MODULAR FORMS EXAMPLE SHEET 2

1. Let L and L' be lattices in $\mathbb{C}$ such that $G_4(L) = G_4(L')$ and $G_6(L) = G_6(L')$. Show that $L = L'$.

As j is a rational function in G_4 and G_6, we also have $j(L) = j(L')$. Since j induces a bijection from lattices modulo rescaling to $\mathbb{C}$, we must have $L = \lambda L' = \lambda' L_z$ for some $\lambda \in \mathbb{C}$, $z \in \mathbb{H}$. Subsituting into $G_4(L) = G_4(L')$ and $G_6(L) = G_6(L')$ we find that $(\lambda^6 - 1)G_6(z)$ and $(\lambda^4 - 1)G_4(z)$ both vanish. If $G_4(z)$ and $G_6(z)$ are both nonzero we can conclude that $\lambda^2 = 1$, so $\lambda = \pm 1$ and thus $L = L'$. If $G_4(z) = 0$, then z is in the orbit of ρ, so $L_z = L_\rho$. Then $G_6(z)$ is nonzero (the only zeroes of G_6 are in the orbit of i), so λ is a 6th root of unity; i.e. $\lambda \in \{\pm 1, \pm\rho, \pm\rho^2\}$. For such λ we have $\lambda L_\rho = L_\rho$, so $L = L'$. Finally, if $G_6(z) = 0$, then z is in the orbit of i, $L_z = L_i$, and λ is a fourth root of unity; we again have $\lambda L_i = L_i$ so $L = L'$.

2. Define $G_2(z)$, the Eisenstein series of weight 2, to be the series:

$$G_2(z) = \Sigma_{m \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n)^2}.$$

(Note that this is not absolutely convergent.)

2a. Define $G_2'(z)$ by:

$$G_2'(z) = \Sigma_{n \in \mathbb{Z}} \Sigma_{m \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n)^2}.$$

Show that $G_2(-z^{-1}) = z^2 G_2'(z)$.

We have

$$G_2(-z^{-1}) = \Sigma_{m \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(m(-z)^{-1}+n)^2}$$

$$= z^2 \Sigma_{m \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(m+nz)^2} = z^2 G_2'(z)$$

2b. Similarly define H and H' to be the sums:

$$H(z) = \Sigma_{m \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}, (m,n) \neq (0,0),(0,1)} \frac{1}{(mz+n-1)(mz+n)}$$

$$H'(z) = \Sigma_{n \in \mathbb{Z}} \Sigma_{m \in \mathbb{Z}, (m,n) \neq (0,0),(0,1)} \frac{1}{(mz+n-1)(mz+n)}$$

and show that $H(z) = 2$, whereas $H'(z)$ converges to $2 - \frac{2\pi i}{z}$.

Write $\frac{1}{(mz+n-1)(mz+n)} = \frac{1}{mz+n-1} - \frac{1}{mz+n}$. Now for $H(z)$, note that when $m \neq 0$, the inner sum

$$\Sigma_{n\in\mathbb{Z}}\frac{1}{mz+n-1} - \frac{1}{mz+n} = \lim_{N\to\infty}\Sigma_{n=-N}^{N}\frac{1}{mz+n-1} - \frac{1}{mz+n}.$$

Successive terms cancel in this truncated sum, leaving $\frac{1}{mz-N-1} - \frac{1}{mz+N}$, which tends to zero as N tends to infinity. When $m = 0$ one has similar cancellation, except that the "missing" terms $(0,0)$ and $(0,1)$ prevent everything from cancelling; one checks that in this case the inner sum is 2.

For $H'(z)$ we are not so lucky. Instead, we need to use the series:

$$\pi\cot\pi z = \Sigma_{n\in\mathbb{Z}}\frac{1}{z+n}$$

to compute $H'(\frac{1}{z})$. We have:

$$H'(\frac{1}{z}) = \Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)}\frac{1}{\frac{m}{z}+n-1} - \frac{1}{\frac{m}{z}+n}$$

$$z\Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)}\frac{1}{m+(n-1)z} - \frac{1}{m+nz}.$$

Let us first compute the sum:

$$\Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)}\frac{1}{m+(n-1)z}.$$

If $n \neq (0,1)$ the inner sum is $\pi\cot(n-1)\pi z$. When $n = 0$ the inner sum is "almost" the series for $\pi\cot(-\pi z)$, except that it is missing the $m = 0$ term $-\frac{1}{z}$. So when $n = 0$ the inner sum is $\frac{1}{z} + \pi\cot(-\pi z)$. When $n = 1$ the inner sum vanishes.

We thus have

$$\Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)}\frac{1}{m+(n-1)z} = \frac{1}{z} + \Sigma_{n\in\mathbb{Z},n\neq 1}\pi\cot(n-1)\pi z.$$

This latter sum is the limit, as $N \to \infty$, of $\Sigma_{n=-N-1,n\neq 0}^{N-1}\pi\cot n\pi z$. As cotangent is odd, every term cancels except for $n = -N-1$, $n--N$, so this sum is equal to the limit, as $N \to \infty$, of $\pi\cot -N\pi z + \pi\cot -(N+1)\pi z$. It follows from the identity $cotz = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$, together with the fact that z is in the upper half plane, that $\cot -N\pi z$ approaches $-i$ as N approaches infinity. Thus as N approaches infinity, this series approaches $\frac{1}{z} - 2\pi i$.

