WHITTAKER MODELS AND THE INTEGRAL BERNSTEIN CENTER FOR GL_n

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ABSTRACT. We establish integral analogues of results of Bushnell and Henniart [BH] for spaces of Whittaker functions arising from the groups $\mathrm{GL}_n(F)$ for F a p-adic field. We apply the resulting theory to the existence of representations arising from the conjectural "local Langlands correspondence in families" of [EH], and reduce the question of the existence of such representations to a natural conjecture relating the integral Bernstein center of $\mathrm{GL}_n(F)$ to the deformation theory of Galois representations.

1. INTRODUCTION

The *Bernstein center* is a central tool in the study of the smooth complex representations of a p-adic group G. Introduced by Bernstein and Deligne in [BD], the Bernstein center is a commutative ring that acts naturally on every smooth complex representation of G. Its primitive idempotents thus decompose the category $\mathrm{Rep}_{\mathbb{C}}(G)$ of smooth complex representations of G into full subcategories known as "blocks"; any object of $\mathrm{Rep}_{\mathbb{C}}(G)$ then has a canonical decomposition as a product of factors, one from each block. Moreover, Bernstein and Deligne give a description of each block, and show that the action of the Bernstein center on each block factors through a finite type $\mathbb{C}$-algebra (the center of the block), that can be described in a completely explicit fashion.

One advantage of this approach is that it allows one to give purely algebraic proofs of results that were classically proven using difficult technqiues from Harmonic analysis. For example, one can regard a (not necessarily admissible) smooth representation of G as a sheaf on the spectrum of the Bernstein center; doing so provides a purely algebraic analogue of the Fourier decomposition of this representation as a "direct integral" of irreducible representations over a suitable measure space. A clear benefit of this algebraic approach is that it applies even when one considers representations over fields (or even rings) other than the complex numbers.

In [BH], Bushnell and Henniart applied the above ideas to the study of Whittaker models in the complex representation theory of p-adic groups. In particular, they study the space $\text{c-Ind}_U^G \Psi$, where U is the unipotent radical of a Borel subgroup B of G, and Ψ is a "generic character" of U. This space is dual to the space of functions in which Whittaker models live. They establish two key technical results about this space: first, if e is an idempotent of the Bernstein center corresponding to a particular block of $\mathrm{Rep}_{\mathbb{C}}(G)$, they show that $e\, \text{c-Ind}_U^G \Psi$ is finitely generated as a $\mathbb{C}[G]$-module. Second, they show that the center of the block corresponding to e acts faithfully on $e\, \text{c-Ind}_U^G \Psi$, and that in many clases (in particular $G = \mathrm{GL}_n$),

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this center is the full endomorphism ring of $e\,\text{c-Ind}_U^G \Psi$. As an application, they give a purely algebraic proof of a vanishing theorem, originally due to Jacquet, Piatetski-Shapiro, and Shalika ([JPS], Lemma 3.5), for functions in $\text{c-Ind}_U^G \Psi$.

The theory of the Bernstein center for categories of representations over coefficient rings other than $\mathbb{C}$ presents significant technical difficulties beyond the complex case; in particular the approach of Bernstein and Deligne makes heavy use of certain properties of cuspidal representations that only hold over fields of characteristic zero. Dat [Da1] has established some basic structural results for general p-adic groups in this context, but for general groups not much more is known.

For $\mathrm{GL}_n(F)$, however, previous work of the author [H1] establishes a much more detailed structure theory for the center of the category $\mathrm{Rep}_{W(k)}(\mathrm{GL}_n(F))$ of smooth $W(k)[\mathrm{GL}_n(F)]$-modules, where k is an algebraically closed field of characteristic ℓ. Although it does not seem possible to give as complete and thorough description of the center in this context, the results of [H1] are detailed and precise enough that one might hope for applications in both representation theory and arithmetic. In particular it is natural to ask whether one can apply the theory of [H1] to analogues of the questions studied in [BH].

The purpose of this paper is twofold: first, to establish integral versions of the results of [BH] discussed above in the case where $G = \mathrm{GL}_n(F)$, using the results and techniques of [H1]. The second is to apply these techniques to questions concerning the local Langlands correspondence of [EH].

Rather than attempting to mimic the arguments of [BH], our approach to the first goal is by a quite different argument. We make heavy use of the fact that the results of [BH] hold over $\mathbb{C}$, and therefore also over the field $\overline{\mathcal{K}}$, where $\mathcal{K}$ is the field of fractions of $W(k)$. Our approach here relies on the Bernstein-Zelevinski theory of the derivative, developed over $W(k)$ in [EH], and in particular on computing the derivatives of certain projective objects of $\mathrm{Rep}_{W(k)}(G)$ first considered in [H1]. This computation makes heavy use of the notion of an essentially AIG representation, introduced in [EH].

Our main results about $\text{c-Ind}_U^G \Psi$ are established in section 5. As an application we prove a basic fact about essentially AIG representations that was conjectured in [EH].

We now discuss our second goal in more detail. In [EH], Emerton and the author introduce a conjectural "local Langlands correspondence in families". The main result states roughly that given a Galois representation $\rho : G_F \to \mathrm{GL}_n(A)$, for A a suitable complete local $W(k)$-algebra, there is at most one admissible $A[\mathrm{GL}_n(F)]$-module $\pi(\rho)$, satisfying a short list of technical conditions, such that at characteristic zero points $\mathfrak{p}$ of $\operatorname{Spec} A$, the representations $\pi(\rho)_x$ and ρ_x are related by a variant of the local Langlands correspondence. (We refer the reader to Theorem 7.1 for a precise statement).

Emerton has shown [E] that certain spaces that arise by considering the completed cohomology of the tower for modular curves have a natural tensor factorization, and that the tensor factors that arise in this way are isomorphic to $\pi(\rho)$ for certain representations ρ of $G_{\mathbb{Q}_p}$ over Hecke algebras. This result is crucial to his approach to the Fontaine-Mazur conjecture.

However, the results of [EH] do not address the question of whether the representations $\pi(\rho)$ exist in general. They also leave several fundamental questions about the structure of $\pi(\rho)$ unanswered.

Our approach to these questions revolves around the concept of a "co-Whittaker" module, introduced in section 6. The key point is that the families $\pi(\rho)$, when they exist, are co-Whittaker modules. Moreover, there is a natural connection between co-Whittaker modules and the module $\text{c-Ind}_U^G \Psi$, which can be regarded as a "universal" co-Whittaker module, in a sense we make precise in Theorem 6.3.

In section 7 we apply our structure theory to the local Langlands correspondence in families. In particular, we reduce the construction of the families $\pi(\rho)$, where ρ is a deformation of a given Galois representation $\overline{\rho}$ over k, to the question of the existence of a certain map from the integral Bernstein center to the universal framed deformation ring of $\overline{\rho}$ (Theorem 7.9). Moreover, we show that the converse also holds (conditionally on a result from the forthcoming work [H2].)

There are several advantages to reformulating the question of the existence of $\pi(\rho)$ in this way. The first is that the new conjectures that arise in this way (Conjecture 7.5 and the stronger variant Conjecture 7.6) are interesting in their own right. Indeed, one can regard Conjecture 7.6 as a geometric reformulation of the correspondence between supercuspidal support (for admissible representations of G) and semisimplification of Galois representations (see remark 7.7.)

Moreover, it seems to be substantially easier in practice to construct maps from the Bernstein center to universal framed deformation rings than it is to directly construct representations $\pi(\rho)$. For example, it is not difficult to show that both conjectures hold after inverting ℓ, as well as in the case where ℓ is a *banal* prime; that is, when the order q of the residue field of F has the property that $1, q, \dots, q^n$ are distinct mod ℓ. (We sketch a proof of this in Example 7.11; the details, as well as deeper results about the conjectures in question, will appear in the forthcoming work [H2].) In particular, this approach provides the only currently known construction of $\pi(\rho)$ in the banal setting for $n > 2$.

For small n, it is in principle possible to construct suitable families $\pi(\rho)$ by ad-hoc methods. It seems unlikely, however, that these methods can be pushed much beyond the case $n = 2$. For instance, when $n = 2$, ℓ is odd, and q is congruent to -1 modulo ℓ, it is possible to explicitly construct the representation $\pi(\rho)$, where ρ is the universal framed deformation ring of $1 \oplus \omega$ (here ω is the mod ℓ cyclotomic character). This construction requires a delicate analysis of congruences between lattices in cuspidal and Steinberg representations of $\mathrm{GL}_2(F)$. By contrast, if one knows Conjecture 7.5 in this setting, the construction is clean and straightforward. In a sense, the map in Conjecture 7.5 encodes all of these congruences in a sufficiently systematic way that one does not need to work with them directly.

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2. The integral Bernstein center

We now fix notation and summarize some of the basic properties of the integral Bernstein center from [H1] that will be used throughout the paper. Let p and ℓ be distinct primes, and let F be a finite extension of $\mathbb{Q}_p$. We let G denote the group $\mathrm{GL}_n(F)$.

Let k be an algebraically closed field of characteristic ℓ, and let $\mathcal{K}$ be the field of fractions of $W(k)$. We will be concerned with the categories $\mathrm{Rep}_k(G)$, $\mathrm{Rep}_{\mathcal{K}}(G)$, and

$\mathrm{Rep}_{W(k)}(G)$ of smooth $k[G]$-modules, smooth $\mathcal{K}[G]$-modules, and smooth $W(k)[G]$-modules, respectively.

The blocks of $\mathrm{Rep}_{W(k)}[G]$ are parameterized by equivalence classes of pairs (L, π), where L is a Levi subgroup of G and π is an irreducible supercuspidal representation of L over k. Two pairs (L, π) and (L', π') are said to be *inertially equivalent* if there is an element g of G such that $L' = gLg^{-1}$ and π' is a twist of π^g by an unramified character of L'.

Given a pair $[L, \pi]$, up to inertial equivalence, we can consider the full subcategory $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ of $\mathrm{Rep}_{W(k)}(G)$ consisting of smooth $W(k)[G]$-modules Π such that every simple subquotient of Π has *mod ℓ inertial supercuspidal support* (in the sense of [H1], Definition 4.10) given by the pair (L, π). By [H1], Theorem 10.8, $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ is a block of $\mathrm{Rep}_{W(k)}(G)$. For any smooth $W(k)[G]$-module Π, we may thus speak of the factor $\Pi_{[L,\pi]}$ of Π that lies in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$.

We may also consider the center $A_{[L,\pi]}$ of $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$. We have the following basic structure theory of $A_{[L,\pi]}$:

Theorem 2.1 ([H1], Theorem 12.1)**.** *The ring $A_{[L,\pi]}$ is a finitely generated, reduced, ℓ-torsion free $W(k)$-algebra.*

Let κ be a $W(k)$-algebra that is a field; then any smooth representation Π of G over κ lies in $\mathrm{Rep}_{W(k)}(G)$. If Π is absolutely irreducible then it lies in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ for some (L, π); by Schur's Lemma the action of $A_{[L,\pi]}$ on Π is via a homomorphism $f_\Pi : A_{[L,\pi]} \to \kappa$.

Theorem 2.2. *Let Π_1 and Π_2 be two absolutely irreducible representations of G over κ that lie in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, and let f_1 and f_2 be the maps: $A_{[L,\pi]} \to \kappa$ giving the action of $A_{[L,\pi]}$ on Π_1 and Π_2 respectively. Then $f_1 = f_2$ if, and only if, Π_1 and Π_2 have the same supercuspidal support.*

Proof. When κ has characteristic zero this follows from the classical theory of Bernstein and Deligne. When κ has characteristic ℓ this is (a slight generalization of) Theorem 12.2 of [H1]; the proof of that theorem works in the generality claimed here. □

This result has the following useful corollary:

Corollary 2.3. *Let $f : A_{[L,\pi]} \to \kappa$ be a map of $W(k)$-algebras. There exists a unique irreducible generic representation of G over κ in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ on which $A_{[L,\pi]}$ acts via f.*

Proof. It is an easy consequence of work of Vigneras (see for instance section V of [V1]) that there is a unique irreducible generic representation with given supercuspidal support. The corollary is thus immediate from Theorem 2.2. □

3. Essentially AIG representations

Our attempts to generalize results from [BH] will rely on a version of the Bernstein-Zelevinski theory of the derivative and its related functors that makes sense for integral representations. We refer the reader to [EH], section 3.1, for the details of this theory over $W(k)$. Here we will content ourselves with the theory of the "top derivative".

