M3P8 ASSESSED PROBLEMS 2 - DUE FRIDAY, NOVEMBER 29, 2019

This is the second assessed coursework for M3P8; unlike the discussion problems, all rules for assessed coursework at Imperial apply. It is due Friday, November 29, at 4PM in the undergraduate office, and is worth 5 percent of your total marks for the course. There are 20 total marks.

1. (2 points) Let $K \subseteq F$ be a field extension, and let $P(X) \in K[X]$ be a polynomial of degree d with d distinct roots $\alpha_1, \ldots, \alpha_d$ in F. Show that the degree of $K(\alpha_1, \ldots, \alpha_d)$ over K is at most $d!$.

2. (2 points) List, with proof, the irreducible polynomials of degree 4 over $\mathbb{F}_2$.

3. Let R be a ring and let V be a free R-module. Give a proof or counterexample to each of the following statements:

3a. (2 points) Any generating set for V contains a basis.

3b. (2 points) Any linearly independent subset of V can be extended to a basis for V.

4. (2 points) Show that if R and S are rings, then any $R \times S$-module V is of the form $M \times N$, where M is an R-module, N is an S-module, and $(r, s)(m, n) = (rm, sn)$ for $(r, s) \in R \times S$ and $(m, n) \in M \times N$.

5. (2 points) Let L be the subgroup of $\mathbb{Z}^3$ generated by $(3, 2, 1)$, $(8, 4, 2)$, $(7, 6, 2)$, and $(9, 6, 1)$. Describe the quotient $\mathbb{Z}^3/L$ in terms of the classification of finitely generated abelian groups.

6. (2 points) Find the Smith normal form of the matrix $\begin{pmatrix} t^2 - 3t + 2 & t - 2 \\ (t-1)^3 & t^2 - 3t + 2 \end{pmatrix}$ with entries in $\mathbb{Q}[t]$.

7. Let $L : \mathbb{C}^3 \to \mathbb{C}^3$ be the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$.

7a. (2 points) Find a presentation matrix A for the $\mathbb{C}[T]$-module M_L attached to L (that is, the three-dimensional $\mathbb{C}$-vector space on which T acts via the matrix L.) What is the Smith normal form of this matrix?

7b. (2 points) Find the rational canonical form and Jordan normal form of L.

8. (2 points) Let A be an n by n matrix over the field $\mathbb{C}$ such that A^m is the identity matrix. Show that A is diagonalizable.