MODULAR FORMS EXAMPLE SHEET 4

1. Let L be a lattice, with basis ω_1, ω_2. Use the parameterization of $y^2 = 4x^3 - g_4(L)x - g_6(L)$ by $\wp$ and $\wp'$ to show that the roots of the cubic $4x^3 - g_4(L)x - g_6(L)$ are the numbers $\wp(\omega_1/2, L)$, $\wp(\omega_2/2, L)$, and $\wp((\omega_1 + \omega_2)/2, L)$.

We have seen in class that this cubic has distinct roots, and moreover that the map:

$$z \mapsto (\wp(z, L), \wp'(z, L))$$

is a bijection from $\mathbb{C}/L \setminus \{0\}$ to the set of solutions of $y^2 = 4x^3 - g_4(L)x - g_6(L)$. Thus, the z such that $\wp'(z, L) = 0$ are precisely those such that $\wp(z, L)$ is a root of $4x^3 - g_4(L)x - g_6(L)$, and there are exactly three such z modulo translation by L. On the other hand, since $\wp'(z, L)$ is odd, and $\omega_1/2$, $\omega_2/2$, and $(\omega_1 + \omega_2)/2$ all differ from their negations by elements of L, $\wp'(z, L)$ must equal zero for such z.

2. Let L be a lattice with basis ω_1, ω_2, let $z \in \mathbb{C}$, and let D_z be the region $\{z + r_1\omega_1 + r_2\omega_2 : 0 \leq r_1, r_2 \leq 1\}$. Show that if f is an L-elliptic function, and f has no zeros or poles on the boundary of D_z, then we have:

$$\Sigma_{p \in D_z} \operatorname{ord}_p(f) = 0.$$

Let C_z be the boundary of D_z. Then C_z is a parallelogram, and by the residue theorem we have:

$$\Sigma_{p \in D_z} \operatorname{ord}_p(f) = \frac{1}{2\pi i} \int_{C_z} \frac{df}{f}.$$

But because f is L-periodic, the integrals on opposite sides of the parallelogram C_z cancel, proving the claim.

3. Let d be a positive integer, and L a lattice with basis ω_1, ω_2. Show that we have:

$$\wp(dz, L) = C + \frac{1}{d^2} \Sigma_{a=0}^{d-1} \Sigma_{b=0}^{d-1} \wp(z - \frac{a\omega_1 + b\omega_2}{d}, L)$$

for some constant C.

The Laurent expansion for $\wp(z, L)$ shows that $\wp(z, L) = \frac{1}{z^2}$ plus a holomorphic function in a neighborhood of zero (and analogously near any point of L.) Thus near any point x of $\frac{1}{d}L$, the function $\wp(dz, L)$ looks like $\frac{1}{d^2(z-x)^2}$ plus a holomorphic function. It follows that the difference:

$$\wp(dz, L) - \frac{1}{d^2} \Sigma_{a=0}^{d-1} \Sigma_{b=0}^{d-1} \wp(z - \frac{a\omega_1 + b\omega_2}{d}, L)$$

is holomorphic everywhere, and $\frac{1}{d}L$-periodic. In particular it is a bounded holomorphic function and hence constant.

4. Let $\mathbb{T}$ be the $\mathbb{Z}$-subalgebra of $\mathrm{End}(S_k)$ generated by the Hecke operators T_n for all n (that is, the subring of the ring of linear maps $S_k \to S_k$ consisting of linear maps that can be expressed as polynomials in the T_n with integer coefficients.) Recall that $S_k(\mathbb{Z})$ is the sublattice of S_k consisting of cusp forms whose q-expansions have integral coefficients. Show that the map: $\mathbb{T} \times S_k(\mathbb{Z}) \to \mathbb{Z}$ defined by $(T, f) = a_1(Tf)$ (where $a_1(Tf)$ is the leading coefficient of Tf) is a bilinear pairing, and is *perfect*; that is, induces an isomorphism of $S_k(\mathbb{Z})$ with $\mathrm{Hom}(\mathbb{T}, \mathbb{Z})$. (Hint, use the basis of $S_k(\mathbb{Z})$ constructed in example sheet 2.)

Consider the basis $g_1, \dots, g_{d-1}$ of $S_k(\mathbb{Z})$ constructed in example sheet 2. Recall from our formula in class for the q-expansion of $T_n(g)$ that $a_1(T_n g)$ is the nth q-expansion coefficient of g. Thus for $1 \leq i, j \leq d-1$ we have $a_1(T_i g_j) = 1$ if $i = j$ and 0 otherwise. It follows that $T_1, \dots T_{d-1}$ are linearly independent in $\mathbb{T}$.

On the other hand, let $\mathbb{T}_{\mathbb{C}}$ be the $\mathbb{C}$-subalgebra of $\mathrm{End}(S_k)$ generated by the Hecke operators T_n. By problem 1 on example sheet 3, $\mathbb{T}_{\mathbb{C}}$ is a product of copies of $\mathbb{C}$, one for each common eigenspace of the elements of $\mathbb{T}$. In particular, $\dim_{\mathbb{C}} \mathbb{T}_{\mathbb{C}}$ is at most $d-1$. Thus $T_1, \dots, T_{d-1}$ are a basis for $\mathbb{T}_{\mathbb{C}}$.

Now if $T = c_1 T_1 + \cdots + c_{d-1} T_{d-1}$ is any element of $\mathbb{T}_{\mathbb{C}}$, then $c_i = a_1(T g_i)$. In particular if T lies in $\mathbb{T}$ then the c_i are integers, so $T_1, \dots, T_{d-1}$ is a $\mathbb{Z}$-basis of $\mathbb{T}$.

The map $S_k(\mathbb{Z})$ to $\mathrm{Hom}(\mathbb{T}, \mathbb{Z})$ sends g to the map $T \mapsto a_1(Tg)$. In particular $\Sigma a_i g_i$ maps to the element of $\mathrm{Hom}(\mathbb{T}, \mathbb{Z})$ that takes T_i to a_i for $1 \leq i \leq d-1$. Since the T_i are a basis, every element of $\mathrm{Hom}(\mathbb{T}, \mathbb{Z})$ has this form and we are done.