)$, we have a character $\tilde{\psi}_\alpha$ of $U_{\mathfrak{A}}^{[\frac{n}{2}]+1}$ defined by

$$\tilde{\psi}_\alpha(x) = \tilde{\psi}(\mathrm{tr}(\alpha(x-1))).$$

(c.f. [BH], 12.5). Let ψ_α be the mod l reduction of $\tilde{\psi}_\alpha$. We then have:

Theorem 4.2 ([BH], 15.5)**.** *Let $\tilde{\pi}$ be an irreducible, cuspidal representation of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{Q}}_l$. Then $\tilde{\pi} = (\text{c-Ind}_J^{\mathrm{GL}_2(F)} \tilde{\Lambda}') \otimes (\chi \circ \det)$, where χ is a character of $F^\times$ and either:*

1. *$J = F^\times \mathrm{GL}_2(\mathcal{O}_F)$, and the restriction of $\tilde{\Lambda}'$ to $\mathrm{GL}_2(\mathcal{O}_F)$ is the inflation of an irreducible, cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$, or*
2. *there exists a stratum $(\mathfrak{A}, n, \alpha)$ such that n is odd, $J = E^\times U_{\mathfrak{A}}^{\frac{n+1}{2}}$, and $\tilde{\Lambda}'$ is an irreducible representation of J whose restriction of to $U_{\mathfrak{A}}^{\frac{n+1}{2}}$ is the character $\tilde{\psi}_\alpha$, or*
3. *there exists a stratum $(\mathfrak{A}, n, \alpha)$ such that n is even, $J = E^\times U_{\mathfrak{A}}^{\frac{n}{2}}$, and $\tilde{\Lambda}'$ is an irreducible representation of J whose restriction to $U_{\mathfrak{A}}^{\frac{n}{2}+1}$ is a multiple of $\tilde{\psi}_\alpha$.*

Here E is the field $F[\alpha]$; it is a quadratic extension of F.

Results of Vignéras ([Vi2], Theorems 1.1 and 1.2) allow us to turn the above description into a characterization of supercuspidal representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}}_l$. In particular, every such representation π arises by reduction mod l from a supercuspidal representation $\tilde{\pi}$ over $\overline{\mathbb{Q}}_l$ whose central character is integral. More precisely there is a unique homothety class of lattices in such a $\tilde{\pi}$, and the mod l reduction of any such lattice is isomorphic to π. Thus if we set $\tilde{\pi} = (\text{c-Ind}_J^{\mathrm{GL}_2(F)} \tilde{\Lambda}') \otimes (\chi \circ \det)$, we can find a $\mathrm{GL}_2(F)$-stable lattice in $\tilde{\pi}$ by choosing a J-stable lattice in $\tilde{\Lambda}' \otimes (\chi \circ \det)$. Reducing this lattice mod l we find:

Corollary 4.3. *Let π be an irreducible, supercuspidal representation of* $\mathrm{GL}_2(F)$ *over* $\overline{\mathbb{F}}_l$. *Then* $\pi = (\text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda') \otimes (\phi \circ \det)$, *where ϕ is a character of $F^\times$ and either:*

(1) $J = F^\times \mathrm{GL}_2(\mathcal{O}_F)$, *and the restriction of Λ' to* $\mathrm{GL}_2(\mathcal{O}_F)$ *is the inflation of an irreducible, supercuspidal representation of* $\mathrm{GL}_2(\mathbb{F}_q)$, *or*

(2) *there exists a stratum $(\mathfrak{A}, n, \alpha)$ such that n is odd, $J = E^\times U_{\mathfrak{A}}^{\frac{n+1}{2}}$, and Λ' is a character of J whose restriction to $U_{\mathfrak{A}}^{\frac{n+1}{2}}$ is ψ_α, or*

(3) *there exists a stratum $(\mathfrak{A}, n, \alpha)$ such that n is even, $J = E^\times U_{\mathfrak{A}}^{\frac{n}{2}}$, and Λ' is an irreducible representation of J whose restriction to $U_{\mathfrak{A}}^{\frac{n}{2}+1}$ is a multiple of ψ_α.*

Fix a representation π as in the corollary. It will be convenient to work with a certain subgroup U of J, defined as follows: in case 1, U is the kernel of the map: $\mathrm{GL}_2(\mathcal{O}_F) \to \mathrm{GL}_2(\mathbb{F}_q)$. In cases 2 and 3, U is $U_{\mathfrak{A}}^{[\frac{n+1}{2}]}$. Let $\Lambda = \Lambda' \otimes (\phi \circ \det)$, and let V be the subspace of π on which U acts via Λ.