Let U be the unipotent radical of a Borel subgroup of G, and let $\Psi : U \to W(k)^\times$ be a generic character. Let W be the module $\text{c-Ind}_U^G \Psi$; W is independent, up to isomorphism, of the generic character Ψ. As U has order prime to ℓ and c-Ind takes projectives to projectives, W is a projective $W(k)[G]$-module. The Bernstein factor $W_{[L,\pi]}$ is then a projective object of $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, which is closely related to the theory of Whittaker models.

The module $W_{[L,\pi]}$ will be crucial for our approach to the conjectures of [EH] on the local Langlands correspondence for families of Galois representations. First, however, we need some basic results on the structure of $W_{[L,\pi]}$. If we invert ℓ, this was studied systematically by Bushnell-Henniart. From our perspective a key result of theirs is:

Theorem 3.1. *The natural map*

$$A_{[L,\pi]} \otimes_{W(k)} \mathcal{K} \to \mathrm{End}_{\mathcal{K}[G]}(W_{[L,\pi]} \otimes_{W(k)} \mathcal{K})$$

is an isomorphism. In particular, $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$ *is commutative.*

Proof. We can check the first statement after base change from $\mathcal{K}$ to $\overline{\mathcal{K}}$; it then follows from [BH], Theorem 4.3. The second statement is immediate, as $W_{[L,\pi]}$ is ℓ-torsion free and thus its endomorphism ring embeds in $\mathrm{End}_{\mathcal{K}[G]}(W_{[L,\pi]} \otimes_{W(k)} \mathcal{K})$. □

To go further, we recall that we have an "nth derivative functor" $V \mapsto V^{(n)}$ from $W(k)[G]$-modules to $W(k)$ modules with the following properties:

(1) If V is an irreducible $k[G]$-module, then $V^{(n)}$ is a one-dimensional k-vector space if V is generic, and zero otherwise.
(2) There is a natural isomorphism of functors $\mathrm{Hom}_{W(k)[G]}(W, -) \to (-)^{(n)}$.
(3) The functor $V \mapsto V^{(n)}$ is exact.
(4) If V is an $A[G]$-module for some $W(k)$-algebra A, and B is an A-algebra, $(V \otimes_A B)^{(n)}$ is naturally isomorphic to $V^{(n)} \otimes_A B$.
(5) If V and W are $A[\mathrm{GL}_n(F)]$ and $A[\mathrm{GL}_m(F)]$-modules, respectively, then there is a natural isomorphism:
$$[i_P^{\mathrm{GL}_{n+m}(F)} V \otimes W]^{(n+m)} \cong V^{(n)} \otimes W^{(m)}.$$
(6) If V is an $A[\mathrm{GL}_n(F)]$-module, there is a natural A-linear surjection $V \to V^{(n)}$.

Lemma 3.2. *The top derivative* $W_{[L,\pi]}^{(n)}$ *is free of rank one over* $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$.

Proof. Property (2) of the derivative gives a natural isomorphism:

$$W_{[L,\pi]}^{(n)} \to \mathrm{Hom}_{W(k)[G]}(W, W_{[L,\pi]}),$$

and the latter is clearly isomorphic to $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$. □

Our goal is to apply this structure theory to ideas from [EH]. We first recall the definition of an essentially AIG representation [EH], 3.2.1.

Definition 3.3. Let κ be a $W(k)$-algebra that is a field. A $\kappa[G]$-module V is *essentially AIG* if:

(1) the socle of V is absolutely irreducible and generic,
(2) the quotient $V/\mathrm{soc}(V)$ has no generic subquotients (or, equivalently, the top derivative $(V/\mathrm{soc}(V))^{(n)}$ vanishes), and
(3) V is the sum of its finite length submodules.

We will need the following "dual version" of this (c.f. [EH], Lemma 6.3.5).

Lemma 3.4. *Let κ be a $W(k)$-algebra that is a field, and let V be a finite length admissible $\kappa[G]$-module such that the cosocle of V is absolutely irreducible and generic, and such that $V^{(n)}$ is a one-dimensional κ-vector space. Then the smooth κ-dual of V is essentially AIG.*

It follows easily from the definitions (see [EH], Lemma 3.2.3 for details) that the only endomorphisms of an essentially AIG $\kappa[G]$-module V are scalars. In particular such a V is indecomposable and lies in a single Bernstein component. If V lies in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, then $A_{[L,\pi]}$ acts on V, and this action factors through a map $f_V : A_{[L,\pi]} \to \kappa$.

4. Projective objects and their derivatives

We now recall some facts from [H1] about certain projective objects of $\mathrm{Rep}_{W(k)}(G)$ and the action of the Bernstein center on them. Let (K, τ) be a maximal distingushed cuspidal k-type of G in the sense of [V1], IV.3.1B.

From such a type we can construct a projective $W(k)[G]$-module $\mathcal{P}_{K,\tau}$ by a construction detailed in section 4 of [H1] (see particularly Lemmas 4.7 and 4.8, and the paragraph immediately following them). We recall some of the details of this construction.

Because (K, τ) is a maximal distinguished cuspidal k-type, we have a surjection $K \to \mathrm{GL}_{\frac{n}{ef}}(\mathbb{F}_{q^f})$ (for certain divisors e and f of n) whose kernel K_1 is a pro-p group. Moreover, we have a decomposition $\tau = \xi \otimes \sigma$, where σ is the inflation to K of an irreducible cuspidal representation of $\mathrm{GL}_{\frac{n}{ef}}(\mathbb{F}_{q^f})$, and ξ is a certain representation of K over k whose restriction to K_1 is irreducible. (Note that ξ is the representation denoted by κ in section 4 of [H1]; we have deviated from this notation in order to avoid conflicts later in this section.)

Then Lemma 4.7 of [H1] shows that ξ lifts to a $W(k)[K]$-module $\tilde{\xi}$ that is free as a $W(k)$-module. This $\tilde{\xi}$ depends on choices, but if one forms the tensor product $\tilde{\xi} \otimes \mathcal{P}_\sigma$, where $\mathcal{P}_\sigma$ is the projective envelope of σ in the category of $W(k)[K]$-modules, then Lemma 4.8 of [H1] shows that $\tilde{\xi} \otimes \mathcal{P}_\sigma$ is independent, up to isomorphism, of this choice. We set $\mathcal{P}_{K,\tau} = \text{c-Ind}_K^G \tilde{\xi} \otimes \mathcal{P}_\sigma$.

It will be useful later to note that the surjection of $\mathcal{P}_\sigma$ onto σ induces a surjection of $\tilde{\xi} \otimes \mathcal{P}_\sigma$ onto τ, and hence a natural surjection

$$\mathcal{P}_{K,\tau} \to \text{c-Ind}_K^G \tau.$$

Since (K, τ) is a maximal distinguished cuspidal k-type, any two irreducible cuspidal k-representations of G containing (K, τ) differ by an unramified twist. The supercuspidal support of such a k-representation is thus well-defined up to inertial equivalence; in particular we can represent it by a pair (L, π). Moreover, Vigneras' classification of cuspidal k-representations of G (see for instance [V1], section V) shows that we can take L to be a product of m copies of $\mathrm{GL}_{\frac{n}{m}}(F)$, and π to have the form $(\pi')^{\otimes m}$ for a certain irreducible supercuspidal representation π' of $\mathrm{GL}_{\frac{n}{m}}(F)$. The classification further shows that m lies in the set $\{1, m(\pi'), m(\pi')\ell, m(\pi')\ell^2, \dots\}$ for a certain integer $m(\pi')$ depending only on π'.

There is a close relationship between $\mathcal{P}_{K,\tau}$ and the center of the block corresponding to (L, π). In particular, we have:

Theorem 4.1. *The representation $\mathcal{P}_{K,\tau}$ lies in the block $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$. Moreover, the map*

$$A_{[L,\pi]} \to \mathrm{End}_{W(k)[G]}(\mathcal{P}_{K,\tau})$$

is an isomorphism, and makes $\mathcal{P}_{K,\tau}$ into an admissible $A_{[L,\pi]}$-module.

Proof. This is a composite of results from [H1], in particular Corollary 10.19 and Corollary 11.11. Admissibility follows from Theorem 8.8. □

Let $E_{K,\tau}$ be the ring $\mathrm{End}_{W(k)[G]}(\mathcal{P}_{K,\tau})$. We identify $A_{[L,\pi]}$ with $E_{K,\tau}$ for the remainder of this section.

Lemma 4.2. *Let $\mathfrak{p}$ be a prime of $E_{K,\tau}$, of residue field $\kappa(\mathfrak{p})$. Then $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ has finite length.*

Proof. Admissibility of $\mathcal{P}_{K,\tau}$ over $E_{K,\tau}$ implies that $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ is an admissible $\kappa(\mathfrak{p})[G]$-module. The action of the Bernstein center on $\mathcal{P}_{K,\tau}$ factors through $A_{[L,\pi]}$, and therefore so does the action of the Bernstein center on $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$. In particular the latter is contained in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$. On the other hand, only a finite collection of blocks of $\mathrm{Rep}_{\kappa(\mathfrak{p})}(G)$ map to a given block of $\mathrm{Rep}_{W(k)}(G)$ under the forgetful functor $\mathrm{Rep}_{\kappa(\mathfrak{p})}(G) \to \mathrm{Rep}_{W(k)}(G)$. (Indeed, if $\mathfrak{p}$ has characteristic ℓ, then any irreducible object of $\mathrm{Rep}_{\kappa(\mathfrak{p})}(G)$ that lives in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ has supercuspidal support given, up to a twist, by $[L,\pi]$, and hence lives in $\mathrm{Rep}_{\kappa(\mathfrak{p})}(G)_{[L,\pi]}$. On the other hand, if $\mathfrak{p}$ has characteristic zero, and Π is an irreducible object of $\mathrm{Rep}_{\kappa(\mathfrak{p})}(G)$ that lives in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, then the supercuspidal support of Π (over $\kappa(\mathfrak{p})$) is an irreducible supercuspidal $\kappa(\mathfrak{p})$-representation $\tilde{\pi}$ of a Levi M of G. After an unramified twist, we may assume that this representation is integral and defined over $\overline{\mathcal{K}}$; it then lives in one of the finitely many inertial equivalence classes of pairs $(M,\tilde{\pi})$ whose reduction mod ℓ has inertial supercuspidal support (L,π).)

Thus $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ lives in a finite collection of blocks of $\mathrm{Rep}_{\kappa(\mathfrak{p})}(G)$ and (as an admissible $\kappa(\mathfrak{p})[G]$-module) therefore has finite length. □

Lemma 4.3. *The cosocle of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ is absolutely irreducible.*

Proof. Let $\tilde{C}$ be the cosocle of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$. It suffices to show that $\mathrm{End}_{\kappa(\mathfrak{p})}(\tilde{C})$ is isomorphic to $\kappa(\mathfrak{p})$. Let C be the image of $\mathcal{P}_{K,\tau}$ in $\tilde{C}$. The isomorphism of $E_{K,\tau}$ with $A_{[L,\pi]}$ gives an action of $E_{K,\tau}$ on C. Given an endomorphism of C, one can compose with the surjection of $\mathcal{P}_{K,\tau}$ onto C to obtain a map $\mathcal{P}_{K,\tau} \to C$. Because $\mathcal{P}_{K,\tau}$ is projective, such a map lifts to a map from $\mathcal{P}_{K,\tau}$ to itself; that is, an element of $E_{K,\tau}$. This shows that the map from $E_{K,\tau}$ to $\mathrm{End}_{W(k)[G]}(C)$ is surjective.

