MODULAR FORMS EXAMPLE SHEET 3

1. Let K be a field and let A be a finite dimensional K-algebra; that is, a ring containing K that is finite dimensional as a K-vector space.

1a. Show that every prime ideal of A is maximal.

1b. Show that if A contains no nonzero nilpotent elements, then A is a product of finite extensions of K. [You may use, without proof, the result that an element r of a ring R is nilpotent if, and only if, r is contained in every prime ideal of R. For a proof of this fact, see for instance Eisenbud, *Commutative Algebra*, pp. 70-71.]

1c. Let V be a finite dimensional vector space over an algebraically closed field K, and let $T_1, T_2, \dots$ be a collection of commuting linear operators on V. Show that V admits a basis of common eigenvectors for the T_i if, and only if, there is no polynomial in the T_i (with coefficients in K) that is a nonzero nilpotent endomorphism of V. [Hint: Let A be the subring of $\mathrm{End}(V)$ consisting of all polynomials in the T_i with coefficients in K, and use 1b.]

2. Let $F(z) = q\prod(1-q^n)^{24}$. Show that $\frac{d}{dz}\log F(z) = \frac{6i}{\pi}G_2(z)$, where G_2 is the weight 2 Eisenstein series from the previous example sheet. Deduce that $\log F(-\frac{1}{z}) - \log z^{12}F(z)$ is constant, and show that this constant is zero. Conclude that $F = \Delta$.

3. Let $M_k(\mathbb{Z})$ be the space of modular forms of weight k with integral q-expansions. Show that the graded ring:

$$\bigoplus M_k(\mathbb{Z})$$

is generated over $\mathbb{Z}$ by E_4, E_6, and Δ.

4. Let V be a *three* dimensional real vector space, and let X be the set of lattices L in V. For (a,b) positive integers with a dividing b, define a correspondence $T_{(a,b)} : \mathbb{Z}[X] \to \mathbb{Z}[X]$ that sends L to the sum of $L' \subset L$ such that L/L' is isomorphic to $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$.

4a. Show that if $(b,b') = 1$, then $T_{(a,b)}T_{(a',b')} = T_{(aa',bb')}$.

4b. Fix a prime p, and express $T_{(1,p^2)}, T_{(1,p^3)}$, and $T_{(p,p^2)}$ as polynomials in $T_{(1,p)}, T_{(p,p)}$ and the "rescaling by p" operator R_p.