M4/5P8 MASTERY MATERIAL EXERCISES

1. Let R be an integral domain with field of fractions K, and let I be a nonzero ideal of R. Let $Z_I \subset K$ be the subset $z \in K : zI \subseteq I$.

1a. Show that Z_I is a subring of K that contains R.

1b. Suppose R is Noetherian. Show that Z_I is integral over R. [HINT: adapt the proof from the mastery material that if R is integrally closed and $xI \subseteq I$ for some $x \in K$ then x lies in R.]

1c. Show that if R is Noetherian and integrally closed then $Z_I = R$.

1d. Show that for R an arbitrary integral domain, if I and J are ideals then $Z_I \subseteq Z_{IJ}$. Conclude that if $Z_I \neq R$ then there does not exist an ideal J such that IJ is a principal ideal.

1e. Show that if R is Noetherian, and the nonzero fractional ideals of R form a group under multiplication, then R is integrally closed. [HINT: if R is not integrally closed, use an element x that is integral over R but not contained in R to construct an ideal I with Z_I larger than R; then use 1d to show that no fractional ideal is its multiplicative inverse.]

2. Let F be a field, let $P(X) \in F[X]$ be a squarefree polynomial, and let K be the field of fractions $F(X)$ of $F[X]$. Let L be the quadratic extension of K defined by $K[T]/\langle T^2 - P(X)\rangle$, and let $\mathcal{O}$ be the integral closure of $F[X]$ in L.

2a. Show that if the characteristic of F is not 2, then every element of $\mathcal{O}$ can be expressed uniquely as $A(X) + B(X)\sqrt{P(X)}$, where $A(X)$ and $B(X)$ are elements of $F[X]$. [HINT: mimic the calculation of the ring of integers of a quadratic extension of $\mathbb{Q}$ from the lectures.]

2b. Give an example showing that the description of $\mathcal{O}$ from 2a fails when F has characteristic two.

3. Let R be a Noetherian UFD in which every nonzero prime is maximal.

3a. Show that every nonzero prime ideal of R contains an irreducible element.

3b. Show that R is a principal ideal domain.

4a. Let R be a Dedekind domain, and let $\mathfrak{p}$ be a nonzero prime ideal of R. Show that for all $i \geq 0$, there exists an element π_i of $\mathfrak{p}^i$ that does not lie in $\mathfrak{p}^{i+1}$.

4b. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be a finite collection of distinct nonzero prime ideals of a Dedekind domain R. For each $i \geq 0$, and each $j \in 1, \dots, r$, let $\pi_{i,j}$ be an element of $\mathfrak{p}_j^i \setminus \mathfrak{p}_j^{i+1}$. Show that for any sequence of nonnegative integers $n_1, \dots, n_r$, there exists $a \in R$ such that $a \equiv \pi_{n_j,j}$ in $R/\mathfrak{p}_j^{n_j+1}$ for all j.

4c. Use 4b to show that any Dedekind domain R with only *finitely many* prime ideals is a principal ideal domain. [HINT: given an ideal of the form $\mathfrak{p}_1^{n_1} \dots \mathfrak{p}_r^{n_r}$, use 4b to find a generator.]

5a. Let R be an integrally closed integral domain, and let $S \subset R$ be a multiplicative system. Show that $S^{-1}R$ is integrally closed.

5b. Show that if R is a Dedekind domain then the ideals of $S^{-1}R$ (other than the unit ideal) are in bijection with the ideals J of R such that if $s \in S$, and $j \in R$ such that $sj \in J$, then j lies in J.

5c. Show that if R is a Dedekind domain then so is $S^{-1}R$ for any multiplicative system S.

6a. Let $\mathfrak{p}$ be a prime ideal of a ring R. Show that the complement $R \setminus \mathfrak{p}$ is a multiplicative system.

6b. Let R be a Dedekind domain and $\mathfrak{p}$ a nonzero prime ideal. Let $S = R \setminus \mathfrak{p}$. Show that $S^{-1}R$ is a principal ideal domain. [HINT: use 4c!]

6c. Show further that there is an element $\pi \in R$ (called a *uniformizer* for R) such that every nonzero element of R can be written uniquely as $u\pi^n$ for a unit $u \in R^\times$ and a nonnegative integer n.

7a. Let $K/\mathbb{Q}$ be a finite extension and let $\mathcal{O}_K$ denote the ring of integers of K. The *norm* $N(I)$ of a nonzero ideal I of $\mathcal{O}_K$ is the number of elements in the quotient $\mathcal{O}_K/I$. Show that if I and J are two nonzero ideals, then $N(IJ) = N(I)N(J)$. [HINT: adapt the argument of problem 4 to show that every ideal of $\mathcal{O}_K/IJ$ is principal. Use this to show that I/IJ is isomorphic to J.]

7b. Show that if I is a maximal ideal of $\mathcal{O}_K$ then $N(I)$ is a prime power.

7c. Give an example of a field K and a prime ideal I of $\mathcal{O}_K$ such that $N(I)$ is not prime.

8. Factor the ideals $\langle 14, 5 + 3\sqrt{-5} \rangle$ and $\langle 18, 11 - \sqrt{-5} \rangle$ into products of prime ideals in $\mathbb{Z}[\sqrt{-5}]$. Be sure to prove your ideals are prime! [HINT: consider their norms, and note that every ideal contains its norm.]