On the other hand, $\mathfrak{p}$ annihilates C in $E_{K,\tau}$, so the map $E_{K,\tau} \to \mathrm{End}_{W(k)[G]}(C)$ factors through $E_{K,\tau}/\mathfrak{p}$. Finally, the surjection of $E_{K,\tau}/\mathfrak{p}$ onto $\mathrm{End}_{W(k)[G]}(C)$ must be injective, as if not, there would be an element of $E_{K,\tau}$ outside $\mathfrak{p}$ that annihilates C. This is impossible as C is contained in the $\kappa(\mathfrak{p})$-vector space $\tilde{C}$, on which all such elements act invertibly. We thus have an isomorphism of $E_{K,\tau}/\mathfrak{p}$ with $\mathrm{End}_{W(k)[G]}(C)$.

Tensoring with $\kappa(\mathfrak{p})$ completes the proof. □

Proposition 4.4. *The following are equivalent:*

(1) *$\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ has an absolutely irreducible generic quotient.*
(2) *The smooth $\kappa(\mathfrak{p})$-dual of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ is essentially AIG.*

Proof. It is clear that (2) implies (1). Suppose conversely that (1) holds. It suffices to show that $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ has no irreducible generic subquotients other than its cosocle. Let $\tilde{C}$ denote this cosocle, and suppose we have:

$$N \subseteq \ker \left[\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}) \to \tilde{C}\right],$$

such that N has an irreducible generic quotient $\tilde{C}'$. The action of $A_{[L,\pi']}$ on both $\tilde{C}$ and $\tilde{C}'$ is via the same map

$$A_{[L,\pi']} \cong E_{K,\tau} \to \kappa(\mathfrak{p}).$$

Since $\tilde{C}$ and $\tilde{C}'$ are both generic, they are then isomorphic by Corollary 2.3. The surjection of N onto $\tilde{C}'$ thus gives a surjection of N onto $\tilde{C}$. The surjection

$$\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}) \to \tilde{C}$$

then lifts (by projectivity of $\mathcal{P}_{K,\tau}$) to a map

$$\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}) \to N.$$

Composing this with the inclusion of N in $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ gives a nonzero endomorphism of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ whose composition with the surjection

$$\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}) \to \tilde{C}$$

is the zero map. This is impossible since we have

$$\mathrm{End}_{\kappa(\mathfrak{p})}(\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})) \cong E_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}) = \kappa(\mathfrak{p}).$$

□

This motivates the following definition:

Definition 4.5. A point $\mathfrak{p}$ of $\operatorname{Spec} E_{K,\tau}$ is called an *essentially AIG point* if $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ satisfies the equivalent conditions of Proposition 4.4.

Let $\mathfrak{m}$ and $\mathfrak{p}$ be points of $\operatorname{Spec} E_{K,\tau}$, such that $\mathfrak{m}$ is in the closure of $\mathfrak{p}$. Then there is a discrete valuation ring $\mathcal{O} \subset \kappa(\mathfrak{p})$ that dominates $\mathfrak{m}$; that is, such that $\mathcal{O}$ contains the image of $E_{K,\tau}$ in $\kappa(\mathfrak{p})$, and the preimage of the maximal ideal $\tilde{\mathfrak{m}}$ of $\mathcal{O}$ under the map $E_{K,\tau} \to \mathcal{O}$ is equal to $\mathfrak{m}$.

By Corollary 2.3, there is a unique irreducible generic representation $\tilde{C}$ of G over $\kappa(\mathfrak{p})$ on which $E_{K,\tau}$ acts via the natural map $E_{K,\tau} \to \kappa(\mathfrak{p})$. Then the supercuspidal support of $\tilde{C}$ is $\mathcal{O}$-integral, and so $\tilde{C}$ is $\mathcal{O}$-integral as well.

Consider the smooth $\kappa(\mathfrak{p})$-dual $\tilde{C}^\vee$ of $\tilde{C}$. Then by [EH], Proposition 3.3.2, there is an $\mathcal{O}$-lattice $C^\vee$ in $\tilde{C}^\vee$ such that the reduction $C^\vee \otimes_{\mathcal{O}} \mathcal{O}/\tilde{\mathfrak{m}}$ has an absolutely irreducible generic socle. If we let C be the smooth $\mathcal{O}$-dual of $C^\vee$, then $C \otimes_{\mathcal{O}} \mathcal{O}/\tilde{\mathfrak{m}}$ is an $\mathcal{O}$-lattice in $\tilde{C}$ with an absolutely irreducible generic quotient. Denote this quotient by $\overline{C}$.

Lemma 4.6. *If* $\mathfrak{m}$ *is an essentially AIG point, then so is* $\mathfrak{p}$*.*

Proof. Note that $E_{K,\tau}$ acts on $\overline{C}$ via the map $E_{K,\tau} \to \kappa(\mathfrak{m})$, and so by Corollary 2.3, $\overline{C}$ is the unique absolutely irreducible generic representation of G over k on which $E_{K,\tau}$ acts via this map. Since $\mathfrak{m}$ is an essentially AIG point, we have a nonzero map $\mathcal{P}_{K,\tau} \to \overline{C}$. Projectivity of $\mathcal{P}_{K,\tau}$ lets us lift this to a map $\mathcal{P}_{K,\tau} \to C$, and hence to a map $\mathcal{P}_{K,\tau} \to \tilde{C}$. Thus $\mathfrak{p}$ is an essentially AIG point. □

A partial converse also holds:

Lemma 4.7. *If $\mathfrak{p}$ is an essentially AIG point, and $C \otimes_{\mathcal{O}} \mathcal{O}/\tilde{\mathfrak{m}}$ is absolutely irreducible, then $\mathfrak{m}$ is an essentially AIG point.*

Proof. If $\mathfrak{p}$ is an essentially AIG point, then we have a nonzero map $\mathcal{P}_{K,\tau} \to \tilde{C}$. The image of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \mathcal{O}$ in $\tilde{C}$ is an admissible $\mathcal{O}[G]$-submodule of $\tilde{C}$ and thus a G-stable $\mathcal{O}$-lattice in $\tilde{C}$. As $C/\tilde{\mathfrak{m}}C$ is absolutely irreducible, such an $\mathcal{O}$-lattice must be homothetic to C. We thus get a surjection of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \mathcal{O}$ onto C. Composing with the map $C \to \overline{C}$ yields a nonzero map $\mathcal{P}_{K,\tau} \to \overline{C}$ with kernel $\mathfrak{m}$, and the result follows. □

Of course, neither lemma is of much use without knowing some essentially AIG points of $\operatorname{Spec} E_{K,\tau}$. The following result provides such points:

Lemma 4.8. *Let $\mathfrak{p}$ be a point of $\operatorname{Spec} E_{K,\tau}$, and suppose that there exists an irreducible cuspidal representation V of G over $\kappa(\mathfrak{p})$ on which $E_{K,\tau}$ acts via the natural map $E_{K,\tau} \to \kappa(\mathfrak{p})$. Then $\mathfrak{p}$ is an essentially AIG point.*

Proof. If $\mathfrak{p}$ has characteristic zero, then V is supercuspidal and is therefore the unique irreducible representation with supercuspidal support (G, π), and hence the only irreducible representation on which $E_{K,\tau}$ acts via $E_{K,\tau} \to \kappa(\mathfrak{p})$. It follows that every subquotient of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ is isomorphic to π in this case. In particular $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ has an absolutely irreducible generic quotient, and so by Proposition 4.4, $\mathfrak{p}$ is an essentially AIG point. (Since V is generic, this means that $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ is actually isomorphic to V in this case.)

We may thus assume that $\mathfrak{p}$ has characteristic ℓ. Then V is a cuspidal k-representation of G with supercuspidal support (L, π). It follows that V contains the cuspidal type (K, τ). In particular we have a map

$$\text{c-Ind}_K^G \tau \to V.$$

On the other hand, composing with the natural surjection of $\mathcal{P}_{K,\tau}$ onto $\text{c-Ind}_K^G \tau$ yields a nonzero map of $\mathcal{P}_{K,\tau}$ onto V, and this map is annihilated by $\mathfrak{p}$. Tensoring with $\kappa(\mathfrak{p})$ gives a surjection of $\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p})$ onto V, proving the claim. □

Combining the above observations, we may now prove:

Proposition 4.9. *Every point $\mathfrak{m}$ of $\operatorname{Spec} E_{K,\tau}$ is an essentially AIG point.*

Proof. We will prove this by constructing points $\mathfrak{p}$ and $\mathfrak{m}'$ of $\operatorname{Spec} E_{K,\tau}$, such that $\mathfrak{m}$ and $\mathfrak{m}'$ are in the closure of $\mathfrak{p}$, and the unique irreducible generic representation on which $E_{K,\tau}$ acts via $E_{K,\tau} \to \kappa(\mathfrak{m}')$ is cuspidal. Then $\mathfrak{m}'$ is essentially AIG by Lemma 4.8, and so $\mathfrak{p}$ is an essentially AIG point by Lemma 4.6.

Thus, if we can further arrange that the pair $(\mathfrak{m}, \mathfrak{p})$ satisfies the hypotheses of Lemma 4.7, the claim will follow. To do this we make use of the recent classification of modular representations of G due to Mínguez-Sécherre [MS]. Let $\overline{C}$ be the unique absolutely irreducible generic representation on which $E_{K,\tau}$ acts via $E_{K,\tau} \to \kappa(\mathfrak{m})$.

By [MS], Théorème 9.10, the representation $\overline{C}$ is the parabolic induction of the tensor product

$$L(\Delta_1) \otimes \cdots \otimes L(\Delta_r),$$

where each Δ_i is a *segment* in the sense of [MS], Définition 7.1, and $L(\Delta_i)$ is the representation attached to Δ_i by [MS], Définition 7.5. Moreover, the segments Δ_i and Δ_j are not linked ([MS], Définition 7.3) for any i, j, and for each i, the segment

$\Delta_i = [a_i, b_i]_{\pi_i}$ has length less than $m(\pi_i)$. It follows that the multiset $\{\Delta_1, \dots, \Delta_r\}$ is *aperiodic* in the sense of [MS], Définition 9.7.

Let $\tilde{\kappa}$ be the field of fractions of $\kappa(\mathfrak{m})[T_1, \dots, T_r]$, and let $\chi_1, \dots, \chi_r$ be the unramified characters $F^\times \to \tilde{\kappa}^\times$ such that χ_i takes the value T_i on a uniformizer ϖ_F of F. For each i, let Δ'_i be the twist $\Delta_i \otimes \chi_i$, and let $\tilde{C}$ be the parabolic induction of the tensor product

$$L(\Delta'_1) \otimes \cdots \otimes L(\Delta'_r).$$

The segments Δ'_i and Δ'_j are not linked for any i, j, so $\tilde{C}$ is absolutely irreducible and generic. Let $\mathcal{O}$ be a valuation ring of $\tilde{\kappa}$ that dominates the maximal ideal $\langle T_i - 1 \rangle$ in $\kappa(\mathfrak{m})[T_1, \dots, T_r]$. Then $\tilde{C}$ contains a stable $\mathcal{O}$-lattice C, and the reduction $C \otimes_{\mathcal{O}} \kappa(\mathfrak{m})$ is isomorphic to $\overline{C}$. In particular $\tilde{C}$ lies in the same block as $\overline{C}$ and thus admits an action of $E_{K,\tau}$; let $\mathfrak{p}$ be the kernel of this action. It is clear that $\kappa(\mathfrak{p}) = \tilde{\kappa}$. Our construction shows that $\mathfrak{m}$ and $\mathfrak{p}$ satisfy the hypotheses of Lemma 4.7, so that $\mathfrak{m}$ is an essentially AIG point if $\mathfrak{p}$ is.

