MODULAR FORMS EXAMPLE SHEET 4

1. Let L be a lattice, with basis ω_1, ω_2. Use the parameterization of $y^2 = 4x^3 - g_4(L)x - g_6(L)$ by $\wp$ and $\wp'$ to show that the roots of the cubic $4x^3 - g_4(L)x - g_6(L)$ are the numbers $\wp(\omega_1/2, L)$, $\wp(\omega_2/2, L)$, and $\wp((\omega_1 + \omega_2)/2, L)$.

2. Let L be a lattice with basis ω_1, ω_2, let $z \in \mathbb{C}$, and let D_z be the region $\{z + r_1\omega_1 + r_2\omega_2 : 0 \leq r_1, r_2 \leq 1\}$. Show that if f is an L-elliptic function, and f has no zeros or poles on the boundary of D_z, then we have:

$$\Sigma_{p \in D_z} \operatorname{ord}_p(f) = 0.$$

3. Let d be a positive integer, and L a lattice with basis ω_1, ω_2. Show that we have:

$$\wp(dz, L) = C + \frac{1}{d^2}\Sigma_{a=0}^{d-1}\Sigma_{b=0}^{d-1}\wp(z - \frac{a\omega_1 + b\omega_2}{d}, L)$$

for some constant C.

4. Let $\mathbb{T}$ be the $\mathbb{Z}$-subalgebra of $\operatorname{End}(S_k)$ generated by the Hecke operators T_n for all n (that is, the subring of the ring of linear maps $S_k \to S_k$ consisting of linear maps that can be expressed as polynomials in the T_n with integer coefficients.) Recall that $S_k(\mathbb{Z})$ is the sublattice of S_k consisting of cusp forms whose q-expansions have integral coefficients. Show that the map: $\mathbb{T} \times S_k(\mathbb{Z}) \to \mathbb{Z}$ defined by $(T, f) = a_1(Tf)$ (where $a_1(Tf)$ is the leading coefficient of Tf) is a bilinear pairing, and is *perfect*; that is, induces an isomorphism of $S_k(\mathbb{Z})$ with $\operatorname{Hom}(\mathbb{T}, \mathbb{Z})$. (Hint, use the basis of $S_k(\mathbb{Z})$ constructed in example sheet 2.)