M3P8 ASSESSED PROBLEMS 1 - DUE FRIDAY, NOVEMBER 1, 2019

This is the first assessed coursework for M3P8; unlike the discussion problems, all rules for assessed coursework at Imperial apply. It is due Friday, November 1, at 4PM in the undergraduate office, and is worth 5 percent of your total marks for the course.

1. Let R be a ring. A nilpotent element of R is an element $r \in R$ such that $r^n = 0$ for some nonnegative integer n.

1a. (1 point) Show that the nilpotent elements of R form an ideal I of R.

1b. (1 point) Show that if r is a nilpotent element of R, then for any unit $u \in R^{\times}$, $u + r$ is also a unit of R.

2. (1 point) For I, J ideals of a ring R, define the ideal product IJ to be the subset of R consisting of all elements of R expressible as

$$i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$$

for some positive integer n and elements $i_1, \ldots, i_n$ of I and $j_1, \ldots, j_n$ of J. Show that if P is a prime ideal of R that contains IJ, then P contains I or P contains J.

3a. (1 point) Show that the ring $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

3b. (1 point) Using the Euclidean algorithm on $\mathbb{Z}[\sqrt{-2}]$, or otherwise, compute a greatest common divisor of 85 and $10 + \sqrt{-2}$.

4. (1 point) Show that if R is a principal ideal domain then every nonzero prime ideal of R is maximal.

5. (1 point) Show that the elements $ax + b$ for $a, b \in \{0, 1, 2\}$ form a set of representatives for the equivalence classes in the quotient $\mathbb{Z}/3\mathbb{Z}[x]/\langle x^2 + x - 1\rangle$, and write down the multiplication table for this ring with respect to these elements.

6. (1 point) Let R be an integral domain with only finitely many elements. Show that R is a field.

7. (1 point) Give an example of an ideal of $\mathbb{Z}[\sqrt{-3}]$ that is not principal.

8. (1 point) Let R be an integral domain. Show that for all n, we have $R[X_1, \ldots, X_n]^{\times} = R^{\times}$.