It remains to construct an essentially AIG point $\mathfrak{m}'$ in the closure of $\mathfrak{p}$. The segments Δ_i have the form $[a_i, b_i]_{\pi_i}$, where π_i is a cuspidal representation. In the notation of [MS], section 6.2, the cuspidal representation π_i of $\mathrm{GL}_{n_i}(F)$ has the form $\mathrm{St}(\pi'_i, m)$ for some integer m_i and supercuspidal representation π'_i of $\mathrm{GL}_{\frac{n_i}{m_i}}(F)$.

The supercuspidal support of $\mathrm{St}(\pi'_i, m_i)$, considered as a multiset of supercuspidal representations, is given by

$$\pi'_i, \pi'_i \otimes \nu, \dots, \pi'_i \otimes \nu^{m_i - 1},$$

where ν is the absolute value character. It follows that the supercuspidal support of $L(\Delta_i)$ is given by the sum of the multisets

$$\pi'_i \otimes \chi'_i \otimes \nu^j, \pi'_i \otimes \chi'_i \otimes \nu^{j+1}, \dots, \pi'_i \otimes \chi'_i \otimes \nu^{j+m_i-1}$$

as j ranges from a_i to b_i.

On the other hand, the parabolic induction of the $L(\Delta_i)$ lies in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, and its supercuspidal support is thus inertially equivalent to (L, π). As a multiset, this means its supercuspidal support is a sum of unramified twists of π'. (Recall that we have $\pi = (\pi')^{\otimes m}$.) Thus π'_i is an unramified twist of π' for all i.

Thus, if we choose suitable unramified characters $\chi''_i : F^\times \to \kappa(\mathfrak{m})^\times$, and set $\Delta''_i = \Delta_i \otimes \chi''_i$ we can arrange that, when considered as multisets of supercuspidal representations, the sum of the supercuspidal supports of the $L(\Delta''_i)$ coincides with the supercuspidal support of the cuspidal representation $\mathrm{St}(\pi', m)$.

It then follows (by the correspondence between $\kappa(\mathfrak{m})$-points of $E_{K,\tau}$ and supercuspidal supports) that $E_{K,\tau}$ acts on the parabolic induction of the tensor product

$$L(\Delta''_1) \otimes \cdots \otimes L(\Delta''_r)$$

by the map $E_{K,\tau} \to \kappa(\mathfrak{m})$ that gives the action of $E_{K,\tau}$ on $\mathrm{St}(\pi', m)$.

Observe that the natural map $E_{K,\tau} \to \kappa(\mathfrak{p})$ factors through the inclusion:

$$\kappa(\mathfrak{m})[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to \kappa(\mathfrak{p}).$$

Let $c_i = \chi''_i(\varpi_F)$, and let $\mathfrak{m}'$ be the preimage of the maximal ideal $\langle T_i - c_i \rangle$ of $\kappa(\mathfrak{m})[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$ in $E_{K,\tau}$. Then $\mathfrak{m}'$ is the kernel of the map giving the action of $E_{K,\tau}$ on $\mathrm{St}_{\pi',m}$; that is $\mathfrak{m}'$ satisfies the hypotheses of Lemmma 4.8. In particular $\mathfrak{m}'$ is an essentially AIG point in the closure of $\mathfrak{p}$, and the result follows. □

Corollary 4.10. *The space $\mathcal{P}^{(n)}_{K,\tau}$ is locally free of rank one over $E_{K,\tau}$.*

Proof. Admissibility of $\mathcal{P}_{K,\tau}$ as an $E_{K,\tau}[G]$-module (and hence as an $A_{[L,\pi']}[G]$-module) implies by [EH], 3.1.14, that $\mathcal{P}_{K,\tau}^{(n)}$ is finitely generated over $E_{K,\tau}$. On the other hand, for any prime $\mathfrak{p}$ of $E_{K,\tau}$ we have

$$\mathcal{P}_{K,\tau}^{(n)} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}) \cong (\mathcal{P}_{K,\tau} \otimes_{E_{K,\tau}} \kappa(\mathfrak{p}))^{(n)}$$

by property (4) of the derivative. As $\mathfrak{p}$ is an essentially AIG point the right-hand side is a one-dimensional $\kappa(\mathfrak{p})$ vector space; the result follows. □

5. Admissibility of $W_{L,\pi}$

Now let L be an arbitrary Levi subgroup of GL_n, and let π be a supercuspidal representation of L. Our goal is to use the results of the previous section to show that the module $W_{L,\pi}$ is admissible over $A_{L,\pi}$. We must first relate $A_{L,\pi}$ to the spaces $\mathcal{P}_{K,\tau}$ studied above; we do so by invoking results of [H1] which we now recall:

Section 11 of [H1] attaches to the pair (L,π) a Levi subgroup $M^{\max}$ of G, containing L and an irreducible cuspidal k-representation $\pi^{\max}$ of $M^{\max}$, whose definitions we now recall. Twisting by an unramified character if necessary, we may assume that π has the form

$$\pi = \bigotimes_i \pi_i^{\otimes m_i}$$

where π_i is an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ and π_i is not an unramified twist of π_j for $i \neq j$.

Suppose we have a cuspidal k-representation π_{ij} of $\mathrm{GL}_{jn_i}(F)$ whose supercuspidal support is an unramified twist of $\pi_i^{\otimes j}$. Vigneras' classification shows that then j lies in the set $\{1, m(\pi_i), \ell m(\pi_i), \ell^2 m(\pi_i), \dots\}$, and that, once j and π_i are fixed, π_{ij} is determined up to an unramified twist.

For each i, let $\nu_i = \{\nu_{i,1}, \dots, \nu_{i,k_i}\}$ be the coarsest partition of m_i whose elements lie in the set $\{1, m(\pi_i), \ell m(\pi_i), \ell^2 m(\pi_i), \dots\}$. For each j with $1 \leq j \leq k_i$, we have a cuspidal representation $\pi_{i\nu_{i,j}}$ of $\mathrm{GL}_{\nu_{i,j}n_i}(F)$. Let $\pi^{\max}$ be the tensor product:

$$\pi^{\max} = \bigotimes_i \bigotimes_{j=1}^{k_i} \pi_{i\nu_{i,j}}.$$

(This is well-defined only up to unramified twist, but this poses no problem for our purposes.) We may regard $\pi^{\max}$ as a cuspidal representation of a Levi $M^{\max}$ of G whose block sizes are given by the $\nu_{i,j}n_i$. The pair $(M^{\max}, \pi^{\max})$ is the cuspidal support of the induction $\times_i \mathrm{St}(\pi_i, m_i)$. The supercuspidal support of this pair is an unramified twist of (L,π).

Morally (c.f. the paragraph before Proposition 11.10 in [H1],) $M^{\max}, \pi^{\max}$ is "farthest from supercuspidal" among the pairs (M,π') such π' is a cuspidal representation of some Levi M and the supercuspidal support of (M,π') lies in the inertial equivalence class of (L,π).

Following section 11 of [H1], we attach to $(M^{\max}, \pi^{\max})$ a maximal projective $W(k)[G]$-module as follows. Each $\pi_{i\nu_{i,j}}$ contains a maximal distinguished k-type (K_{ij}, τ_{ij}). We can thus form the representation $\otimes_{i,j}\mathcal{P}_{K_{ij},\tau_{ij}}$ of $M^{\max}$. Parabolically inducing from $M^{\max}$ to G then yields a projective module

$$\mathcal{P}_{M^{\max},\pi^{\max}} := i_P^G \otimes_{i,j} \mathcal{P}_{K_{ij},\tau_{ij}}$$

for a suitable parabolic P of G with Levi $M^{\max}$. The resulting object is a projective object of $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$.

To simplify our notation, from this point forward we linearly order the pairs (K_{ij}, τ_{ij}), relabeling them as $(K_1, \tau_1), \dots (K_r, \tau_r)$ in any way whatsoever.

We now consider the algebra $B_{M^{\max},\pi^{\max}}$ introduced after the proof of Lemma 11.2 in [H1]. By definition, this is the subalgebra of $A_{[L,\pi]} \otimes \overline{\mathcal{K}}$ that preserves $\mathcal{P}_{M^{\max},\pi^{\max}}$ inside $\mathcal{P}_{M^{\max},\pi^{\max}} \otimes \overline{\mathcal{K}}$. Note that $B_{M^{\max},\pi^{\max}}$ contains $A_{[L,\pi]}$.

One of the key points of [H1] is then:

Theorem 5.1 ([H1], Corollary 11.11)**.** *The inclusion of $A_{[L,\pi]}$ in $B_{M^{\max},\pi^{\max}}$ is an isomorphism.*

Moreover, one has a very nice description of the image of $A_{[L,\pi]}$ in the endomorphism ring of $\mathcal{P}_{M^{\max},\pi^{\max}}$. We first need a few remarks about this ring:

Let E_i be the endomorphism ring E_{K_i,τ_i}. Then the tensor product $\otimes_i E_i$ acts on $\otimes_i \mathcal{P}_{K_i,\tau_i}$ and thus, by functoriality of parabolic induction, on $\mathcal{P}_{M^{\max},\pi^{\max}}$. Moreover, we have:

Proposition 5.2. *The action of $A_{[L,\pi]}$ on $\mathcal{P}_{M^{\max},\pi^{\max}}$ factors uniquely through the map*

$$\otimes_i E_i \to \mathrm{End}_{W(k)[G]}(\mathcal{P}_{M^{\max},\pi^{\max}}),$$

and the resulting map $A_{[L,\pi]} \to \otimes_i E_i$ is an injection. Moreover, if x is an element of $\otimes_i E_i$ such that $\ell^r x$ is in the image of $A_{[L,\pi]}$, then x is also in the image of $A_{[L,\pi]}$.

Proof. By Lemma 11.4 of [H1], the action of $B_{M^{\max},\pi^{\max}}$ on $\mathcal{P}_{M^{\max},\pi^{\max}}$ factors uniquely through the map

$$\otimes_i E_i \to \mathrm{End}_{W(k)[G]}(\mathcal{P}_{M^{\max},\pi^{\max}}).$$

Corollary 11.11 of [H1] then allows us to conclude that the same holds for $A_{[L,\pi]}$, and that the resulting map is an injection.

For the last statement, note that it is clear from the definition of $B_{M^{\max},\pi^{\max}}$ that if x is any endomorphism of $\mathcal{P}_{M^{\max},\pi^{\max}}$, and ℓx lies in $B_{M^{\max},\pi^{\max}}$, then so does x. The result follows. $\square$

The module $\mathcal{P}_{M^{\max},\pi^{\max}}$ turns out to be well-suited to understanding the endomorphisms of $W_{[L,\pi]}$. In particular, the top derivative $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$ has an action of this endomorphism ring that we will show extends the action of $A_{[L,\pi]}$ in a natural way. On the other hand, Proposition 5.2 and Corollary 4.10 give us control over the endomorphisms of $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$. Combining these two allows us to prove:

Theorem 5.3. *The natural map $A_{[L,\pi]} \to \mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$ is an isomorphism.*

Proof. By Theorem 3.1 this map becomes an isomorphism after tensoring with $\overline{\mathcal{K}}$, and hence after inverting ℓ. But $A_{[L,\pi]}$ is ℓ-torsion free by Theorem 2.1, and $W_{[L,\pi]}$ is ℓ-torsion free by definition. It follows that the map in question is injective. It thus suffices to show surjectivity.

We have an isomorphism:

$$\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}} \cong \mathrm{Hom}_{W(k)[G]}(W_{[L,\pi]}, \mathcal{P}_{M^{\max},\pi^{\max}}),$$

by property (2) of the derivative, and therefore an action of $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$ on $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$. We also have actions of $A_{[L,\pi]}$ on $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$ (via the action of $A_{[L,\pi]}$ on $\mathcal{P}_{M^{\max},\pi^{\max}}$) and on $W_{[L,\pi]}$. The latter action induces an action of $A_{[L,\pi]}$

on $\mathrm{Hom}_{W(k)[G]}(W_{[L,\pi]}, \mathcal{P}_{M^{\max},\pi^{\max}})$, and thus (via the identification of this Hom-space with $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$), a second action of $A_{[L,\pi]}$ on $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$. Since any map in $\mathrm{Hom}_{W(k)[G]}(W_{[L,\pi]}, \mathcal{P}_{M^{\max},\pi^{\max}})$ is $A_{[L,\pi]}$-equivariant these two actions coincide. In particular the action of $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$ on $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$ extends the action of $A_{[L,\pi]}$ on $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$, where the latter action is induced by the action of $A_{[L,\pi]}$ on $\mathcal{P}_{M^{\max},\pi^{\max}}$.