As J normalizes $\mathrm{Res}_U^J \Lambda$, V is a J-stable subspace of π. In fact, we have:

Lemma 4.4. *As representations of J, we have $V \cong \Lambda$.*

Proof. Clearly Λ is a J-subrepresentation of π, and hence also of V. On the other hand, as a representation of U, V is a direct sum of copies of $\mathrm{Res}_U^J \Lambda$. By Mackey's induction-restriction formula ([Vi1], I.5.5), we have:

$$\mathrm{Res}_U^{\mathrm{GL}_2(F)} \pi = \oplus_{JgU} \text{c-Ind}_{U \cap gJg^{-1}}^U (\mathrm{Res}_U^J \Lambda)^g.$$

For each g, let $U_g = U \cap gJg^{-1}$. Then we have

$$\mathrm{Hom}_U(\text{c-Ind}_{U_g}^U (\mathrm{Res}_U^J \Lambda)^g, \mathrm{Res}_U^J \Lambda) = \mathrm{Hom}_{U_g}(\mathrm{Res}_{U_g}^J (\Lambda)^g, \mathrm{Res}_{U_g}^J \Lambda),$$

and the latter is nonzero if and only if g intertwines Λ. By [BH], 15.1, g is then an element of J. Thus π contains exactly one copy of $\mathrm{Res}_U^J \Lambda$, so Λ must be all of V as required. □

Proposition 4.5. *Let A be a finite length $W(\overline{\mathbb{F}}_l)$-algebra A, and let π_A be a lift of π to a representation over A. The map*

$$\Lambda_A \mapsto \text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda_A$$

gives a bijection between A-deformations of Λ and A-deformations of π.

Proof. Let π_A be an A-deformation of π. Since U is a p-group, the representation $\mathrm{Res}_U^J \Lambda$ lifts *uniquely* to A; let $\tilde{V}$ be the A-submodule of π_A on which U acts via this lift. As an A-module $\tilde{V}$ is a direct summand of π_A; its reduction mod l is V. Moreover, $\tilde{V}$ is stable under J, and therefore defines an irreducible representation Λ_A of J lifting Λ. We have a natural map

$$\text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda_A \to \pi_A$$

that reduces modulo m_A to the isomorphism

$$\text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda \to \pi.$$

As π_A is a free A-module it follows by the long exact sequence for Tor that this map is injective; on the other hand, Nakayama's lemma for admissible representations (see for instance [Em], Lemma 2.7) shows that the image of this map is all of π_A. □

The deformations of Λ are not difficult to understand, as Λ is "not far from" a pro-p group. We will be most interested in the case in which Λ arises from an *admissible pair* (mod l) (c.f. [BH], 5.19).

Definition 4.6. (c.f. [BH], 18.2). An admissible pair (over $\overline{\mathbb{F}}_l$) is a pair (E, χ), where E is a tamely ramified quadratic extension of F, and $\chi : E^\times \to \overline{\mathbb{F}}_l^\times$, such that:

- χ does not factor through the norm $N_{E/F} : E^\times \to F^\times$.
- Let U_E^1 be the group of units in $\mathcal{O}_E$ congruent to 1 mod $\mathfrak{p}$, where $\mathfrak{p}$ is a uniformizer of $\mathcal{O}_E$. If the restriction of χ to U_E^1 factors through $N_{E/F}$, then E is unramified over F.

Two pairs (E, χ) and (E', χ') are isomorphic if there is an isomorphism of $j : E \to E'$ such that $\chi' \circ j = \chi$.

We now describe a "mod l" version of the parametrization of tame cuspidal representations, which associates a supercuspidal representation to an admissible pair. We follow [BH], 5.19; little needs to be changed, but we will need the explicit description of the parameterization in order to properly understand the deformation theory.

Given (E, χ), we first choose a character ϕ of $F^\times$ such that $\chi = (\phi \circ N_{E/F})\chi'$, with χ' a *minimal* character of $E^\times$ in the sense of [BH], 5.18. There are then three cases:

(1) χ' has level 0. In this case E/F is unramified, and the restriction of χ' to $\mathcal{O}_E^\times$ is inflated from a character θ of $\mathbb{F}_{q^2}^\times$, that satisfies $\theta^q \neq \theta$. The representation π_θ of $\mathrm{GL}_2(\mathbb{F}_q)$ then inflates to a representation of $\mathrm{GL}_2(\mathcal{O}_F)$; we extend this to a representation Λ' of $J = F^\times \mathrm{GL}_2(\mathcal{O}_F)$ be letting $F^\times$ act via χ'. We take $\Lambda_\chi = \Lambda' \otimes (\phi \circ \det)$, and $\pi_\chi = \text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda_\chi$.

