MODULAR FORMS EXAMPLE SHEET 2

1. Let L and L' be lattices in $\mathbb{C}$ such that $E_4(L) = E_4(L')$ and $E_6(L) = E_6(L')$. Show that $L = L'$. For which pairs (x, y) does there exist a lattice L such that $E_4(L) = x$ and $E_6(L) = y$?

As j is a rational function in E_4 and E_6, we also have $j(L) = j(L')$. Since j induces a bijection from lattices modulo rescaling to $\mathbb{C}$, we must have $L = \lambda L' = \lambda' L_z$ for some $\lambda \in \mathbb{C}$, $z \in \mathbb{H}$. Subsituting into $E_4(L) = E_4(L')$ and $E_6(L) = E_6(L')$ we find that $(\lambda^6 - 1)E_6(z)$ and $(\lambda^4 - 1)E_4(z)$ both vanish. If $E_4(z)$ and $E_6(z)$ are both nonzero we can conclude that $\lambda^2 = 1$, so $\lambda = \pm 1$ and thus $L = L'$. If $E_4(z) = 0$, then z is in the orbit of ρ, so $L_z = L_\rho$. Then $E_6(z)$ is nonzero (the only zeroes of E_6 are in the orbit of i), so λ is a 6th root of unity; i.e. $\lambda \in \{\pm 1, \pm\rho, \pm\rho^2\}$. For such λ we have $\lambda L_\rho = L_\rho$, so $L = L'$. Finally, if $E_6(z) = 0$, then z is in the orbit of i, $L_z = L_i$, and λ is a fourth root of unity; we again have $\lambda L_i = L_i$ so $L = L'$.

Given any lattice L, $\Delta(L)$ is nonzero (since Δ has no zeroes on the upper half plane.) Thus, if $x = E_4(L)$, and $y = E_6(L)$, then $x^3 - y^2$ is nonzero. Conversely, given any x and y, such that $x^3 - y^2$ is nonzero, choose L such that $j(L) = \frac{1728x^3}{x^3 - y^2}$. Then $\Delta(L)$ is nonzero, so rescaling L if necessary we may assume that $\Delta(L) = \frac{x^3 - y^2}{1728}$.

We then have $E_4(L)^3 = \frac{j(L)\Delta(L)}{1728} = x^3$, and $E_6(L)^2 = y^2$. Rescaling L by an appropriate twelfth root of unity, we can arrange that $E_4(L) = x$ and $E_6(L) = y$.

2. Define $G_2(z)$, the Eisenstein series of weight 2, to be the series:

$$G_2(z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz + n)^2}.$$

(Note that this is not absolutely convergent, so this sum depends on the order of summation. In order to fix such an order, we adopt the convention that $\Sigma_{n \in \mathbb{Z}} f(n)$ is given by $\lim_{N \to \infty} \sum_{n=-N}^{N} f(n)$.)

2a. Define $G_2'(z)$ by:

$$G_2'(z) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz + n)^2}.$$

Show that $G_2(-z^{-1}) = z^2 G_2'(z)$.

We have

$$G_2(-z^{-1}) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(m(-z)^{-1}+n)^2}$$
$$= z^2 \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(m+nz)^2} = z^2 G_2'(z)$$

2b. Similarly define H and H' to be the sums:

$$H(z) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{(mz+n-1)(mz+n)}$$
$$H'(z) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{(mz+n-1)(mz+n)}$$

and show that $H(z) = 2$. (Hint: write $\frac{1}{(mz+n-1))(mz+n)} = \frac{1}{mz+n-1} - \frac{1}{mz+n}$. Then there is a lot of cancellation in the sums!)

Write $\frac{1}{(mz+n-1)(mz+n)} = \frac{1}{mz+n-1} - \frac{1}{mz+n}$. Now for $H(z)$, note that when $m \neq 0$, the inner sum

$$\Sigma_{n\in\mathbb{Z}} \frac{1}{mz+n-1} - \frac{1}{mz+n} = \lim_{N\to\infty} \Sigma_{n=-N}^{N} \frac{1}{mz+n-1} - \frac{1}{mz+n}.$$

Successive terms cancel in this truncated sum, leaving $\frac{1}{mz-N-1} - \frac{1}{mz+N}$, which tends to zero as N tends to infinity. When $m = 0$ one has similar cancellation, except that the "missing" terms $(0,0)$ and $(0,1)$ prevent everything from cancelling; one checks that in this case the inner sum is 2.