On the other hand, the multiplicativity of the top derivative with respect to parabolic induction (property (5) above) yields an isomorphism of $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$ with the tensor product of the invertible E_i-modules $\mathcal{P}^{(n_i)}_{K_i,\tau_i}$. The action of $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$ on $\mathcal{P}^{(n)}_{M^{\max},\pi^{\max}}$ thus yields a map

$$\mathrm{End}_{W(k)[G]}(W_{[L,\pi]}) \to \otimes_i E_i$$

that extends the inclusion of $A_{L,\pi}$ into $\otimes_i E_i$. Such an extension is necessarily injective, as $\otimes_i E_i$ is ℓ-torsion free.

We thus have inclusions:

$$A_{[L,\pi]} \subseteq \mathrm{End}_{W(k)[G]}(W_{[L,\pi]}) \subseteq \otimes_i E_i.$$

Now let x be an element of $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$. Then for some positive integer a, $\ell^a x$ lies in $A_{[L,\pi]}$. But then by Proposition 5.2, x must have been an element of $A_{[L,\pi]}$ to start with. The result follows. □

Proposition 5.4. *The module $W_{[L,\pi]}$ is admissible as an $A_{[L,\pi]}[G]$-module.*

Proof. Theorem 5.3, together with Lemma 3.2, show that $W^{(n)}_{[L,\pi]}$ is free of rank one over $A_{[L,\pi]}$. Let x be an element of $W^{(n)}_{[L,\pi]}$ that generates $W^{(n)}_{[L,\pi]}$ as an $A_{[L,\pi]}$-module, and let $\tilde{x}$ be an element of $W_{[L,\pi]}$ that maps to x via the natural surjection:

$$W_{[L,\pi]} \to W^{(n)}_{[L,\pi]}$$

given by property (6) of the derivative.

Let W' be the $W(k)[G]$-submodule of $W_{[L,\pi]}$ generated by $\tilde{x}$. The inclusion of W' in $W_{[L,\pi]}$ gives an inclusion of $(W')^{(n)}$ in $W^{(n)}_{[L,\pi]}$ whose image contains x; it follows that $(W')^{(n)}$ is equal to $W^{(n)}_{[L,\pi]}$. Thus $(W_{[L,\pi]}/W')^{(n)} = 0$. But the latter is naturally isomorphic to $\mathrm{Hom}_{W(k)[G]}(W_{[L,\pi]}, W_{[L,\pi]}/W')$, so we must have $W' = W_{[L,\pi]}$. In particular $W_{[L,\pi]}$ is finitely generated over $W(k)[G]$ and is thus (by [H1], Corollary 12.4) admissible over $A_{[L,\pi]}$. □

We now have the following "structure theory" for essentially AIG $k[G]$-modules (or, more precisely, their duals):

Proposition 5.5. *Let κ be a $W(k)$-algebra that is a field, and let $f : A_{[L,\pi]} \to \kappa$ be a map of $W(k)$-algebras. Then $[W_{[L,\pi]} \otimes_{A_{[L,\pi]},f} \kappa]^{\vee}$ is essentially AIG, where the superscript $\vee$ denotes κ-dual.*

Conversely, let V be an object of $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ that is the smooth κ-dual of an essentially AIG $\kappa[G]$-module. Then V is a quotient of $W_{[L,\pi]} \otimes_{A_{[L,\pi]},f} \kappa$ for some $f : A_{[L,\pi]} \to \kappa$.

Proof. By Theorem 5.3, together with the fact that $W_{[L,\pi]}^{(n)}$ is free of rank one over $\mathrm{End}_{W(k)[G]}(W_{[L,\pi]})$ (Lemma 3.2) we find that $[W_{[L,\pi]} \otimes_{A_{[L,\pi]},f} \kappa]^{(n)}$ is a one-dimensional κ-vector space. Moreover, if π is any nonzero quotient of $W_{[L,\pi]} \otimes_{A_{[L,\pi]},f} \kappa$, then $\mathrm{Hom}_{W(k)[G]}(W, \pi)$ is nonzero, so that $\pi^{(n)}$ is nonzero by property (2) of the derivative. Thus, by [EH], Lemma 6.3.5, the κ-dual of $W_{[L,\pi]} \otimes_{A_{[L,\pi]},f} \kappa$ is essentially AIG.

Now let V be a $\kappa[G]$ module in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, such that V is the smooth dual of an essentially AIG module. Then V has an absolutely irreducible generic cosocle V_0, and $A_{[L,\pi]}$ acts on V_0 by a map $f : A_{[L,\pi]} \to \kappa$. As the only endomorphisms of V are scalars, $A_{[L,\pi]}$ acts on V via f, as well.

As V_0 is irreducible and generic, and is an object of $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$, we have a nonzero map of $W_{[L,\pi]}$ onto V_0; projectivity of $W_{[L,\pi]}$ lifts this to a map $W_{[L,\pi]} \to V$. Let $\mathfrak{p}$ be the kernel of f; then $\mathfrak{p}$ acts by zero on V, so the map $W_{[L,\pi]} \to V$ descends to a nonzero map:

$$W_{[L,\pi]}/\mathfrak{p}W_{[L,\pi]} \to V;$$

as V is a κ-vector space, and $A/\mathfrak{p}$ is contained in κ, this extends to a map:

$$W_{[L,\pi]} \otimes_A \kappa \to V$$

whose composition with the map $V \to V_0$ is nonzero. This map is necessarily surjective (if not, its image would be contained in the kernel of the map $V \to V_0$, as V_0 is the cosocle of V). □

Corollary 5.6. *Every essentially AIG $\kappa[G]$-module has finite length.*

Proof. Let V be an essentially AIG $\kappa[G]$-module. It is then indecomposable, and hence contained in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ for some (L, π). The smooth κ-dual of V is then a quotient of the representation $W_{[L,\pi]} \otimes_{A_{[L,\pi]},f} \kappa$ for some $f : W_{[L,\pi]} \to \kappa$. As the latter is an admissible $\kappa[G]$-module, it has finite length. □

This verifies a conjecture of [EH] (see the remarks after Lemma 3.2.8). In particular, we can conclude from this that for any absolutely irreducible generic $\kappa[G]$-module π, its essentially AIG envelope (in the sense of [EH], Definition 3.2.6) has finite length. It thus follows from the corollary that one has an upper bound on the length of any essentially AIG module with socle π entirely in terms of the isomorphism class of π.

6. co-Whittaker modules

We now describe a sense in which $W_{[L,\pi]}$ can be thought of as a "universal object" over $A_{[L,\pi]}$. This will turn out to be crucial to our reinterpretation of the results of [EH].

Let A be a Noetherian $W(k)$-algebra; then any smooth $A[G]$-module carries an action of the center of $\mathrm{Rep}_{W(k)}(G)$, and hence admits a Bernstein decomposition. Thus $\mathrm{Rep}_A(G)$ decomposes as a product of Bernstein components $\mathrm{Rep}_A(G)_{[L,\pi]}$.

Definition 6.1. An object V of $\mathrm{Rep}_A(G)_{[L,\pi]}$ is a *co-Whittaker $A[G]$-module* if the following hold:

1. V is admissible as an $A[G]$-module.
2. $V^{(n)}$ is free of rank one over A.
3. If $\mathfrak{p}$ is a prime ideal of A, with residue field $\kappa(\mathfrak{p})$, then the $\kappa(\mathfrak{p})$-dual of $V \otimes_A \kappa(\mathfrak{p})$ is essentially AIG.

If V and V' are co-Whittaker $A[G]$-modules, we say that V dominates V' if there exists a surjection from V to V'.

One motivation for this definition is that the families of admissible representations that correspond to Galois representations in the sense of section 6.2 of [EH] are co-Whittaker modules. We will discuss this further in section 7.

Proposition 6.2. *Let V be a co-Whittaker $A[G]$-module. Then the natural map*

$$A \to \mathrm{End}_{A[G]}(V)$$

is an isomorphism.

Proof. By localizing at each prime ideal of A, it suffices to consider the case in which A is a local ring. Lemma 6.3.2 of [EH] then shows that V is generated by a certain submodule $\mathfrak{J}(V)$ that is stable under any endomorphism of V. On the other hand, $\mathfrak{J}$ of V is stable under the mirabolic subgroup P_n of V, and, by Proposition 3.1.16 of [EH], together with the fact that $V^{(n)}$ is free of rank one over A, we have that the natural map $A \to \mathrm{End}_{A[P_n]}(\mathfrak{J}(V))$ is an isomorphism. This map factors as:

$$A \to \mathrm{End}_{A[G]}(V) \to \mathrm{End}_{A[P_n]}(\mathfrak{J}(V))$$

and the right-hand map is injective because V is generated by $\mathfrak{J}(V)$ over $A[G]$. The result follows. □

If V is a co-Whittaker $A[G]$-module of $\mathrm{Rep}_A(G)_{[L,\pi]}$, then it admits an action of $A_{[L,\pi]}$, which must arise from a unique map $f_V : A_{[L,\pi]} \to A$.

Theorem 6.3. *Let A be a Noetherian $A_{[L,\pi]}$-algebra. Then $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A$ is a co-Whittaker $A[G]$-module. Conversely, if V is a co-Whittaker $A[G]$-module in $\mathrm{Rep}_A(G)_{[L,\pi]}$, then A is an $A_{[L,\pi]}$-algebra via $f_V : A_{[L,\pi]} \to A$, and $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A$ dominates V.*

Proof. The module $W_{[L,\pi]}$ is admissible over $A_{[L,\pi]}[G]$ by Proposition 5.4, and therefore $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A$ is admissible over A. As $W^{(n)}_{[L,\pi]}$ is free of rank one over $A_{[L,\pi]}$ by Theorem 5.3, and the derivative operator commutes with base change, $[W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A]^{(n)}$ is free of rank one over A. Finally, by Proposition 5.5, $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} \kappa(\mathfrak{p})$ is dual to an essentially AIG representation for any prime ideal $\mathfrak{p}$ of $A_{[L,\pi]}$. It follows that $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A$ is a co-Whittaker $A[G]$-module.

For the converse, choose a generator of $V^{(n)}$ as an A-module; such a generator corresponds to a map $W_{[L,\pi]} \to V$. This map induces a map $W \otimes_{A_{[L,\pi]}} A \to V$ of $A[G]$-modules which we must prove is surjective. Let V' be the cokernel. We have an exact sequence:

$$0 \to (W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A)^{(n)} \to V^{(n)} \to (V')^{(n)} \to 0$$

by property (3) of the derivative, and the first horizontal map is surjective by construction. Thus $(V')^{(n)} = 0$. Assume V' is nonzero; then since V' is admissible over $A[G]$ it has a quotient that is simple as an $A[G]$-module; this is a non-generic quotient of $V \otimes_A \kappa(\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of A. But $V \otimes_A \kappa(\mathfrak{m})$ has an absolutely irreducible generic cosocle so this cannot happen. □

We conclude with a technical lemma which will be useful in section 7.

