M3P8 DISCUSSION PROBLEMS 3

These problems are for discussion in lecture and will NOT be assessed. They are somewhat more difficult than typical assessed coursework problems; do not get discouraged if it is not immediately clear how to approach them. Working on these in groups with your fellow students and trading ideas and approaches is HIGHLY encouraged!

If you have difficulty getting complete solutions to any particular problem it might be helpful to: work out some examples, try to find partial results in special cases, identify related, easier questions, etc. If all else fails, I am happy to give hints in office hours!

1. Let K be a field, V a finite dimensional K-vector space, and $L : V \to V$ a K-linear map. Let M_L be the corresponding $K[T]$-module. We say that an element v of V is a *cyclic vector* for L if $v, Lv, L^2v, \dots$ spans all of V.

1a. Show that there exists a cyclic vector for L if, and only if, M_L is generated by a single element as a $K[T]$-module.

1b. We say L is *regular* if its minimal polynomial equals its characteristic polynomial. Show that L is regular if and only if there exists a cyclic vector for L.

1c. Show that L is regular if, and only if, every K-linear map $L' : V \to V$ that commutes with L is a polynomial in L. [HINT: show that such L' correspond to $K[T]$-module homomorphisms from M_L to M_L.]

2. Show that if R is a Noetherian integral domain, then every element of R is a finite product of irreducible elements.

3a. Let K be a field in which 2 is invertible, and let $P(X) = aX^2+bX+c$ be a quadratic polynomial in $K[X]$, with a nonzero. Show that $P(X)$ factors in $K[X]$ if, and only if, there exists α in K with $\alpha^2 = b^2 - 4ac$, and express the roots of $P(X)$ in terms of α when this is the case.

3b. Now let K be finite of characteristic 2, and let $P(X) = aX^2 + bX + c$ in $K[X]$, with a nonzero. Show that $P(X)$ factors in $K[X]$ if $b = 0$. If b is nonzero, show that $P(X)$ factors if, and only if, there exists $\alpha \in K$ such that $\alpha(\alpha + 1) = \frac{ac}{b^2}$, and express the roots of $P(X)$ in terms of α.

3c. Show that if $K = \mathbb{F}_{2^r}$, the polynomial $X^2 + X = c$ has a root if, and only if, $c + c^2 + c^4 + \cdots + c^{2^{r-1}} = 0$.

4a. Show that for every odd prime $p \in \mathbb{Z}$, at least one of -1, 2, and -2 is a square in $\mathbb{F}_p$.

4b. Show that if there exists $a \in \mathbb{F}_p$ such that $a^2 = 2$, then $X^4 + 1 = (X^2 + aX + 1)(X^2 - aX + 1)$ in $\mathbb{F}_p[X]$. Show similarly that $X^4 + 1$ also factors as a product of two quadratics in $\mathbb{F}_p[X]$ if -1 or -2 is a square in $\mathbb{F}_p$.

4c. Show that $X^4 + 1$ factors over $\mathbb{F}_2$, and is thus reducible mod p for every p.

5. Show that the following polynomials are irreducible in the given rings:

5a. The polynomial $X^4Y^3 + 3X^5Y + X^6 + X^4Y + 27X^2 + XY + 3Y + 6$ in $\mathbb{Q}[X, Y]$.

5b. The polynomial $X^6Y + X^2Y^2 - Y^2 + X^2 + 3X + 2$ in $\mathbb{C}[X, Y]$.