(2) χ' has odd level n. Fix a stratum $(\mathfrak{A}, n, \alpha)$, and let $J = E^\times U_{\mathfrak{A}}^{\frac{n+1}{2}}$. We take Λ' to be the character of J whose restriction to $E^\times$ is χ and whose restriction to $U_{\mathfrak{A}}^{\frac{n+1}{2}}$ is ψ_α. As above, we set $\Lambda_\chi = \Lambda' \otimes (\phi \circ \det)$, and $\pi_\chi = \text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda_\chi$.

(3) χ' has even positive level $n = 2m$. In this case E/F is unramified. Fix a stratum $(\mathfrak{A}, n, \alpha)$, and let $J = E^\times U_{\mathfrak{A}}^m$. Let $\mathcal{O}_E^{\times,1}$ be the subgroup of $\mathcal{O}_E^\times$ of units that map to 1 in $\mathbb{F}_{q^2}$, and set $J^1 = \mathcal{O}_E^{\times,1} U_{\mathfrak{A}}^m$. In [BH], 19.5.4, Bushnell-Henniart associate to this data an irreducible representation η of J^1 depending on χ, and show that there is a unique irreducible representation Λ' of J such that:
   - $\Lambda' | J^1 = \eta$,
   - $\Lambda' | F^\times$ is a multiple of $\chi | F^\times$, and
   - for every root of unity ζ in $E^\times \setminus F^\times$, of order prime to p, the trace of $\Lambda'(\zeta)$ is $-\chi(\zeta)$.

   We take $\Lambda_\chi = \Lambda' \otimes (\phi \circ \det)$, and $\phi_\chi = \text{c-Ind}_J^{\mathrm{GL}_2(F)} \Lambda_\chi$.

Remark 4.7. The construction described above is precisely the mod l reduction of the construction of Bushnell-Henniart; that is, if $(E, \tilde{\chi})$ is a characteristic zero admissible pair whose mod l reduction is (E, χ), then the characteristic zero representation $\pi_{\tilde{\chi}}$ associated to $(E, \tilde{\chi})$ by Bushnell-Henniart reduces mod l to the representation π_χ described above.

It is fairly straightforward to extend this construction to deformations. We will need the following lemma:

Lemma 4.8. *Let J be a locally profinite group and let J^1 be a compact normal subgroup. Let A be a finite length local $W(\overline{\mathbb{F}}_l)$-algebra, and let η_A be a finite-dimensional representation of J^1 over A, such that $\eta_A \otimes_A \overline{\mathbb{F}}_l$ is irreducible. Suppose η_A extends to an A-representation Λ_A of J. Then any other extension Λ'_A of η_A to a representation of J differs from Λ_A by twisting by a character of J/J^1.*

Proof. Let V and V' be the representation spaces of Λ and Λ' respectively; fix a J^1-equivariant isomorphism between them, so that we can consider $\Lambda_A(\tau)$ and $\Lambda'_A(\tau)$ as elements of $\mathrm{End}(V)$ for any $\tau \in J$. Consider the element $\Lambda_A(\tau)^{-1}\Lambda'_A(\tau)$ of $\mathrm{End}(V)$. This element commutes with $\Lambda_A(\sigma)$ for any $\sigma \in J^1$, and hence is an η_A-equivariant endomorphism of V. By Schur's lemma $\Lambda_A(\tau)^{-1}\Lambda'_A(\tau)$ is equal to c_τ times the identity for a scalar $c_\tau \in A^\times$. One verifies easily that $c_\tau = 1$ if $\tau \in J^1$ and that $c_{\tau\tau'} = c_\tau c_{\tau'}$ for all τ, τ'. In particular $\tau \mapsto c_\tau$ is a character of J/J^1, and twisting Λ_A by this character yields Λ'_A. □

Theorem 4.9. *Let (E, χ) be a (mod l) admissible pair. Then for any finite length $W(\overline{\mathbb{F}}_l)$-algebra A, there are natural bijections between A-deformations χ_A of χ, deformations $\Lambda_{\chi,A}$ of Λ_χ, and deformations $\pi_{\chi,A}$ of π_χ.*

Proof. We have already constructed the bijection between deformations of Λ_χ and π_χ, so it remains to relate these to deformations of χ. We proceed case-by-case. Fix a χ' and ϕ such that $\chi = \chi'(\phi \circ N_{E/F})$ with χ' minimal.

