MODULAR FORMS EXAMPLE SHEET 4

1a. Show that the group $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ (p prime) has order $p(p^2-1)$.

1b. Show by induction on e that $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ has order $p^{3e}(1-\frac{1}{p^2})$.

1c. Show that $\Gamma(N)$ has index $N^3\prod_{p|N}(1-\frac{1}{p^2})$ in $\mathrm{SL}_2(\mathbb{Z})$, where the product is over all prime divisors p of N.

1d. Show that $\Gamma_1(N)$ has index $N^2\prod_{p|N}(1-\frac{1}{p^2})$ in $\mathrm{SL}_2(\mathbb{Z})$.

1e. Show that $\Gamma_0(N)$ has index $N\prod_{p|N}(1+\frac{1}{p})$ in $\mathrm{SL}_2(\mathbb{Z})$.

2. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, containing $\pm I$. Let $x \in \mathbb{Q}\cup\{\infty\}$. Let Z_x denote the stabilizer of x in $\mathrm{SL}_2(\mathbb{Z})$, and let $\Gamma_x = Z_x \cap \Gamma$. The *width* of the cusp x (relative to the congruence subgroup Γ) is the index $[Z_x : \Gamma_x]$, and is denoted $R_\Gamma(x)$.

2a. Show that for $\gamma \in \Gamma$, one has $R_\Gamma(\gamma x) = R_\Gamma(x)$.

2b. For x, y in $\mathbb{Q}\cup\{\infty\}$, let $Z_{x,y}$ denote the set $\{\delta \in \mathrm{SL}_2(\mathbb{Z}) : \delta x \in \Gamma\cdot y\}$. Show that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma x = y$, $Z_{x,y}$ is equal to the "double coset" $\Gamma\gamma Z_x$ of elements of $\mathrm{SL}_2(\mathbb{Z})$ of the form $\gamma'\gamma z$ for $\gamma' \in \Gamma$, and $z \in Z_x$.

2c. Let G be a group, and H and K subgroups of finite index in G. Show that for any $g \in G$, the double coset $HgK = \{hgk : h \in H, k \in K\}$ is the disjoint union of n cosets Hg, where n is the index of $H \cap gKg^{-1}$ in gKg^{-1}.

2d. Show that the sum of $R_\Gamma(x)$, as x runs over a set of representatives for the Γ-orbits in $\mathbb{Q}\cup\infty$, is equal to the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$. [HINT: write $\mathrm{SL}_2(\mathbb{Z})$ as a disjoint union of cosets $\Gamma\gamma_i$, and for each Γ-orbit Γx in $\mathbb{Q}\cup\infty$, count the number of such cosets that take ∞ to a point in Γx.]

3a. Find a set of representatives for the set of cusps of the congruence subgroups $\Gamma_0(4), \Gamma_0(6)$, and $\Gamma_1(5)$, and find their widths.

3b. Show that for N squarefree, $\Gamma_0(N)$ has precisely one cusp of width d for each divisor d of N.

4a. Show that the group $\Gamma_0(4)$ is generated by the matrices $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\pm\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$.

4b. Show that for $n \geq 7$, the matrices $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\pm\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$ do *not* generate $\Gamma_0(n)$.