M3P8 DISCUSSION PROBLEMS 1

These problems are for discussion in lecture and will NOT be assessed. They are somewhat more difficult than typical assessed coursework problems; do not get discouraged if it is not immediately clear how to approach them. Working on these in groups with your fellow students and trading ideas and approaches is HIGHLY encouraged!

If you have difficulty getting complete solutions to any particular problem it might be helpful to: work out some examples, try to find partial results in special cases, identify related, easier questions, etc. If all else fails, I am happy to give hints in office hours!

1. Let R be the ring of continous functions on the unit interval $[0,1]$, where addition and multiplication of functions is defined pointwise.

1a. Show that for any $c \in [0,1]$, the subset $\{f \in R : f(c) = 0\}$ is a maximal ideal M_c of R.

1b. Show that if $b \neq c$, then $M_b \neq M_c$.

1c. Show that if M is any maximal ideal of R, then $M = M_c$ for some c.

1d. Show that M_c is not generated by the element $f(x) = x - c$ of R. Show further that M_c is not even finitely generated!

2. For any multiplicative subset S of $\mathbb{Z}$, the ring $S^{-1}\mathbb{Z}$ is a subring of $\mathbb{Q}$. Is every subring of $\mathbb{Q}$ of this form?

3. Show that when R is an integral domain, and S is a multiplicative system on R, then for any homomorphism $f : R \to R'$ such that $f(S)$ is contained in $(R')^\times$, there is a unique homomorphism $S^{-1}f : S^{-1}R \to R'$ extending f. (This is sometimes called the "universal property of localization".)

4. Let R be a ring that is not necessarily an integral domain, and S a multiplicative subset.

4a. Define a relation $\sim_1$ on the set of expressions of the form $\frac{r}{s}$, with $r \in R, s \in S$, by $\frac{r}{s} \sim_1 \frac{r'}{s'}$ if $rs' = r's$. Give an example to show that if R is not an integral domain, then $\sim_1$ is not an equivalence relation.

4b. Now say that $\frac{r}{s} \sim_2 \frac{r'}{s'}$ if there exists $s'' \in S$ such that $rs's'' = r'ss''$. Show that $\sim_2$ is an equivalence relation, and that the set of equivalence classes of expressions of the form $\frac{r}{s}$ forms a ring (with the same addition and multiplication as in the case when R is an integral domain), which we denote $S^{-1}R$.

4c. Describe the kernel of the homomorphism $R \to S^{-1}R$ that takes R to $\frac{r}{1}$. In particular give an example where this kernel is nonzero.

5a. Let a and b be relatively prime integers, and $i^2 = -1$. Show that the natural map $\mathbb{Z} \to \mathbb{Z}[i]/\langle a+bi \rangle$ induces an isomorphism of $\mathbb{Z}/(a^2+b^2)\mathbb{Z}$ with $\mathbb{Z}[i]/\langle a+bi \rangle$.

5b. Show on the other hand that if a and b are not relatively prime, then the map $\mathbb{Z}/(a^2+b^2)\mathbb{Z} \to \mathbb{Z}[i]/\langle a+bi \rangle$ is neither injective nor surjective.

6. Let R be a ring. The ring $R[[X]]$ of *formal power series* in X with coefficients in R is the ring whose elements are (informally) expressions of the form

$$\sum_{n=0}^{\infty} a_n X^n$$

with $a_n \in R$ for all n. (Alternatively, they can be thought of as infinite sequences $a_0, a_1, \ldots$ of elements of R). We add and multiply these by the usual rules:

$$\sum_{n=0}^{\infty} a_n X^n + \sum_{n=0}^{\infty} b_n X^n = \sum_{n=0}^{\infty} (a_n + b_n) X^n,$$

$$\left(\sum_{n=0}^{\infty} a_n X^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n X^n\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) X^n.$$

6a. Show that the units of $R[[X]]$ are precisely those power series $\sum_{n=0}^{\infty} a_n X^n$ with a_0 a unit in R. In particular, if R is a field, then so is $R[[X]][\frac{1}{X}]$ (in particular, the latter is isomorphic to the field of fractions of $R[[X]]$, which we denote by $R((X))$.

6b. Show that when R is a field, the ideals of $R[[X]]$ are precisely the ideals $\langle X^n \rangle$ for n a nonnegative integer.

6c. Show that $(R[[X]])[[Y]]$ is naturally isomorphic to $R[[Y]][[X]]$ (this ring is often denoted $R[[X, Y]]$.)

6d. Show that when R is a field, the field of fractions of $R[[X, Y]]$ embeds in $[R((X))]((Y))$, but that this embedding is *not* an isomorphism!