# MODULAR FORMS EXAMPLE SHEET 3

1. Let $M_k(\mathbb{Z})$ be the space of modular forms of weight $k$ with integral $q$-expansions. Show that the graded ring:

$$\bigoplus M_k(\mathbb{Z})$$

is generated over $\mathbb{Z}$ by $E_4$, $E_6$, and $\Delta$.

2a. Show that the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective, for any $N > 1$.

2b. Show that the map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is not surjective, for any $N > 6$.

3. Compute the matrix of the Hecke operator $T_2$ acting on $S_{24}$ with respect to the basis $E_4^3\Delta, \Delta^2$ of $S_{24}$, and show that its characteristic polynomial is irreducible. What does this mean about the eigenforms of level 24?

4. Let $V$ be a *three* dimensional real vector space, and let $\mathcal{L}$ denote the space of lattices in $V$. For $a, b$ positive integers with $a$ dividing $b$, define a correspondence $T_{a,b}$ on $\mathcal{L}$ by letting $T_{a,b}L$ be the sum of the sublattices $L' \subset L$ such that $L/L'$ is isomorphic to $\mathbb{Z}/a \times \mathbb{Z}/b$.

4a. Show that if $(b, b') = 1$, then $T_{a,b}T_{a',b'} = T_{aa',bb'}$.

4b. Fix a prime $p$, and express $T_{1,p^2}$, $T_{1,p^3}$, and $T_{p,p^2}$ as polynomials in $T_{1,p}$, $T_{p,p}$, and the "rescaling by $p$" operator $R_p$.