Lemma 6.4. *Let A be a reduced Noetherian $A_{[L,\pi]}$-algebra, and suppose that for each minimal prime $\mathfrak{a}$ of A, we specify a quotient $V_{\mathfrak{a}}$ of $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} \kappa(\mathfrak{a})$. Then there exists a co-Whittaker $A[G]$-module V, unique up to isomorphism, such that V is A-torsion free, and $V \otimes_A \kappa(\mathfrak{a})$ is isomorphic to $V_{\mathfrak{a}}$ for all $\mathfrak{a}$.*

Proof. Let V be the image of the diagonal map:

$$W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A \to \prod_{\mathfrak{a}} V_{\mathfrak{a}}.$$

The $A[G]$-module V is clearly A-torsion free by construction, and $V \otimes_A \kappa(\mathfrak{a})$ is clearly isomorphic to $V_{\mathfrak{a}}$. It thus suffices to show that V is a co-Whittaker module. But V is a quotient of the module $W_{[L,\pi]} \otimes_{A_{L,\pi}} A$, which is co-Whittaker by Theorem 6.3 so it suffices to show that $V^{(n)}$ is free of rank one over A. Certainly $V^{(n)}$ is cyclic, so it suffices to show that $V^{(n)} \otimes_A \kappa(\mathfrak{a})$ is nonzero for all minimal primes $\kappa(\mathfrak{a})$. This is clear because $V \otimes_A \kappa(\mathfrak{a})$ is isomorphic to the essentially AIG representation $V_{\mathfrak{a}}$.

As for the uniqueness claim, any such V is dominated by $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A$, and embeds in $\prod_{\mathfrak{a}} V_{\mathfrak{a}}$. Up to an automorphism of $\prod_{\mathfrak{a}} V_{\mathfrak{a}}$, the composition of $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} A \to V$ with the embedding of V in $\prod_{\mathfrak{a}} V_{\mathfrak{a}}$ is equal to the diagonal map, and thus identifies V with the image of this diagonal map. $\square$

7. The local Langlands correspondence in families

We now apply our results to the local Langlands correspondence in families of [EH], which we now recall:

Theorem 7.1 ([EH], Theorem 6.2.1)**.** *Let A be a reduced complete Noetherian local ℓ-torsion free $W(k)$-algebra, with residue field k, and let $\rho : G_F \to \mathrm{GL}_n(A)$ be a continuous representation of the absolute Galois group G_F of F. Then there is, up to isomorphism, at most one admissible $A[G]$-module $\pi(\rho)$ such that:*

1. *$\pi(\rho)$ is A-torsion free,*
2. *$\pi(\rho)$ is a co-Whittaker $A[G]$-module, and*
3. *for each minimal prime $\mathfrak{a}$ of A, the representation $\pi(\rho)_{\mathfrak{a}}$ is $\kappa(\mathfrak{a})$-dual to the representation that corresponds to $\rho_{\mathfrak{a}}^{\vee}$ via the Breuil-Schneider generic local Langlands correspondence. (For a discussion of this correspondence, see* [EH]*, section 4.2.)*

Remark 7.2. Note that this result is stated slightly differently in [EH], where the language of co-Whittaker modules is not available. In particular condition (2) above is equivalent to considitions (3)-(5) in [EH], Theorem 6.2.1; this follows from Lemma 6.3.1 of [EH].

It is conjectured ([EH], Conjecture 1.1.3) that such a $\pi(\rho)$ exists for any ρ. The goal of this section is to reformulate this conjecture as a relationship between the Bernstein center $A_{[L,\pi]}$ and the deformation theory of Galois representations.

First recall that, given any mod ℓ Galois representation $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$, together with a k-basis for the representation space of $\overline{\rho}$, there exists a complete Noetherian local $W(k)$-algebra $R_{\overline{\rho}}^{\square}$, called the *universal framed deformation ring* of $\overline{\rho}$, and a continous Galois representation

$$\rho^{\square} : G_F \to \mathrm{GL}_n(R_{\overline{\rho}}^{\square}),$$

together with a distinguished basis for the deformation space of $\rho^{\square}$. Moreover, one has an isomorphism:

$$\rho^{\square} \otimes_{R^{\square}_{\overline{\rho}}} k \cong \overline{\rho}$$

taking the base change of distinguished basis of $\rho^{\square}$ to our chosen basis of $\overline{\rho}$.

The pair $(R^{\square}_{\overline{\rho}}, \rho^{\square})$ is characterized by the following universal property: for any complete Noetherian $W(k)$-algebra A, any continous Galois representation $\rho : G_F \to \mathrm{GL}_n(A)$, any choice of basis for the representation space of ρ, and any isomorphism: $\rho \otimes_A k \cong \overline{\rho}$ that takes the base change of the chosen basis for ρ to the chosen basis of $\overline{\rho}$, there exists a unique map:

$$R^{\square}_{\overline{\rho}} \to A$$

and an isomorphism

$$\rho^{\square} \otimes_{R^{\square}_{\overline{\rho}}} A \cong \rho$$

taking the distinguished basis of $\rho^{\square}$ to the chosen basis of ρ. A choice of such data, for a given A, is called a *framed deformation* of ρ.

To motivate our reformulation, it will be useful to invoke the following fact, which will be proved in the forthcoming work [H2]. We do not attempt to prove this fact here, and indeed we use it *strictly for motivation*; it will only be invoked in Proposition 7.4 and the discussion preceding Conjecture 7.5. In particular our main result (Theorem 7.9) does not depend on it. (Note also that one can avoid the use of Fact 7.3 completely by replacing $R^{\square}_{\overline{\rho}}$ everywhere with the quotient of this ring by the ideal generated by its ℓ-torsion and nilpotent elements. This is more or less harmless but makes for somewhat messier statements.)

Fact 7.3. Fix a representation $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$, and let $(R^{\square}_{\overline{\rho}}, \rho^{\square})$ be the universal framed deformation of $\overline{\rho}$. The ring $R^{\square}_{\overline{\rho}}$ is reduced and ℓ-torsion free.

Granting this, the representation $\rho^{\square}$ fits into the framework of Theorem 7.1. Let us suppose that the family $\pi(\rho^{\square})$ exists. (If this is true, then the argument of [EH], Proposition 6.2.10, allows us to construct the family $\pi(\rho)$ for *any* deformation ρ of $\overline{\rho}$, essentially via "base change".) Let $\overline{\pi}$ be an irreducible representation of G over k whose supercuspidal support corresponds to $\overline{\rho}$ under the mod ℓ semisimple local Langlands correspondence of Vignéras [V2], and choose a pair (L, π) such that $\overline{\pi}$ lies in the block $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$. Then every subquotient of $\pi(\rho^{\square}) \otimes_{R^{\square}_{\overline{\rho}}} k$ has the same supercuspidal support as $\overline{\pi}$.

By [EH], Theorem 6.2.1, the endomorphism ring $\mathrm{End}_{W(k)[G]}(\pi(\rho^{\square}))$ is isomorphic to $R^{\square}_{\overline{\rho}}$. As this ring is local (and thus has no nontrivial idempotents) the action of the Bernstein center of $\pi(\rho^{\square})$ must factor through $A_{[L,\pi]}$. We thus obtain a map:

$$A_{[L,\pi]} \to R^{\square}_{\overline{\rho}}.$$

Note that (as $R^{\square}_{\overline{\rho}}$ is complete and local by definition), this map factors through the completion $(A_{[L,\pi]})_{\mathfrak{m}}$, where $\mathfrak{m}$ is the kernel of the action of $A_{[L,\pi]}$ on $\overline{\pi}$; this yields a map:

$$\mathrm{LL} : (A_{[L,\pi]})_{\mathfrak{m}} \to R^{\square}_{\overline{\rho}}.$$

We can easily describe the effect of this map on the characteristic zero points of $\mathrm{Spec}\, R^{\square}_{\overline{\rho}}$. Let κ be a complete field of characteristic zero that contains $W(k)$.

A representation $\rho : G_F \to \mathrm{GL}_n(\kappa)$, determines, via (Tate normalized) local Langlands, an irreducible representation Π in $\mathrm{Rep}_\kappa(G)$. If Π lies in $\mathrm{Rep}_\kappa(G)_{[L,\pi]}$, we let f_ρ denote the map $A_{[L,\pi]} \to \kappa$ giving the action of $A_{[L,\pi]}$ on Π. Note that this map depends only on the supercuspidal support of Π, so that f_ρ depends only on the semisimplification of ρ.

Now let ρ be a deformation of $\overline{\rho}$ over κ. A suitable choice of basis for ρ determines a map $x : R^\square_{\overline{\rho}} \to \kappa$, such that $\rho^\square_x = \rho$. It follows from [EH], Theorem 6.2.6 that every subquotient of $\pi(\rho^\square)_x$ then has supercuspidal support corresponding to ρ under (Tate normalized) local Langlands. On the other hand, $A_{[L,\pi]}$ acts on $\pi(\rho^\square)_x$ via the composition

$$A_{[L,\pi]} \to (A_{[L,\pi]})_\mathfrak{m} \stackrel{\mathrm{LL}}{\to} R^\square_{\overline{\rho}} \stackrel{x}{\to} \kappa.$$

It follows that this composition is equal to f_ρ. Summarizing, we have:

Proposition 7.4. *Suppose that Fact 7.3 holds, and that $\pi(\rho^\square)$ exists, and let κ be a complete field of characteristic zero containing $W(k)$. Then the map:*

$$\mathrm{LL}(\kappa) : (\mathrm{Spec}\, R^\square_{\overline{\rho}})(\kappa) \to (\mathrm{Spec}(A_{[L,\pi]})_\mathfrak{m})(\kappa)$$

takes the point x in $(\mathrm{Spec}\, R^\square_{\overline{\rho}})(\kappa)$ that corresponds to a framed deformation ρ to the κ-point f_ρ of $\mathrm{Spec}(A_{[L,\pi]})_\mathfrak{m}$.

In less formal language, we will say that a map $(A_{[L,\pi]})_\mathfrak{m} \to R^\square_{\overline{\rho}}$ "interpolates the characteristic zero semisimple local Langlands correspondence" over Spec $R^\square_{\overline{\rho}}$ if it satisfies the conclusion of Proposition 7.4.

The following then follow easily from Fact 7.3, together with the fact that $A_{[L,\pi]}$ is reduced and ℓ-torsion free:

- There is at most one map: $(A_{[L,\pi]})_\mathfrak{m} \to R^\square_{\overline{\rho}}$ that interpolates the characteristic zero semisimple local Langlands correspondence; i.e. LL is determined by Proposition 7.4 if it exists.
- If $\overline{\rho}$ is semisimple, then the map LL is injective. (As $A_{[L,\pi]}$ is reduced and ℓ-torsion free, it suffices to show that $\mathrm{LL}(\kappa)$ is surjective for κ a complete algebraically closed field of characteristic zero containing $W(k)$. To see this, note that a point x of $(\mathrm{Spec}\, A_{[L,\pi]})_\mathfrak{m}(\kappa)$ corresponds to a lift, to κ, of the supercuspidal support of $\overline{\pi}$. This in turn determines a lift of the semisimplification of $\overline{\rho}$ to a representation over κ. When $\overline{\rho}$ is semisimple, we obtain (from this lift, plus a choice of frame) a point of $(\mathrm{Spec}\, R^\square_{\overline{\rho}})(\kappa)$ that maps to x.)
- The image of LL in $R^\square_{\overline{\rho}}$ is contained in the set of functions that are "invariant under change of frame". More precisely, if $G^\square$ is the formal completion of $\mathrm{GL}_n / W(k)$ at the k-point corresponding to the identity identity, then $G^\square$ acts on $R^\square_{\overline{\rho}}$ by changing the frame, and the image of LL is contained in the invariants $(R^\square_{\overline{\rho}})^{G^\square}$ of this action.

This motivates the following conjecture, which (if we grant Fact 7.3) is a consequence of the existence of $\pi(\rho^\square)$.

Conjecture 7.5. *For any $\overline{\rho}$, there is a map:*

$$\mathrm{LL} : (A_{[L,\pi]})_\mathfrak{m} \to R^\square_{\overline{\rho}}$$

that interpolates the characteristic zero semisimple local Langlands correspondence.