(1) Suppose that χ' has level zero. Given a lift χ'_A of χ', the restriction of χ'_A to $\mathcal{O}_E^\times$ is inflated from a character θ_A of $\mathbb{F}_{q^2}^\times$ lifting θ. Then π_{θ_A} is an A-deformation of π_θ; inflating it to $\mathrm{GL}_2(\mathcal{O}_F)$ and extending it to $F^\times \mathrm{GL}_2(\mathcal{O}_F)$ by letting $F^\times$ act by χ'_A gives a lift Λ'_A of Λ'. Conversely, given a lift Λ'_A of Λ', the restriction of Λ'_A to $\mathrm{GL}_2(\mathcal{O}_F)$ is inflated from an A-deformation $\pi_{\theta,A}$ of π_θ; by Theorem 3.2 $\pi_{\theta,A}$ arises from a (uniquely determined) character θ_A of $\mathbb{F}_{q^2}^\times$ lifting θ. Let χ'_A be the unique A-deformation of χ' that is inflated from θ_A on $\mathcal{O}_E^\times$ and agrees with Λ'_A on $F^\times$. This recovers χ'_A from Λ'_A and gives a bijection between deformations of χ' and Λ'; twisting we obtain the desired bijection between deformations of χ and deformations of Λ_χ.

(2) If χ' has odd level n, then Λ' is a character of $E^\times U_{\mathfrak{A}}^{\frac{n+1}{2}}$ whose restriction to $E^\times$ is χ', and whose restriction to the pro-p group $U_{\mathfrak{A}}^{\frac{n+1}{2}}$ is a fixed character ψ_α. Given a deformation χ'_A of χ' we let Λ'_A be the character whose restriction to $E^\times$ is χ'_A and whose restriction to $U_{\mathfrak{A}}^{\frac{n+1}{2}}$ is the unique lift of ψ_α to A. This manifestly yields a bijection between deformations of χ' and deformations of Λ'; twisting we obtain a bijection between deformations of χ and deformations of Λ_χ.

(3) Suppose χ' has even level $2m$. Then Λ' is a representation of $J = E^\times U_{\mathfrak{A}}^m$; its restriction to the normal subgroup J^1 of J given by $J^1 = \mathcal{O}_E^{\times,1} U_{\mathfrak{A}}^m$ is an irreducible representation η. If we lift χ' to a character $\tilde{\chi}'$ with values in $W(\overline{\mathbb{F}}_l)^\times$ (which we can always do), then $(E, \tilde{\chi}')$ is an admissible pair, to which Bushnell-Henniart associate a representation $\tilde{\Lambda}'$ of J over $W(\overline{\mathbb{F}}_l)$ lifting Λ'.

Suppose we have an A-deformation Λ'_A of Λ'. The restriction of Λ'_A to J^1 is the unique A-deformation η_A of η, since J^1 is a pro-p group. By the lemma above, Λ'_A is a twist of $\tilde{\Lambda}' \otimes_{W(\overline{\mathbb{F}}_l)} A$ by a uniquely determined character of J/J^1.

Now let χ'_A be an A-deformation of χ'. Define Λ'_A to be the twist of $\tilde{\Lambda}'_A \otimes_{W(\overline{\mathbb{F}}_l)} A$ by the character $\chi'_A(\tilde{\chi}')^{-1}$. This gives a bijection between deformations of χ' and deformations of Λ', and is independent of the choice of $\tilde{\chi}'$. As usual, twisting yields the desired bijection between deformations of χ and deformations of Λ.

$\square$

If E/F is ramified, there is an even simpler classification of deformations of π_χ:

Proposition 4.10. *Let (E, χ) is an admissible pair with E/F ramified, and let ϕ be its restriction to $F^\times$. Then restriction to $F^\times$ is a bijection between A-deformations of χ and A-deformations of ϕ. In particular giving an A-deformation of π_χ is equivalent to giving an A-deformation of its central character.*