A completely analogous computation with the sum

$$\Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0),(0,1)}\frac{1}{m+nz}$$

shows that it is equal to $\frac{1}{z}$. (This time, the lack of a shift for n causes the cotangents to cancel exactly.) Thus $H'(\frac{1}{z}) = 2 - 2\pi iz$. Substituting z for $frac1z$ yields the desired result.

2c. Show that the resulting series for $G_2 - H$ and $G_2' - H'$ are absolutely convergent, and rearrangements of each other, so that $G_2(z) - H(z) = G_2'(z) - H'(z)$. Show that it follows from this that G_2 is convergent, uniformly on compact subsets of the upper half plane. In particular, $G_2(z)$ is holomorphic.

We have (up to a missing term where $m = 0, n = 1$):

$$G_2(z) - H(z) = \Sigma_{m\in\mathbb{Z}}\Sigma_{n\in\mathbb{Z},(m,n)\neq(0,0)}\frac{1}{(mz+n)^2} - \frac{1}{(mz+n-1)(mz+n)}$$
$$= \Sigma_{m\in\mathbb{Z}}\Sigma_{n\in\mathbb{Z},(m,n)\neq(0,0)}\frac{1}{(mz+n-1)(mz+n)^2}.$$
$$G_2'(z) - H'(z) = \Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0)}\frac{1}{(mz+n)^2} - \frac{1}{(mz+n-1)(mz+n)}$$
$$= \Sigma_{n\in\mathbb{Z}}\Sigma_{m\in\mathbb{Z},(m,n)\neq(0,0)}\frac{1}{(mz+n-1)(mz+n)^2}.$$

These are both absolutely convergent, uniformly on compact subsets of the upper half plane, because the denominator goes as $(m^2+n^2)^{frac32}$ as (m,n) go to infinity. Moreover the two series are clearly rearrangements of each other, and thus have the same sum. Thus G_2 differs from the (conditionally) convergent series H by an absolutely convergent series, and is thus convergent. Moreover, since the convergence for H is uniform on compact subsets of $\mathbb{H}$, the same is true for G_2. Thus $G_2(z)$ is holomorphic.

2d. Conclude from 2d that $G_2(-z^{-1}) - z^2G(z) = -2\pi iz$. (In particular, G_2, although holomorphic, is NOT a modular form.

This is immediate from 2b and 2c.

2e. This means there's no lattice function corresponding to G_2. If we try to define a lattice function by setting

$$G_2(L) = \Sigma_{w\in L\backslash 0}w^{-2},$$

what goes wrong?

This sum depends not just on L but also the order of summation, so is not a well-defined function on lattices.

2g. Find the q-expansion of G_2.

We use the series identity from class:

$$\Sigma_{n\in\mathbb{Z}}\frac{1}{z+n} = \pi\cot\pi z = i\pi - 2i\pi\Sigma_{d=0}^{\infty}q^d$$

Differentiating once (as in the calculation of the q-expansions in higher weights) we find

$$\Sigma_{n\in\mathbb{Z}}\frac{1}{(z+n)^2} = -4\pi^2\Sigma_{d=0}^{\infty}dq^d.$$

Separating out the $m = 0$ term in the series for $G_2(z)$, and using that the m and $-m$ sums are the same, we have

$$G_2(z) = \Sigma_{n\in\mathbb{Z},n\neq 0}\frac{1}{n^2} + 2\Sigma_{n=1}^{\infty}\Sigma_{d=0}^{\infty}dq^{nd}.$$

Collecting q^m terms, and using the fact that the sum of the reciprocals of the squares is $\pi^2/6$, we find:

$$G_2(z) = \frac{\pi^2}{3} - 8\pi^2\Sigma_{m=1}^{\infty}\sigma_1(m)q^m.$$

3a. Let $d = \dim M_k$. Show that there is a unique basis for M_k of the form $g_1, \ldots, g_d$, where for all i, the q-expansion of g_i has the form $q^{i-1} + \Sigma_{n=d}^{\infty}c_nq^n$.

We first show that there is a basis $f_1, \ldots, f_d$ such that f_i vanishes to order exactly $i-1$ at ∞. We do this by induction on k; note that if $k < 12$, then d is zero or 1, and if $d = 1$ then taking $f_1 = E_k$ suffices. For the general case, note that $\dim M_{k-12} = d - 1$ and let $h_1, \ldots, h_{d-1}$ be a basis for M_{k-12} such that h_i vanishes to order exactly $i - 1$ for all i. Then we can take $f_1 = E_k$ and $f_i = \Delta h_{i-1}$ for all i and this has the desired property. Moreover, note that the f_i have integer coefficients if the h_i do (and they do in the base case!) so the induction also proves that the f_i are integer coefficients.

To get the g_i, let $g_d = f_d$, obtain f_{d-1} from g_{d-1} by subtracting a multiple of g_d so that the q^{d-1} term vanishes, obtain g_{d-2} from g_{d-2} by subtracting multiples of g_d and g_{d-1} so that the q^{d-2} and q^{d-1} terms vanish, and so forth.

3b. Show further that any element of M_k whose q-expansion has integer coefficients is an integral linear combination of the g_i.

Let f be such an element, write $f = \Sigma a_nq^n$, and consider $h = f - \Sigma_{n=0}^{d-1}a_ig_{i+1}$. Then h lies in M_k and vanishes to order d at infinity. It follows that $k \geq 12d$, by our formula relating the sum of the orders of vanishing of h to the weight of h. This only occurs when $k \equiv 2 \bmod 12$, and then $d = \frac{k-2}{12}$. Then h/Δ^d is a modular form of weight 2 and is therefore zero. So f is an integer linear combination of the g_i as claimed.