It is also tempting, given the naturality of the rings $A_{[L,\pi]}$ and $(R^{\square}_{\overline{\rho}})^{G^{\square}}$, to formulate a stronger version of this conjecture when $\overline{\rho}$ is semisimple:

Conjecture 7.6. *If $\overline{\rho}$ is semisimple, there is a unique isomorphism:*

$$\mathrm{LL} : (A_{[L,\pi]})_{\mathfrak{m}} \to (R^{\square}_{\overline{\rho}})^{G^{\square}}$$

that interpolates the characteristic zero semisimple local Langlands correspondence.

Remark 7.7. Conjecture 7.6 can be thought of as a geometric interpolation of the semisimple local Langlands correspondence in characteristic zero. Indeed, the points of $(A_{[L,\pi]})_{\mathfrak{m}}$ correspond to supercuspidal supports of representations in $A_{[L,\pi]}$. On the other hand, let κ be a complete field of characteristic zero containing $W(k)$, and let x and y be two κ-points of $\operatorname{Spec} R^{\square}_{\overline{\rho}}$. It follows from the theory of pseudocharacters that $f(x) = f(y)$ for all $G^{\square}$-invariant elements f of $R^{\square}_{\overline{\rho}}$ if, and only if the Galois representations ρ_x and ρ_y have the same semisimplification. One can thus regard the κ-points of $\operatorname{Spec}(R^{\square}_{\overline{\rho}})^{G^{\square}}$ as parameterizing "lifts of $\overline{\rho}$ to κ up to semisimplification"; from this point of view the conjectured bijection:

$$(\operatorname{Spec}(R^{\square}_{\overline{\rho}})^{G^{\square}})(\kappa) \to (\operatorname{Spec}(A_{[L,\pi]})_{\mathfrak{m}})(\kappa)$$

is simply the map that takes a Galois representation (up to semisimplification) to the corresponding supercuspidal support.

Remark 7.8. Conjecture 7.6 can also be thought of as an integrality result for certain Bernstein center elements defined by Galois-theoretic considerations. For instance, Chenevier has shown that for any $\tau \in W_F$, there is a unique element f_τ of the center of $\operatorname{Rep}_{\overline{\mathcal{K}}}(G)$ whose action on irreducible admissible representations Π of G over $\overline{\mathcal{K}}$ is given in the following way: let ρ be the Frobenius-semisimple representation of W_F over $\overline{\mathcal{K}}$ corresponding to Π; then f_τ acts on Π by the trace of $\rho(\tau)$. (We refer the reader to [C], Proposition 3.11 for details.)

Conjecture 7.6 would imply that Bernstein center elements defined in this way actually live in the completion of the center of $\operatorname{Rep}_{W(k)}(G)$ at *every* maximal ideal $\mathfrak{m}$, and thus lie in the center of $\operatorname{Rep}_{W(k)}(G)$.

The existence of $\pi(\rho^{\square})$, together with Fact 7.3, would imply Conjecture 7.5 for $\overline{\rho}$.

Our main result is that the converse also holds; that is, that Conjecture 7.5 *implies* to the existence of $\pi(\rho)$ for all deformations ρ of $\overline{\rho}$ over suitable coefficient rings.

Theorem 7.9. *Fix an A and a ρ as in Theorem 7.1, and let $\overline{\rho} = \rho \otimes_A k$. Suppose that Conjecture 7.5 holds for $\overline{\rho}$. Then $\pi(\rho)$ exists.*

Proof. A choice of basis for ρ yields a map $R^{\square}_{\overline{\rho}} \to A$. Composition with the map LL makes A into an $A_{[L,\pi]}$-algebra. By Lemma 6.4, to construct $\pi(\rho)$, it suffices to show that for all $\mathfrak{a}$, the representation π_η that is $\kappa(\mathfrak{a})$-dual to the representation attached to $\rho^{\vee}_{\mathfrak{a}}$ by the Breuil-Schneider correspondence is a quotient of $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} \kappa(\eta)$. By Proposition 5.5, this amounts to showing that π_η is dual to an essentially AIG representation, and that its supercuspidal support corresponds to that of $W_{[L,\pi]} \otimes_{A_{[L,\pi]}} \kappa(\eta)$. The representation π_η is dual to an essentially AIG representation because any representation arising from the Breuil-Schneider correspondence is essentially AIG; see [EH], Corollary 4.3.3. It has the correct supercuspidal support because the map LL interpolates the semisimple local Langlands correspondence. □

Remark 7.10. In [H2], we will show that both Conjecture 7.5 and Conjecture 7.6 hold after inverting ℓ. We will also establish Conjecture 7.5 when $n = 2$ and ℓ is odd. (This last result relies on forthcoming work of Paige [P] on the structure of projective envelopes of representations of $\mathrm{GL}_n(\mathbb{F}_q)$.)

In particular the results of [H2], together with Theorem 7.9, will establish the existence of $\pi(\rho)$ for any two-dimensional representation ρ of G_F over a complete local ring A of odd residue characteristic that satisfies the conditions of Theorem 7.1.

Example 7.11. In section 8 we will give a proof of Conjecture 7.5 in the case when ℓ is banal; that is, when the integers $\{1, q, q^2, \dots, q^n\}$ are distinct modulo ℓ. (In fact, one can show that the stronger Conjecture 7.6 holds in this setting as well, but this requires techniques beyond the scope of this paper. A proof of this stronger claim will appear in [H2].)

The local Langlands correspondence in families follow immediately in the banal setting from this result. By constrast, providing a direct construction of $\pi(\rho^{\Box})$ by elementary methods seems to be very hard, even when ℓ is banal. In particular it is easy to construct $\pi(\rho^{\Box})$ over each irreducible component of $\mathrm{Spec}\, R_{\overline{\rho}}^{\Box}$, but "gluing" these branches of the family together in the unique way that satisfies condition (2) of Theorem 7.1 requires verifying the existence of suitable compatibilities between these branches over the intersections of the irreducible components. In general these intersections can be quite singular, and the compatibilities become increasingly difficult to verify as the number of components grows, rendering this approach essentially hopeless for large n.

Example 7.12. Now fix an $\ell > n$, and suppose that q has exact order n modulo ℓ. Let $\overline{\rho}$ be the representation $1 \oplus \omega \oplus \cdots \oplus \omega^{n-1}$, where ω is the mod ℓ cyclotomic character. In this setting there is an interesting phenomenon that does not occur in the banal case: the block $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ contains both principal series and supercuspidal representations.

This example (and somewhat more general ones) are considered in detail in section 5 of [P]. In particular, Paige shows that one has an isomorphism:

$$A_{[L,\pi]} \cong W(k)[\Gamma, T_1, \dots, T_n, T_n^{-1}]/\langle P(\Gamma)\rangle + (\Gamma - n)\langle T_1, \dots, T_{n-1}\rangle.$$

Here $P(\Gamma)$ is the squarefree polynomial in $W(k)[X]$ whose roots are of the form:

$$\zeta + \zeta^q + \cdots + \zeta^{q^{n-1}}$$

for ζ such that $\zeta^{\ell^a} = 1$, where ℓ^a is the largest power of ℓ dividing $q^n - 1$. This isomorphism depends on a choice of Frobenius element Fr for F, and a generator σ for the pro-ℓ quotient of the inertia group of W_F.

On the other hand, there are precisely two kinds of lifts of $\overline{\rho}$ to a representation ρ over an algebraically closed field κ of characteristic zero. One possibility is that ρ is a direct sum of characters lifting the characters in the direct sum decomposition of $\overline{\rho}$. Alternatively, ρ could have the form $\mathrm{Ind}_{W_E}^{W_F} \xi$, where E is the unramified extension of F of degree n, and ξ is a character of $E^{\times}$ that has trivial reduction mod ℓ, and whose conjugates (under the action of W_F on the set of characters of

W_E) are distinct. (Such characters exist precisely when q has exact order n modulo ℓ.) It is not hard to verify that every lift of $\overline{\rho}$ to κ has one of the two above forms.

From this classification of lifts of $\overline{\rho}$ it is not difficult to see that the map

$$W(k)[\Gamma, T_1, \dots, T_n, T_n^{-1}] \to R_{\overline{\rho}}^{\square}$$

that sends Γ to the trace of $\rho^{\square}(\sigma)$ and T_i to $(-1)^i$ times the coefficient of X^{n-i} in the characteristic polynomial of $\rho^{\square}(\mathrm{Fr})$ is zero on $P(\Gamma)$ and $(\Gamma - n)T_i$ for $1 \leq i \leq n-1$. In particular it descends to a map

$$A_{[L,\pi]} \to R_{\overline{\rho}}^{\square}.$$

Furthermore, one has an explicit description of the action of the generators Γ and T_i of $A_{[L,\pi]}$ on objects of $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ (which we omit here), and from this it is easy to verify that this map is compatible with local Langlands. We refer the reader to [P] for the details.

8. The banal case

The goal of this section is to prove Conjecture 7.5 when ℓ is banal. Assume this is so, and fix a pair (L, π), where π is a supercuspidal representation of L over k. Following section 5, we may assume that π has the form

$$\pi \cong \bigotimes_i \pi_i^{\otimes m_i}$$

with no π_i an unramified twist of π_j for $j \neq i$.

The banal hypothesis implies that, for each i, the integer $m(\pi_i)$ coming from Vigneras' classification is greater than n. In particular, the partition ν_i of m_i considered in section 5 consists entirely of ones, and so the pair $(M^{\max}, \pi^{\max})$ described in section 5 is equal to (L, π). (In fact, for this to be true, it suffices to know that for all i, the integers $1, q, q^2, \dots, q^{m_i \deg(\pi_i)}$ are distinct modulo n, and all the claims of this section hold under this weaker assumption as well.)

Let (K_i, τ_i) be a maximal distinguished cuspidal k-type contained in π_i; in this case (K_i, τ_i) is a *supercuspidal* k-type. The ring E_{K_i,τ_i} was first computed by Dat [Da2] in this setting; when ℓ is banal, we have an isomorphism $E_{K_i,\tau_i} \cong W(k)[T, T^{-1}]$, depending on a choice of uniformizer ϖ_F of F. One can characterize this isomorphism as follows: fix a supercuspidal lift $\tilde{\pi}_i$ of π_i to a representation over $\mathcal{K}$. Then for any unramified character $\tilde{\chi}$ of $F^{\times}$, the element T of E_{K_i,τ_i} acts on $\tilde{\pi}_i \otimes \tilde{\chi}$ by $\tilde{\chi}(\varpi_F^{a_i})$, where a_i is the order of the group of unramified characters $\tilde{\chi}$ such that $\tilde{\pi}_i \otimes \tilde{\chi}$ is isomorphic to $\tilde{\pi}_i$.

In this case it is easy to describe the algebra $B_{M^{\max},\pi^{\max}}$: set

$$\tilde{\pi} = \bigotimes_i \tilde{\pi}_i^{\otimes m_i}.$$

Then $(L, \tilde{\pi})$ is only inertial equivalence class in characteristic zero that reduces mod ℓ to a class with supercuspidal support inertially equivalent to (L, π). Therefore, Proposition 11.5 of [H1] identifies $B_{M^{\max},\pi^{\max}}$ with the ring of $W(\tilde{\pi})$-invariants of

$$\bigotimes_i W(k)[T_i, T_i^{-1}]^{\otimes m_i},$$

where $W(\tilde{\pi})$ is the subgroup of the Weyl group of G that preserves L and $\tilde{\pi}$ under conjugation. Explicitly, $B_{M^{\max},\pi^{\max}}$ is isomorphic to the tensor product, over

i, of the rings $\left[W(k)[T_i, T_i^{-1}]^{\otimes m_i}\right]^{S_{m_i}}$, where the symmetric group S_{m_i} acts by permuting the tensor factors.

