MODULAR FORMS EXAMPLE SHEET 1

1. Let V be a finite dimensional real vector space, and recall that a lattice in V is a closed discrete additive subgroup of V. Show that every lattice in V has the form $\mathbb{Z} \cdot v_1 + \mathbb{Z} \cdot v_2 + \cdots + \mathbb{Z} \cdot v_n$ for some linearly independent subset $v_1, \ldots v_n$ of V.

2. Show directly from the definition that there are no nonzero weakly modular functions of weight k for odd k, and that every modular form of weight zero is constant.

3. Let f be a weakly modular function, and g the unique function on the unit disk such that $f(z) = g(e^{2\pi i z})$. Show that g is meromorphic at zero if, and only if, there exists an integer N and a positive constant c such that such that $|f(z)| < ce^{N(\operatorname{Im} z)}$ for $\operatorname{Im} z >> 0$. Show that g is holomorphic at zero if we can take N to be zero, and that in this case $f(z)$ approaches $g(0)$ as $\operatorname{Im} z$ approaches ∞.

4. A lattice L in $\mathbb{C}$ is said to have *complex multiplication* if there is an $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ such that $\alpha L \subseteq L$. Show that the lattice $L_{1,z}$ has complex multiplication if, and only if, z satisfies a quadratic polynomial P with integral coefficients. Show further that if this is the case, then the set of $\alpha \in \mathbb{C}$ with $\alpha L \subseteq L$ is an order in the number field $\mathbb{Q}(z)$.

5a. Use the equations $E_8 = E_4^2$ and $E_{10} = E_4 E_6$ to deduce identities relating σ_3 and σ_7 in the first case, and σ_3, σ_5, and σ_9 in the second.

5b. Find constants c_1, c_2 such that $E_4^3 = c_1 E_{12} + c_2 \Delta$. Conclude that if $\Delta(q) = \Sigma \tau(n) q^n$, then $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$. This is called "Ramanujan's congruence".

6. Show (using the q-expansions for E_4 and E_6, and the identity $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, that the q-expansion of Δ has integral coefficients.