MODULAR FORMS EXAMPLE SHEET 2

1. Let L and L' be lattices in $\mathbb{C}$ such that $G_4(L) = G_4(L')$ and $G_6(L) = G_6(L')$. Show that $L = L'$.

2. Define $G_2(z)$, the Eisenstein series of weight 2, to be the series:

$$G_2(z) = \Sigma_{m \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n)^2}.$$

(Note that this is not absolutely convergent.)

2a. Define $G_2'(z)$ by:

$$G_2'(z) = \Sigma_{n \in \mathbb{Z}} \Sigma_{m \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n)^2}.$$

Show that $G_2(-z^{-1}) = z^2 G_2'(z)$.

2b. Similarly define H and H' to be the sums:

$$H(z) = \Sigma_{m \in \mathbb{Z}} \Sigma_{n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n-1)(mz+n)}$$

$$H'(z) = \Sigma_{n \in \mathbb{Z}} \Sigma_{m \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(mz+n-1)(mz+n)}$$

and show that $H(z) = 2$, whereas $H'(z)$ converges to $2 - \frac{2\pi i}{z}$.

2c. Show that the resulting series for $G_2 - H$ and $G_2' - H'$ are absolutely convergent, and rearrangements of each other, so that $G_2(z) - H(z) = G_2'(z) - H'(z)$. Show that it follows from this that G_2 is convergent, uniformly on compact subsets of the upper half plane. In particular, $G_2(z)$ is holomorphic.

2d. Conclude from 2d that $G_2(-z^{-1}) - z^2 G(z) = -2\pi i z$. (In particular, G_2, although holomorphic, is NOT a modular form.

2e. This means there's no lattice function corresponding to G_2. If we try to define a lattice function by setting

$$G_2(L) = \Sigma_{w \in L \setminus 0} w^{-2},$$

what goes wrong?

2g. Find the q-expansion of G_2.

3a. Let $d = \dim M_k$. Show that there is a unique basis for M_k of the form $g_1, \ldots, g_d$, where for all i, the q-expansion of g_i has the form $q^{i-1} + \Sigma_{n=d}^{\infty} c_n q^n$.

3b. Show further that any element of M_k whose q-expansion has integer coefficients is an integral linear combination of the g_i.