Proof. Let ϕ_A be an A-deformation of ϕ. The group $\mathcal{O}_E^\times$ is generated by the roots of unity in $E^\times$ (all of which lie in $F^\times$), and the pro-p group $\mathcal{O}_E^{\times,1}$. Thus ϕ_A extends uniquely to a character of $F^\times\mathcal{O}_E^\times$ that lifts χ. If $\mathfrak{p}$ is a uniformizer of $\mathcal{O}_E$, then $\mathfrak{p}$ and $F^\times\mathcal{O}_E^\times$ generated $E^\times$, and $\mathfrak{p}^2$ lies in $F^\times\mathcal{O}_E^\times$, so an extension of ϕ_A to a character χ_A of $E^\times$ lifting χ is determined uniquely by a choice of $\phi_A(\mathfrak{p})$ compatible with the (already determined) value of $\phi_A(\mathfrak{p}^2)$. In particular $\phi_A(\mathfrak{p})$ must be a square root of $\phi_A(\mathfrak{p})^2$ that reduces to $\chi_A(\mathfrak{p})$ modulo l; as l is odd there is a unique choice of $\phi_A(\mathfrak{p})$. Thus ϕ_A extends uniquely to a character χ_A of $E^\times$ lifting χ, as required. $\square$

Remark 4.11. We will also be concerned with the deformations of supercuspidal representations π over $\overline{\mathbb{F}}_l$ that correspond via mod l local Langlands to primitive representations- that is, to representations of G_F that are not induced from characters. These do not come from admissible pairs via the above construction. On the other hand, they only occur when q is a power of 2, and have a very specific form: by the construction in the proof of [BH], Theorem 50.3, they arise (up to a twist) from ramified simple strata of odd level; that is, up to twisting every such π is of the form $\text{c-Ind}_J^{\mathrm{GL}_2(F)}\Lambda$, where Λ is a character of J, $J = E^\times U_{\mathfrak{A}}^{\frac{n+1}{2}}$ for some stratum (U, n, α), and E/F is *ramified*. Deformations of such π are in bijection with deformations of Λ and (by the same argument as in case 2 of Theorem 4.9) therefore in bijection with deformations of the restriction of Λ to $E^\times$. The above proposition then shows that such deformations are determined by their restriction to $F^\times$. It follows that in this case deforming π is equivalent to deforming the central character of π.

Finally, we will need to understand the deformation theory of a particular admissible representation that is cuspidal but not supercuspidal. Such representations only occur when q is congruent to -1 mod l, and are equal to a twist by $(\phi \circ \det)$ of the "Weil representation" $\pi(1)$ described in [Vi1], II.2.5. From our perspective, $\pi(1)$ is most conveniently characterised as follows: let π_1 be the cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ associated to the trivial character of $\mathbb{F}_{q^2}^\times$, and let Λ be the representation of $J = F^\times \mathrm{GL}_2(\mathcal{O}_F)$ such that $F^\times$ acts trivially and $\mathrm{GL}_2(\mathcal{O}_F)$ acts via π_1. Then $\pi(1)$ is given by $\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\Lambda$.

By Theorem 3.3, the universal deformation of π_1 is defined over the ring $R = W(\overline{\mathbb{F}}_l)[[t]]/Q(t)$. Define a representation Λ^{univ} of J over $R[[x]]$ for which $F^\times$ acts via the unramified character that takes a uniformizer of $\mathcal{O}_F$ to $1+x$, and $\mathrm{GL}_2(\mathcal{O}_F)$ acts via the universal deformation of π_1.

Proposition 4.12. *The ring $R^{\mathrm{univ}}_{\pi(1)}$ is isomorphic to $W(\mathbb{F}_l)[[x,t]]/Q(t)$. The univeral representation π^{univ} is given by $\mathrm{Ind}_J^{\mathrm{GL}_2(F)} \Lambda^{\mathrm{univ}}$. Under these identifications, the central character of $\rho^{\mathrm{univ}}_{\pi(1)}$ is the unramified character that takes a uniformizer of $\mathcal{O}_F$ to $1+x$. Moreover, if U is the kernel of the map $\mathrm{GL}_2(\mathcal{O}_F) \to \mathrm{GL}_2(\mathbb{F}_q)$, then $\mathrm{GL}_2(\mathbb{F}_q)$ acts on the U-invariants of $\rho^{\mathrm{univ}}_{\pi(1)}$ by the deformation $\pi_{1,t}$ of π_1 over $R^{\mathrm{univ}}_{\pi(1)}$.*

Proof. The same argument as in the supercuspidal case shows that giving an A-deformation of $\pi(1)$ is the same as giving an A-deformation of Λ. It thus suffices to show that Λ^{univ} is the universal deformation of Λ. By the same argument as in the level 0 case of Theorem 4.9, deforming Λ is the same as giving a deformation of its (trivial) central character and a deformation of π_1. The representation Λ^{univ} is clearly universal for such a pair of deformations. Moreover, the U-invariants of Λ^{univ} are by construction $\pi_{1,t}$. □

5. Representations of G_F

We now turn to the Galois side of the local Langlands correspondence. We begin by studying the first-order deformations of two-dimensional representations of G_F over $\overline{\mathbb{F}}_l$. Let $\tilde{\omega}$ be the cyclotomic character of G_F with values in $W(\mathbb{F}_l)$, and let ω be its reduction mod l.

