MODULAR FORMS EXAMPLE SHEET 2

1. Let L and L' be lattices in $\mathbb{C}$ such that $E_4(L) = E_4(L')$ and $E_6(L) = E_6(L')$. Show that $L = L'$. For which pairs (x, y) does there exist a lattice L such that $E_4(L) = x$ and $E_6(L) = y$?

2. Define $G_2(z)$, the Eisenstein series of weight 2, to be the series:

$$G_2(z) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n)^2}.$$

(Note that this is not absolutely convergent, so this sum depends on the order of summation. In order to fix such an order, we adopt the convention that $\Sigma_{n\in\mathbb{Z}} f(n)$ is given by $\lim_{N\to\infty} \sum_{n=-N}^{N} f(n)$.)

2a. Define $G_2'(z)$ by:

$$G_2'(z) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n)^2}.$$

Show that $G_2(-z^{-1}) = z^2 G_2'(z)$.

2b. Similarly define H and H' to be the sums:

$$H(z) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n-1)(mz+n)}$$

$$H'(z) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(mz+n-1)(mz+n)}$$

and show that $H(z) = 2$. (Hint: write $\frac{1}{(mz+n-1)(mz+n)} = \frac{1}{mz+n-1} - \frac{1}{mz+n}$. Then there is a lot of cancellation in the sums!)

2c. Show that $H'(z)$ converges to $2 - \frac{2\pi i}{z}$. This is much harder! Here are some hints:

- Show that $H'(\frac{1}{z})$ is given by the series:

$$H'(\frac{1}{z}) = z \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z} - \frac{1}{m+nz}.$$

- Use the series $\pi \cot \pi z = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}$ to show that

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+(n-1)z} = \frac{1}{z} + \sum_{n\in\mathbb{Z},n\neq 1} \pi \cot(n-1)\pi z.$$

- Show that this sum is equal to $\frac{1}{z} + \lim_{N\to\infty} \pi\cot(-N\pi z) + \pi\cot(-(N+1)\pi z)$.
- Use the identity $\cot z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$ to show that this limit is $\frac{1}{z} - 2\pi i$.
- Use a similar analysis to show that

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)} \frac{1}{m+nz}$$

  is equal to $\frac{1}{z}$.
- Conclude that $H'(z) = 2 - \frac{2\pi i}{z}$ as claimed.

2d. Show that the resulting series for $G_2 - H$ and $G_2' - H'$ are absolutely convergent, uniformly on compact subsets of the upper half plane. and rearrangements of each other, so that $G_2(z) - H(z) = G_2'(z) - H'(z)$. Show that it follows from this that G_2 is convergent, uniformly on compact subsets of the upper half plane. In particular, $G_2(z)$ is holomorphic.

2e. Conclude from 2d that $G_2(-z^{-1}) - z^2G(z) = -2\pi iz$. (In particular, G_2, although holomorphic, is NOT a modular form.

2f. This means there's no lattice function corresponding to G_2. If we try to define a lattice function by setting

$$G_2(L) = \sum_{w\in L\setminus 0} w^{-2},$$

what goes wrong?

2g. Find the q-expansion of G_2.

3. Let η denote the function $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, where as usual $q = e^{2\pi iz}$.

3a. Show that $\eta(z+1) = e^{\frac{\pi i}{12}}\eta(z)$.

3b. Show that $\frac{d}{dz}\log\eta(z) = \frac{\pi i}{12}E_2(z)$, where $E_2(z)$ is the unique scalar multiple of $G_2(z)$ whose q-expansion has constant term 1.

3c. Use the identity $E_2(\frac{-1}{z}) = z^2E_2(z) - \frac{6iz}{\pi}$ to show that $\eta(-\frac{1}{z}) - \sqrt{-iz}\eta(z)$.

3d. Show that $\eta^{24} = \Delta$. In particular $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$; do not use this in your proof!

4a. Let $d = \dim M_k$. Show that there is a unique basis for M_k of the form $g_1, \ldots, g_d$, where for all i, the q-expansion of g_i has the form $q^{i-1} + \Sigma_{n=d}^{\infty} c_n q^n$.

4b. Show further that any element of M_k whose q-expansion has integer coefficients is an integral linear combination of the g_i.