2c. Show that $H'(z)$ converges to $2 - \frac{2\pi i}{z}$. This is much harder! Here are some hints:

- Show that $H'(\frac{1}{z})$ is given by the series:
$$H'(\frac{1}{z}) = z \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z} - \frac{1}{m+nz}.$$
- Use the series $\pi \cot \pi z = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}$ to show that
$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z} = \frac{1}{z} + \sum_{n\in\mathbb{Z},n\neq 1} \pi \cot(n-1)\pi z.$$
- Show that this sum is equal to $\frac{1}{z} + \lim_{N\to\infty} \pi \cot(-N\pi z) + \pi \cot(-(N+1)\pi z)$.
- Use the identity $\cot z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$ to show that this limit is $\frac{1}{z} - 2\pi i$.
- Use a similar analysis to show that
$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+nz}$$
is equal to $\frac{1}{z}$.

- Conclude that $H'(z) = 2 - \frac{2\pi i}{z}$ as claimed.

We have:

$$H'(\frac{1}{z}) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{\frac{m}{z}+n-1} - \frac{1}{\frac{m}{z}+n}$$

$$z\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z} - \frac{1}{m+nz}.$$

Let us first compute the sum:

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z}.$$

If $n \neq (0,1)$ the inner sum is $\pi\cot(n-1)\pi z$. When $n = 0$ the inner sum is "almost" the series for $\pi\cot(-\pi z)$, except that it is missing the $m = 0$ term $-\frac{1}{z}$. So when $n = 0$ the inner sum is $\frac{1}{z} + \pi\cot(-\pi z)$. When $n = 1$ the inner sum vanishes.

We thus have

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z} = \frac{1}{z} + \sum_{n\in\mathbb{Z},n\neq 1} \pi\cot(n-1)\pi z.$$

This latter sum is the limit, as $N \to \infty$, of $\sum_{n=-N-1,n\neq 0}^{N-1} \pi\cot n\pi z$. As cotangent is odd, every term cancels except for $n = -N-1$, $n - -N$, so this sum is equal to the limit, as $N \to \infty$, of $\pi\cot -N\pi z + \pi\cot -(N+1)\pi z$. It follows from the identity $cotz = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$, together with the fact that z is in the upper half plane, that $\cot -N\pi z$ approaches $-i$ as N approaches infinity. Thus as N approaches infinity, this series approaches $\frac{1}{z} - 2\pi i$.

A completely analogous computation with the sum

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+nz}$$

shows that it is equal to $-\frac{1}{z}$. (This time, the lack of a shift for n causes the cotangents to cancel exactly.) Thus $H'(\frac{1}{z}) = 2 - 2\pi i z$. Substituting z for $\frac{1}{z}$ yields the desired result.

2d. Show that the resulting series for $G_2 - H$ and $G_2' - H'$ are absolutely convergent, and rearrangements of each other, so that $G_2(z) - H(z) = G_2'(z) - H'(z)$. Show that it follows from this that G_2 is convergent, uniformly on compact subsets of the upper half plane. In particular, $G_2(z)$ is holomorphic.

We have (up to a missing term where $m = 0, n = 1$):

$$G_2(z) - H(z) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n)^2} - \frac{1}{(mz+n-1)(mz+n)}$$

$$= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n-1)(mz+n)^2}.$$

$$G_2'(z) - H'(z) = \sum_{n\in\mathbb{Z}} \Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n)^2} - \frac{1}{(mz+n-1)(mz+n)}$$

$$= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n-1)(mz+n)^2}.$$

These are both absolutely convergent, uniformly on compact subsets of the upper half plane, because the denominator goes as $(m^2+n^2)^{frac32}$ as (m, n) go to infinity. Moreover the two series are clearly rearrangements of each other, and thus have the same sum. Thus G_2 differs from the (conditionally) convergent series H by an absolutely convergent series, and is thus convergent. Moreover, since the convergence for H is uniform on compact subsets of $\mathbb{H}$, the same is true for G_2. Thus $G_2(z)$ is holomorphic.

2e. Conclude from 2d that $G_2(-z^{-1}) - z^2G(z) = -2\pi iz$. (In particular, G_2, although holomorphic, is NOT a modular form.

This is immediate from 2b, 2c and 2d.

2f. This means there's no lattice function corresponding to G_2. If we try to define a lattice function by setting

$$G_2(L) = \sum_{w\in L\setminus 0} w^{-2},$$

what goes wrong?

This sum depends not just on L but also the order of summation, so is not a well-defined function on lattices.

2g. Find the q-expansion of G_2.