Lemma 5.1. *Let ρ be an irreducible representation of G_F over $\overline{\mathbb{F}}_l$. Suppose q is not congruent to 1 mod l. Then:*

- *$H^1(G_F,\rho)$ is one-dimensional if ρ is trivial or $\rho = \omega$, and is zero otherwise.*
- *$H^2(G_F,\rho)$ is one-dimensional if $\rho = \omega$ and is zero otherwise.*

If q is congruent to 1 mod l, then:

- *$H^1(G_F,\rho)$ is two-dimensional if ρ is trivial and is zero otherwise.*
- *$H^2(G_F,\rho)$ is one-dimensional if ρ is trivial and is zero otherwise.*

Proof. This is an easy application of inflation-restriction. □

Proposition 5.2. *Let ρ be an irreducible two-dimensional representation of G_F over $\overline{\mathbb{F}}_l$. Then either:*

- *there exists a character ξ of G_E, where E is the unique unramified quadratic extension of F, such that $\rho = \mathrm{Ind}_{G_E}^{G_F} \xi$, or*
- *the map $\rho_A \mapsto \det \rho_A$ gives a bijection between A-deformations $\tilde{\rho}$ of ρ and deformations of $\det \rho$.*

Proof. It suffices to show that if $H^1(G_F, \mathrm{Ad}^0 \rho)$ is nonzero, then there exists an ξ as above. By the lemma, this can only happen if $\mathrm{Ad}^0 \rho$ contains a character as a Jordan-Holder constituent. Since l is odd, $\mathrm{Ad}^0 \rho$ is naturally a direct summand of $\mathrm{Ad}\,\rho$ and is therefore a 3-dimensional self-dual representation of G_F. In particular, if $\mathrm{Ad}^0 \rho$ contains a character as a Jordan-Holder consituent then it contains a character ν as a subrepresentation. Then ν is a subrepresentation of $\mathrm{Ad}\,\rho$; by Schur's lemma ν is a nontrivial character of G_F. We therefore have an isomorphism

$\rho \cong \rho \otimes \nu$. Considering determinants we find that ν^2 is trivial. Let E' be the quadratic extension of F corresponding to the kernel of ν. Then the restriction of $\mathrm{Ad}\,\rho$ to $G_{E'}$ contains two copies of the trivial character, so ρ becomes reducible when restricted to $G_{E'}$. If we let ξ' be a character of $G_{E'}$ contained in the restriction of ρ, then $\rho \cong \mathrm{Ind}_{G_{E'}}^{G_F} \xi'$.

We then have a direct sum decomposition:

$$\mathrm{Ad}^0 \rho = \nu \oplus \mathrm{Ind}_{G_{E'}}^{G_F} \frac{(\xi')^\sigma}{\xi'}$$

where σ generates $\mathrm{Gal}(E'/F)$. If the second direct summand is irreducible, then ν must be equal to ω in order for $\mathrm{Ad}^0 \rho$ to have nontrivial cohomology. In particular E' is unramified over F, so $E' = E$. (Note that then q must be congruent to -1 mod l.)

On the other hand, if the second direct summand is reducible, then $\mathrm{Ad}^0 \rho$ is the direct sum of three characters, which must be nontrivial by Schur's lemma. Again, one of these characters must be ω in order for $\mathrm{Ad}^0 \rho$ to have nontrivial cohomology. It follows that $\rho \cong \rho \otimes \omega$, and thus (as with ν), ω has exact order 2. The kernel of ω is thus G_E; it follows that ρ is induced from a character of G_E as required. □

Proposition 5.3. *Let ξ be a character of G_E, and let $\rho = \mathrm{Ind}_{G_E}^{G_F} \xi$. Suppose ρ is absolutely irreducible. Then*

$$\xi_A \mapsto \mathrm{Ind}_{G_E}^{G_F} \xi_A$$

induces a bijection between A-deformations of ξ and A-deformations of ρ. (This gives a natural isomorphism $R_\xi^{\mathrm{univ}} \cong R_\rho^{\mathrm{univ}}$.)

