M3P8 LECTURE NOTES 8: NOETHERIAN RINGS AND R-MODULES

1. Definitions and Basic Properties

Let R be a ring and let M be an R-module. We say M is *Noetherian* if every increasing infinite chain

$$N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$$

of R-submodules N_i of M is eventually constant. (That is, for any such chain, we have $N_i = N_{i+1}$ for all sufficiently large i. A ring R is Noetherian if R is Noetherian as an R-module.

Since the R-submodules of R are just the ideals of R, a ring R is Noetherian if every increasing infinite chain:

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$$

of ideals I_j of R is eventually constant.

The following result about Noetherian R-modules is fundamental:

Theorem 1.1. *An R-module M is Noetherian if, and only if, every R-submodule of M is finitely generated.*

Proof. Suppose first that M is Noetherian, and let N be an R-submodule of M. Choose an element n_0 of N, and let N_0 be the R-submodule of N generated by N_0. If N_0 is all of N, then N is finitely generated. Otherwise, choose n_1 in $N \setminus N_0$, and let N_1 be the R-submodule of N generated by n_0 and n_1. If N is not finitely generated, we may continue this process indefinitely, choosing for each i an n_i in $N \setminus N_{i-1}$ (which is nonempty since N is not finitely generated), and letting N_i be generated by $n_0, \dots, n_i$. In this way we obtain a strictly increasing infinite chain

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots$$

of submodules of M, contradicting the fact that M is Noetherian.

Conversely, suppose that every R-submodule of M is finitely generated, and let

$$N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$$

be an increasing chain. We must show that this chain is eventually constant. Let N be the union of the submodules N_i; note that N is an R-submodule of M. Thus N is finitely generated, say by $n_0, \dots, n_r$. Since N is the union of the N_i, there exists $j_1, \dots, j_r$ such that n_i is in N_{j_i} for all i. Let j be the largest of the j_i. Then N_j contains $n_0, \dots, n_r$ so it contains N. In particular for any $i \geq j$ we have $N_j \subset N_i \subset N \subset N_j$, so $N = N_i = N_j$ for all such i and the chain is constant after N_j. □

Corollary 1.2. *Let R be a Principal Ideal Domain. Then R is Noetherian.*

Proof. Every ideal of R is principal, hence finitely generated. □

2. Finitely Generated Modules over Noetherian Rings

The goal of this section is to prove the following crucial theorem:

Theorem 2.1. *Any finitely generated module over a Noetherian ring is Noetherian.*

We proceed in several steps. First note:

Proposition 2.2. *Let M be a Noetherian R-module. Then for any submodule N of M, both N and M/N are Noetherian.*

Proof. Since M is Noetherian, any submodule of M is finitely generated, and thus any submodule of N is finitely generated. Given a submodule J of M/N, let $\tilde{J}$ be its preimage in N under the natural map $M \to M/N$. Then $\tilde{J}$ is finitely generated, and the image of a generating set for $\tilde{J}$ in J is a generating set for J. □

Proposition 2.3. *Let M be an R-module, let N be a Noetherian submodule of M, and suppose that M/N is Noetherian. Then M is Noetherian.*

Proof. Let J be a submodule of M. Then $J \cap N$ is a submodule of N, hence finitely generated. Let $j_1, \dots, j_n$ generate $J \cap N$. Let $\overline{J}$ denote the image of J in M/N; this is a submodule of M/N and thus finitely generated. Let $\overline{j}_{n+1}, \dots, \overline{j}_m$ generate $\overline{J}$, and choose elements $j_{n+1}, \dots, j_m$ of J mapping to $\overline{j}_{n+1}, \dots, \overline{j}_m$, respectively.

We now show that $j_1, \dots, j_m$ is a generating set for J, proving the claim. Given any $j \in J$, let $\overline{j}$ be its image in M/N. Then we can write $\overline{j}$ as a sum $r_{n+1}\overline{j}_{n+1} + \dots + r_m\overline{j}_m$. Let $j' = j - r_{n+1}j_{n+1} - r_{n+2}j_{n+2} + \dots + r_m j_m$. Then the image of j' in M/N is zero, so j' lies in $J \cap N$. We can thus write $j' asr_1 j_1 + \dots + r_n j_n$. We then have

$$j = r_1 j_1 + \dots + r_n J_n + r_{n+1} j_{n+1} + \dots + r_m j_m,$$

proving the claim. □

Corollary 2.4. *If M and N are Noetherian R-modules, then so is $M \oplus N$.*

Proof. We have a surjection $M \oplus N \to N$ taking (m, n) to n. Its kernel K is the set of pairs of the form $(m, 0)$, which is isomorphic to M, and hence Noetherian. The surjection $M \oplus N \to N$ descends to an isomorphism $(M \oplus N)/K \cong N$, so that $(M \oplus N)/K$ is Noetherian. Thus $M \oplus N$ is Noetherian. □

Corollary 2.5. *If R is Noetherian, then any free R-module of finite rank is Noetherian.*

Proof. A free R-module of rank s is the direct sum of s copies of R, each of which is Noetherian as an R-module when R is Noetherian. □

Proof of the Theorem: Let M be a finitely generated R-module, and let $m_1, \ldots, m_s$ be a set of generators for M. Then if F is a free R-module of rank s, with generators $e_1, \ldots, e_s$, we have a surjection of F onto M taking e_i to m_i for all i. Let K be the kernel. Then M is isomorphic to F/K, and F is a Noetherian R-module, so M is Noetherian as well.