In particular, $B_{M^{\max},\pi^{\max}}$ (and hence $A_{[L,\pi]}$) is freely generated by elements $\Theta_{i,j}$, where $\Theta_{i,j}$ is the jth symmetric polynomial in the T_i, together with Θ_{i,m_i}^{-1}. In terms of these generators, the action of $A_{[L,\pi]}$ on an irreducible characteristic zero $\overline{\mathcal{K}}$-representation of G in $\mathrm{Rep}_{W(k)}(G)_{[L,\pi]}$ is quite explicit: such a representation has supercuspidal support of the form

$$\tilde{\pi}_1 \otimes \tilde{\chi}_{1,1}, \dots, \tilde{\pi}_1 \otimes \tilde{\chi}_{1,m_1}, \tilde{\pi}_2 \otimes \tilde{\chi}_{2,1}, \dots \tilde{\pi}_2 \otimes \tilde{\chi}_{2,m_2}, \dots, \tilde{\pi}_s \otimes \tilde{\chi}_{s,1}, \dots, \tilde{\pi}_s \otimes \tilde{\chi}_{s,m_s},$$

where the $\tilde{\chi}_{i,j}$ are unramified. On a representation with supercuspidal support of this form, the element $\Theta_{i,j}$ acts via the jth symmetric polynomial in the elements $\tilde{\chi}_{i,1}(\varpi_F^{a_i}), \dots, \tilde{\chi}_{i,m_i}(\varpi_F^{a_i})$.

Now fix $\overline{\rho}$ such that $\overline{\rho}^{\mathrm{ss}}$ has the form:

$$\overline{\rho}^{\mathrm{ss}} \cong \bigoplus_i \bigoplus_{j=1}^{m_i} \overline{\rho}_i \otimes \chi_{i,j},$$

where χ_i is an unramified character of G_F and $\overline{\rho}_i$ is the irreducible representation corresponding to π_i under mod ℓ local Langlands. To give a map from $A_{[L,\pi]}$ to $R^{\square}_{\overline{\rho}}$, it suffices to specify the images of the $\Theta_{i,j}$, subject only to the condition that the images of the Θ_{i,m_i} must be invertible.

Let ρ be a lift of $\overline{\rho}$ to a representation over $\overline{\mathcal{K}}$, and let Π correspond to ρ under local Langlands. Then the supercuspidal support of Π will have the form:

$$\tilde{\pi}_1 \otimes \tilde{\chi}_{1,1}, \dots, \tilde{\pi}_1 \otimes \tilde{\chi}_{1,m_1}, \tilde{\pi}_2 \otimes \tilde{\chi}_{2,1}, \dots \tilde{\pi}_2 \otimes \tilde{\chi}_{2,m_2}, \dots, \tilde{\pi}_s \otimes \chi_{s,1}, \dots, \tilde{\pi}_s \otimes \tilde{\chi}_{s,m_s}.$$

We will thus have an isomorphism:

$$\rho^{\mathrm{ss}} \cong \bigoplus_i \bigoplus_{j=1}^{m_i} \rho_i \otimes \tilde{\chi}_{i,j},$$

where ρ_i corresponds to $\tilde{\pi}_i$ under local Langlands. A map $A_{[L,\pi]} \to R^{\square}_{\overline{\rho}}$ will be compatible with local Langlands if, and only if, for all i, j, the value of the image of $\Theta_{i,j}$ at ρ is equal to the jth symmetric polynomial in the $\tilde{\chi}_{i,1}(\varpi_F^{a_i}), \dots, \tilde{\chi}_{i,m_i}(\varpi_F^{a_i})$.

It thus suffices to construct, for each i and j, an element $\Xi_{i,j}$ of $R^{\square}_{\overline{\rho}}$ whose value at any ρ with

$$\rho^{\mathrm{ss}} \cong \bigoplus_i \bigoplus_{j=1}^{m_i} \rho_i \otimes \tilde{\chi}_{i,j}$$

is equal to the jth symmetric polynomial in the $\tilde{\chi}_{i,1}(\varpi_F^{a_i}), \dots, \tilde{\chi}_{i,j}(\varpi_F^{a_i})$.

We will construct the $\Xi_{i,j}$ via an approach due to Clozel-Harris-Taylor. We summarize the key results of theirs we need:

Proposition 8.1 ([CHT], Lemmas 2.4.11-2.4.13)**.** *Let $I_F^{(\ell)}$ denote the prime to ℓ part of the inertia group of F, and let T_F be the quotient $G_F/I_F^{(\ell)}$, so that T_F is a semidirect product of $\hat{\mathbb{Z}}$ and $\mathbb{Z}_\ell$. Let $\overline{\tau}$ be an irreducible representation of $I_F^{(\ell)}$ over k, and let $G_{\overline{\tau}}$ be the subgroup of G_F consisting of those σ in G_F that preserve $\overline{\tau}$ under conjugation. Then:*

1. *$\overline{\tau}$ lifts uniquely to a representation τ of $I_F^{(\ell)}$ over $W(k)$.*
2. *τ extends uniquely to a representation of $I_F \cap G_{\overline{\tau}}$ of determinant prime to ℓ.*

(3) *τ extends (non-uniquely) to a representation of $G_{\overline{\tau}}$.*

If we fix a representation τ of $G_{\overline{\tau}}$ as in part (3), we obtain an action of $T_{\overline{\tau}}$ on $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho)$ for any G_F-module ρ, where $T_{\overline{\tau}}$ is the image of $G_{\overline{\tau}}$ in T_F. Moreover, we have a direct sum decomposition of G_F-modules:

$$\rho = \bigoplus_{[\overline{\tau}]} \mathrm{Ind}_{G_{\overline{\tau}}}^{G_F} [\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho) \otimes \tau],$$

where $\overline{\tau}$ runs over W_F-conjugacy classes of irreducible representations of I_F^{ℓ} over k.

Fix, for each i, an irreducible $I_F^{(\ell)}$-representation $\overline{\tau}_i$ in the restriction of $\overline{\rho}_i$ to $I_F^{(\ell)}$, and let τ_i be an extension of a lift of $\overline{\tau}_i$ to $G_i = G_{\overline{\tau}_i}$. (This determines τ_i up to an unramified twist; we will determine τ_i more precisely in a moment.)

Lemma 8.2. *The group G_i contains I_F and has index a_i in G_F (that is, the fixed field F_i of G_i is unramified over F of degree a_i.) Moreover, we have an isomorphism:*

$$\rho_i = \mathrm{Ind}_{G_i}^{G_F} \tilde{\chi} \otimes \tau_i$$

for an unramified character $\tilde{\chi}$. (In particular, $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho_i)$ is one-dimensional, and $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho_j)$ is zero for $i \neq j$.

Proof. We have

$$\rho_i = \mathrm{Ind}_{G_i}^{G_F} \mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho_i) \otimes \tau_i$$

as ρ_i is irreducible. In particular, the action of G_i on $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_i)$ is irreducible, and factors through $G_i/I_F^{(\ell)}$.

On the other hand, the dimension of ρ, and thus each, ρ_i, is less than n, and (since we are in the banal setting) hence also less than ℓ. In particular the index of G_i in G_F cannot be divisible by ℓ. On the other hand G_i contains the subgroup $I_F^{(\ell)}$, and must thus contain all of I_F.

Since $G_i/I_F^{(\ell)}$ is the extension of an infinite cyclic group by a cyclic group of ℓ-power order, its irreducible representations are either characters, or have dimension divisible by ℓ. In particular $\mathrm{Hom}_{I_F^{(\ell)}}(\tau, \rho_i)$ must be a character of G_i trivial on $I_F^{(\ell)}$. Such a character corresponds by local class field theory to a character of $F_i^{\times}$ whose image on inertia has ℓ-power order. Since we are in the banal setting, and F_i is an unramified extension of F of degree less than n, there are no ℓ-power roots of unity in $F_i^{\times}$, so such a character is in fact unramified as claimed.

Suppose some ρ_j contains τ_i. Then the argument above also applies to ρ_j, and we find that ρ_j is an unramified twist of $\mathrm{Ind}_{G_i}^{G_F} \tau_i$, and hence an unramified twist of ρ_i. This cannot happen as π_i and π_j are not unramified twists of each other for $i \neq j$.

Finally, note that the number of unramified characters χ such that $\rho_i \cong \rho_i \otimes \chi$ is equal on the one hand to the index of G_i in G_F, but on the other (via compatiblility of local Langlands with twists) to a_i. Thus G_i has index a_i in G_F as claimed. $\square$

We choose our extension τ_i of τ so that the unramified twist appearing above is trivial; that is, so that $\rho_i = \mathrm{Ind}_{G_i}^{G_F} \tau_i$. For each i, set $V_i = \mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho^{\square})$. It is an $R_{\overline{\rho}}^{\square}[G_i]$-module on which $I_F^{(\ell)}$ acts trivially.

Lemma 8.3. *For each i, the space V_i is free of rank m_i over $R^{\square}_{\overline{\rho}}$. Moreover, we have a direct sum decomposition:*

$$\rho^{\square} = \bigoplus_i \mathrm{Ind}_{G_i}^{G_F} V_i \otimes \tau_i$$

Proof. The results of Clozel-Harris-Taylor show that the right-hand side is a direct summand of the left-hand side; it is immediate from this, together with the fact that $\rho^{\square}$ is free of finite rank over the local ring $R^{\square}_{\overline{\rho}}$, that each summand on the left-hand side is free, and hence that $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho^{\square})$ is free over $\mathbb{R}^{\square}_{\overline{\rho}}$ for each i.

To determine the rank we reduce modulo the maximal ideal, and note that $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \overline{\rho}) = \mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \overline{\rho}^{\mathrm{ss}})$ since the restriction of $\overline{\rho}$ to $I_F^{(\ell)}$ is semisimple. On the other hand, $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho_j)$ is one-dimensional if $i = j$ and zero-dimensional otherwise, so the dimension of $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \overline{\rho}^{\mathrm{ss}})$ is equal to the number of summands of $\overline{\rho}^{\mathrm{ss}}$ that are unramified twists of ρ_i. This is precisely m_i.

The claimed direct sum decomposition now follows by by comparing ranks of the left-hand and right-hand sides. □

Now let Fr_i be a Frobenius element of G_i corresponding to the uniformizer ϖ_F of F_i. The characteristic polynomial of Fr_i on V_i is a polyomial of degree m_i in $R^{\square}_{\overline{\rho}}[X]$. Let $\Xi_{i,j}$ be equal to $(-1)^j$ times the coefficient of $X^{m_i - j}$ in this characteristic polynomial. We then have:

Lemma 8.4. *Let ρ be a representation of G_F over $\overline{\mathcal{K}}$ lifting $\overline{\rho}$, and suppose that we have*

$$\rho^{\mathrm{ss}} \cong \bigoplus_i \bigoplus_j \rho_i \otimes \chi_{i,j}$$

for unramified characters $\chi_{i,j}$. Then the image of $\Xi_{i,j}$ under the map $R^{\square}_{\overline{\rho}} \rightarrow \overline{\mathcal{K}}$ corresponding to ρ is the jth symmetric polynomial in $\chi_{i,1}(\varpi_F^{a_i}), \dots, \chi_{i,m_i}(\varpi_F^{a_i})$.

Proof. The image of $\Xi_{i,j}$ is equal to $(-1)^j$ times the coefficient of $X^{m_i - j}$ in the characteristic polynomial of Fr_i on $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho)$. On the other hand, the semisimplification of $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho)$ is isomorphic to $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho^{\mathrm{ss}})$, and the latter is the direct sum of the restrictions of $\chi_{i,1}, \dots, \chi_{i,m_i}$ to G_i. Thus the eigenvalues of Fr_i on $\mathrm{Hom}_{I_F^{(\ell)}}(\tau_i, \rho)$ are $\chi_{i,1}(\mathrm{Fr}_i), \dots, \chi_{i,m_i}(\mathrm{Fr}_i)$. Since G_i has index a_i in G_F, and the norm from F_i to F of ϖ_F is $\varpi_F^{a_i}$, we have $\chi_{i,j}(\mathrm{Fr}_i) = \chi_{i,j}(\varpi_F^{a_i})$, and the claim follows. □

An immediate consequence is that the map $A_{[L,\pi]} \rightarrow R^{\square}_{\overline{\rho}}$ that takes $\Theta_{i,j}$ to $\Xi_{i,j}$ is compatible with the local Langlands correspondence, as claimed.

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