Proof. By induction on the length of A, it suffices to prove this for first-order deformations. The first order deformations of ξ are a torsor for $H^1(G_E, 1)$, where 1 denotes the trivial character of G_E. The map

$$H^1(G_E, 1) \to H^1(G_F, \mathrm{Ad}\,\rho)$$

that induces a first order deformation of ξ from G_E to G_F is clearly injective. The dimension of $H^1(G_F, \mathrm{Ad}\,\rho)$ is two if $\mathrm{Ad}\,\rho$ contains ω and one otherwise. As in the proof of the previous proposition, if $\mathrm{Ad}\,\rho$ contains ω then ω is trivial on G_E.

On the other hand, by the lemma $H^1(G_E, 1)$ is two-dimensional if ω is trivial on G_E, and one-dimensional otherwise, so the result follows. □

The deformations of a character of the Galois group of a local field are easy to describe: let ξ be a character of G_F with values in $\overline{\mathbb{F}}_l^\times$, and let $\tilde{\xi} : G_F \to W(\overline{\mathbb{F}}_l)^\times$ be its Teichmuller lift. Then R_ξ^{univ} is isomorphic to $W(\overline{\mathbb{F}}_l)[[t]][\zeta]/\langle \zeta^{l^n} - 1\rangle$, where n is equal to $\mathrm{ord}_l(q-1)$. Define a deformation ξ^{univ} of ξ by $\xi^{univ} = \tilde{\xi}\xi'$, where ξ' is the character that takes a Frobenius element Fr to $1 + t$ and a generator σ of the l-part of I_F to ζ. Local class field theory easily shows that ξ^{univ} is the universal deformation of ξ.

6. The deformation-theoretic correspondence

We are now in a position to relate the deformation theory of an admissible representation π over a finite field of characteristic l to the representation ρ attached to π by the local Langlands correspondence. Throughout we use the Tate normalization for the local Langlands correspondence; this has the advantage that π and ρ have the same field of definition. This normalization differs from the more usual normalization by a twist by an unramified character; it has the property that the central character of π corresponds to $\tilde{\omega} \det \rho$ via local class field theory.

Since we are primarily interested in deformation theory, we will work with continuous representations of G_F, rather than the Weil-Deligne representations usually considered in the Langlands correspondence, as the latter do not behave well from a deformation-theoretic perspective. Over $\overline{\mathbb{F}}_l$ this makes no difference, as any Weil-Deligne representation arises from a unique representation of G_F in this setting. In characteristic zero this is not true, but we will (somewhat abusively) say that an admissible representation $\tilde{\pi}$ and a continuous Galois representation $\tilde{\rho}$ "correspond under local Langlands" if the Weil-Deligne representation arising from $\tilde{\rho}$ corresponds to $\tilde{\pi}$ under the more usual notion of the local Langlands correspondence. (This, of course, does *not* give a bijection between irreducible Galois representations and cuspidal admissible representations, as not every Weil-Deligne representation comes from a representation of G_F. On the other hand, *integral* irreducible representations of W_F over a local field of residue characteristic l- those that contain a lattice- do arise from representations of G_F.)

Bushnell-Henniart give the following description of the characteristic zero local Langlands correspondence for representations arising from admissible pairs: (for details, see [BH], section 34.) If an admissible representation $\tilde{\pi}$ over $\overline{\mathbb{Q}_l}$ arises from an admissible pair $(E, \tilde{\chi})$ then the corresponding Galois representation $\tilde{\rho}$ is equal to $\mathrm{Ind}_{G_E}^{G_F} \tilde{\xi}$ for some character $\tilde{\xi}$ of G_E. Moreover every such $\tilde{\rho}$ arises in this way. If E/F is unramified then there is a character $\tilde{\Delta}$ of $E^\times$, independent of $\tilde{\chi}$, such that $\tilde{\xi}$ corresponds to $\tilde{\chi}\tilde{\Delta}$ via class field theory. (If E/F is ramified, there is still an explicit description of $\tilde{\xi}$ in terms of $\tilde{\chi}$, but it is more complicated and depends on $\tilde{\chi}$; we will not make use of this.)

The mod l local Langlands correspondence is compatible with reduction mod l for cuspidal representations, and so is the process that associates a representation of $\mathrm{GL}_2(F)$ to an admissible pair. Thus if π is a mod l admissible representation attached to an admissible pair (E, χ), we can choose a lift $(E, \tilde{\chi})$ to characteristic zero; this gives a characteristic zero representation $\tilde{\pi}$ lifting π. The Galois representation $\tilde{\rho}$ corresponding to $\tilde{\pi}$ is induced from a character $\tilde{\xi}$ of G_E with values in $\overline{\mathbb{Q}}_l^\times$. Then the representation ρ attached to π is just the reduction mod l of $\tilde{\rho}$, so ρ is induced from a character ξ of G_E with values in $\overline{\mathbb{F}}_l^\times$. In this manner we see that the statements of the previous paragraph hold mutatis mutandis for the mod l local Langlands correspondence.