We use the series identity from class:

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \pi\cot\pi z = i\pi - 2i\pi\Sigma_{d=0}^{\infty} q^d$$

Differentiating once (as in the calculation of the q-expansions in higher weights) we find

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} = -4\pi^2\Sigma_{d=0}^{\infty} dq^d.$$

Separating out the $m = 0$ term in the series for $G_2(z)$, and using that the m and $-m$ sums are the same, we have

$$G_2(z) = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n^2} + 2\sum_{n=1}^{\infty} \Sigma_{d=0}^{\infty} dq^{nd}.$$

Collecting q^m terms, and using the fact that the sum of the reciprocals of the squares is $\pi^2/6$, we find:

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{m=1}^{\infty} \sigma_1(m) q^m.$$

3. Let η denote the function $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, where as usual $q = e^{2\pi i z}$.

3a. Show that $\eta(z+1) = e^{\frac{\pi i}{12}}\eta(z)$.

Since q is invariant under $z \mapsto z+1$, and $q^{\frac{1}{24}}$ transforms to $e^{\frac{\pi i}{12}} q^{\frac{1}{24}}$ under this translation, the claim is immediate.

3b. Show that $\frac{d}{dz}\log\eta(z) = \frac{\pi i}{12} E_2(z)$, where $E_2(z)$ is the unique scalar multiple of $G_2(z)$ whose q-expansion has constant term 1.

We have:

$$\begin{aligned}
\frac{d}{dz}\log\eta(z) &= \frac{2\pi i}{24} + \sum_{n=1}^{\infty} \frac{d}{dz}\log(1-q^n) \\
&= \frac{2\pi i}{24} - (2\pi i)\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \\
&= \frac{(2\pi i)}{24}\left[1 - 24\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nq^{jn}\right] \\
&= \frac{2\pi i}{24}\left[1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^m\right].
\end{aligned}$$

3c. Use the identity $E_2(\frac{-1}{z}) = z^2 E_2(z) - \frac{6iz}{\pi}$ to show that $\eta(-\frac{1}{z}) - \sqrt{-iz}\eta(z)$.

We have:

$$\frac{d}{dz}\log[\eta(-\frac{1}{z})] - \frac{d}{dz}\log[\sqrt{-iz}\eta(z)] = z^{-2}\frac{\pi i}{12}E_2(-\frac{1}{z}) - \frac{1}{2z} - \frac{\pi i}{12}E_2(z)$$
$$= -\frac{1}{2z} + \frac{\pi i}{12}[z^{-2}E_2(-\frac{1}{z}) - E_2(z)] = -\frac{1}{2z} + \frac{\pi i}{12}[-z^{-2}\frac{6iz}{\pi}] = 0.$$

Thus $\eta(-\frac{1}{z})$ is a scalar multiple of $\sqrt{iz}\eta(z)$. To conclude equality we may compare both sides for $z = i$; it suffices to show that $\eta(i)$ is nonzero. This is clear from the product definition of η.

3d. Show that $\eta^{24} = \Delta$. In particular $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$; do not use this in your proof!

Note that $\eta^{24}(z+1) = \eta^{24}(z)$, and $\eta^{24}(-\frac{1}{z}) = z^{-12}\eta(z)$. Thus η^{24} is a modular form of weight 12 and level 1 with q-expansion beginning $q + \dots$. Since the space of cusp forms of weight 12 and level 1 is one-dimensional, and Δ and η^{24} have the same leading term, we must have $\Delta = \eta^{24}$.

4a. Let $d = \dim M_k$. Show that there is a unique basis for M_k of the form $g_1, \dots, g_d$, where for all i, the q-expansion of g_i has the form $q^{i-1} + \Sigma_{n=d}^{\infty} c_n q^n$.

We first show that there is a basis $f_1, \dots, f_d$ such that f_i vanishes to order exactly $i-1$ at ∞. We do this by induction on k; note that if $k < 12$, then d is zero or 1, and if $d = 1$ then taking $f_1 = E_k$ suffices. For the general case, note that $\dim M_{k-12} = d-1$ and let $h_1, \dots, h_{d-1}$ be a basis for M_{k-12} such that h_i vanishes to order exactly $i-1$ for all i. Then we can take $f_1 = E_k$ and $f_i = \Delta h_{i-1}$ for all i and this has the desired property. Moreover, note that the f_i have integer coefficients if the h_i do (and they do in the base case!) so the induction also proves that the f_i are integer coefficients.

To get the g_i, let $g_d = f_d$, obtain f_{d-1} from g_{d-1} by subtracting a multiple of g_d so that the q^{d-1} term vanishes, obtain g_{d-2} from g_{d-2} by subtracting multiples of g_d and g_{d-1} so that the q^{d-2} and q^{d-1} terms vanish, and so forth.

4b. Show further that any element of M_k whose q-expansion has integer coefficients is an integral linear combination of the g_i.

Let f be such an element, write $f = \Sigma a_n q^n$. We can also write $f = \Sigma b_i g_i$. Comparing the coefficients of q^{i-1} in the two sums we have $a_i = b_i$.