This allows us to prove:

Theorem 6.1. *Let π be a supercuspidal representation of $\mathrm{GL}_2(F)$ over a finite field of characteristic l and let k be its field of definition. Let ρ be the Galois representation corresponding to π under mod l local Langlands. Then there is a unique isomorphism of $W(k)$-algebras:*

$$R_\pi^{\mathrm{univ}} \to R_\rho^{\mathrm{univ}}$$

with the property that the induced map $R_\pi^{\mathrm{univ}}(\overline{\mathbb{Q}}_l) \to R_\rho^{\mathrm{univ}}(\overline{\mathbb{Q}}_l)$ is the characteristic zero local Langlands correspondence.

Proof. Such an isomorphism is clearly unique if it exists. We begin by working with deformations over $W(\overline{\mathbb{F}}_l)$ rather than over $W(k)$ so we can apply the results of section 4.

First assume π comes from an admissible pair (E, χ) with E/F unramified. Then by Theorem 4.9 we have an isomorphism of $W(\overline{\mathbb{F}}_l)$-algebras $R_\pi^{\mathrm{univ}} \cong R_\chi^{\mathrm{univ}}$. On the other hand, ρ is the representation $\mathrm{Ind}_{G_E}^{G_F} \xi$, where ξ corresponds to $\chi\Delta$ via local class field theory, and Δ is the mod l reduction of $\tilde{\Delta}$. By Proposition 5.3, R_ρ^{univ} is isomorphic over $W(\overline{\mathbb{F}}_l)$ to R_ξ^{univ}. Twisting by $\tilde{\Delta}$ and applying the local class field theory isomorphism gives an isomorphism of R_χ^{univ} with R_ξ^{univ}; composing these isomorphisms gives the desired isomorphism of R_π^{univ} with R_ρ^{univ}. That the induced map on $\overline{\mathbb{Q}}_l$-points is the characteristic zero local Langlands correspondence is clear from the construction.

Otherwise, π either comes from an admissible pair (E, χ) with E/F ramified or does not come from an admissible pair at all. In either case deformations of π correspond to deformations of the central character of π. By Proposition 5.2 deformations of the representation ρ correspond to deformations of its determinant. In particular for any A-deformation π_A of π, there is a unique A-deformation ρ_A of ρ such that the central character of π_A corresponds to $\tilde{\omega} \det \rho_A$ under local class field theory. This gives the desired isomorphism of R_π^{univ} with R_ρ^{univ}. Let $\tilde{\pi}$ be a $\overline{\mathbb{Q}}_l$-point of R_π^{univ}; i.e. a lift of π to $\overline{\mathbb{Q}}_l$. Then the above isomorphism takes $\tilde{\pi}$ to the unique lift $\tilde{\rho}$ of ρ such that $\tilde{\omega}^{-1} \det \tilde{\rho}$ corresponds to the central character of $\tilde{\pi}$. Since the representation attached to $\tilde{\pi}$ by (Tate normalized) local Langlands is also a lift of $\tilde{\rho}$ with this determinant, the two coincide.

It remains to show the above isomorphism descends to an isomorphism over $W(k)$. Let σ be an automorphism of $W(\overline{\mathbb{F}}_l)$ over $W(k)$. Then the twist $(\pi_A)^\sigma$ of an A-deformation π_A of π is a deformation of π^σ over the $W(\overline{\mathbb{F}}_l)$-algebra A^σ whose underlying ring is A but whose $W(\overline{\mathbb{F}}_l)$-module structure is twisted by σ. The descent data isomorphism $\pi \cong \pi^\sigma$ allows us to identify $(\pi_A)^\sigma$ with an A^σ-deformation of π. This gives an action of σ on R_π^{univ}. As ρ is also defined over k we get a similar action on R_ρ^{univ}. It suffices to show that conjugating the isomorphism $R_\pi^{\mathrm{univ}} \cong R_\rho^{\mathrm{univ}}$ by σ induces the same isomorphism; by uniqueness it is enough to check this on $\overline{\mathbb{Q}}_l$-points. But this is simply the (standard) fact that if $\tilde{\pi}$ and $\tilde{\rho}$ correspond under (Tate normalized) local Langlands, then so do $\tilde{\pi}^\sigma$ and $\tilde{\rho}^\sigma$. $\square$

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