M3P14 EXAMPLE SHEET 4

1. Find the continued fraction expansions of the following rational numbers: $\frac{40}{29}$, $\frac{144}{89}$, $\frac{414}{93}$.

$$\frac{40}{29} = 1 + \frac{11}{29} = 1 + \frac{1}{2 + \frac{7}{11}} = 1 + \frac{1}{2 + \frac{1}{1+\frac{4}{7}}}$$
$$= 1 + \frac{1}{2 + \frac{1}{1+\frac{1}{1+\frac{3}{4}}}} = 1 + \frac{1}{2 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{3}}}}}.$$

144 and 89 are successive Fibonnaci numbers F_{11} and F_{12} (see 2a), so $\frac{144}{89} = [1; 1, \ldots, 1]$, where there are 10 1's to the right of the semicolon.

$$\frac{414}{93} = \frac{138}{31} = 4 + \frac{14}{31} = 4 + \frac{1}{2 + \frac{3}{14}} = 4 + \frac{1}{2 + \frac{1}{4+\frac{2}{3}}}$$
$$= 4 + \frac{1}{2 + \frac{1}{4+\frac{1}{1+\frac{1}{2}}}}.$$

2a. Define the Fibonacci numbers F_n by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $i \geq 2$. Describe, for all $n > 1$, the continued fraction expansion of $\frac{F_n}{F_{n-1}}$.

Let $f_n = \frac{F_n}{F_{n-1}}$. Then We have $f_n = \frac{F_n}{F_{n-1}} = \frac{F_{n-1}+F_{n-2}}{F_{n-1}} = 1 + \frac{1}{f_{n-1}}$. Since $f_2 = 1$, it follow by induction that $f_n = [1; 1, \ldots, 1]$, where there are $n-2$ ones to the right of the semicolon.

2b. Find the continued fraction expansion of $\frac{1+\sqrt{5}}{2}$.

Let $\phi = \frac{1+\sqrt{5}}{2}$. Then $\frac{1}{\phi} = \frac{-1+\sqrt{5}}{2}$. We have $\phi = 1 + \frac{-1+\sqrt{5}}{2} = 1 + \frac{1}{\phi} = 1 + \frac{1}{1+\frac{1}{\phi}} \ldots$. So the continued fraction expansion is $[1; 1, 1, 1, \ldots]$.

2c. Show that the limit, as n goes to infinity, of $\frac{F_n}{F_{n-1}}$ is $\frac{1+\sqrt{5}}{2}$.

The f_n are the convergents to the continuted fraction expansion of ϕ, so they approach ϕ as n goes to infinity.

3a. Show that a positive integer n is expressible as $x^2 - xy + y^2$, with x and y integers if, and only if, for every prime p congruent to 2 mod 3, the exponent of p in the prime factorization of n is even. [Hint: use unique factorization in the Eisenstein integers.]

If $z = x + y\omega$, where $\omega = \frac{-1+\sqrt{-3}}{2}$, we saw in class that $z\overline{z} = x^2 - xy + y^2$. Thus n is expressible as $x^2 - xy + y^2$ if, and only if, n factors as $z\overline{z}$ in $\mathbb{Z}[\omega]$.

We showed in class that if p is a prime congruent to 2 mod 3, then it remains prime in $\mathbb{Z}[\omega]$. In particular, if such a prime divides n, then it divides $z\overline{z}$, and hence divides one of z or $\overline{z}$. Replacing z with its conjugate if necessary, we can assume it divides z. Let p^k be the highest power of p dividing z. Then, since p is its own conjugate, p^k is also the highest power of p dividing $\overline{z}$. Thus p^{2k} is the highest power of p dividing n.

Conversely, we showed in class that if $p = 3$, or p is congruent to 1 mod 3, then p is of the form $x^2 - xy + y^2$. Our assumption implies that n is a square times a product of such primes, so n is also of the form $x^2 - xy + y^2$.

3b. Find x and y such that $x^2 - xy + y^2 = 91$.

We factor 91 as 13 times 7. By experimentation, $7 = 3^2 - 3 + 1 = (3+\omega)(3-\omega)$, and $13 = 4^2 - 4 + 1 = (4+\omega)(4-\omega)$. Thus $91 = (3+\omega)(4+\omega)(3-\omega)(4-\omega)$. We have $(3+\omega)(4+\omega) = 12 + 7\omega + \omega^2 = 11 + 6\omega$. So $x = 11, y = 6$ is a solution.

4a. Find all solutions to the equation $x^2 - 5y^2 = 1$. Explicitly list all solutions with $x < 200$ and $x, y > 0$.

We look for a fundamental solution; this has $x, y > 0$ and y as small as possible. For $y = 1, 2, 3$ one checks explicitly that there are no solutions, but $x = 9, y = 4$ is a solution. Thus the fundamental 1-unit in $\mathbb{Z}[\sqrt{5}]$ is $9 + 4\sqrt{5}$. All 1-units $x + y\sqrt{5}$ with $x, y > 0$ are positive powers of the fundamental 1-unit. We have $(9 + 4\sqrt{5})^2 = 161 + 72\sqrt{5}$, and $(9 + 4\sqrt{5})^3 = (9 * 161 + 1440) + (9 * 72 + 4 * 161)\sqrt{5}$. Since x is already larger than 200 in the latter, $(9, 4)$ and $(161, 72)$ are the only solutions with $x < 200$ and $x, y > 0$.

4b. Find all solutions to the equation $x^2 - 5y^2 = -1$.

One solution is $(2, 1)$. If (x, y) is any solution, then $\frac{x+y\sqrt{5}}{2+\sqrt{5}}$ is a 1-unit in $\mathbb{Z}[\sqrt{5}]$, and thus has the form $\pm(9 + 4\sqrt{5})^n$ for some integer n. Thus the complete set of solutions are those (x, y) such that $x + y\sqrt{5} = \pm(2+\sqrt{5})(9 + 4\sqrt{5})^n$.

5a. Find the value of the continued fraction $[1; 2, 2, 2, \dots]$.

We first find the value of the continued fraction $[2; 2, 2, 2, \dots]$. If $\alpha = [2; 2, 2, 2, \dots]$, then $\alpha = 2 + \frac{1}{\alpha}$. So $\alpha^2 - 2\alpha - 1 = 0$. Thus $\alpha = 1 \pm \sqrt{2}$. Since α is clearly positive we must have $\alpha = 1 + \sqrt{2}$. Thus $[1; 2, 2, 2, \dots] = \alpha - 1 = \sqrt{2}$.

5b. Find the value of the continued fraction $[1; 3, 5, 1, 3, 5, \dots]$.

Let $\beta = [1; 3, 5, 1, 3, 5, 1, 3, 5, \dots]$. We have

$$\beta = 1 + \frac{1}{3 + \frac{1}{5 + \frac{1}{\beta}}} = 1 + \frac{1}{3 + \frac{\beta}{5\beta+1}}$$

$$= 1 + \frac{1}{\frac{16\beta+3}{5\beta+1}} = 1 + \frac{5\beta+1}{16\beta+3} = \frac{21\beta+4}{16\beta+3}.$$

So $16\beta^2 + 3\beta = 21\beta + 4$. Thus β is the positive root of $16\beta^2 - 18\beta - 4 = 0$.

6a. Show that, for n a positive integer, we have $\sqrt{n^2+1} = [n; 2n, 2n, 2n, \dots]$.

Let $\beta = [2n; 2n, 2n, \dots]$. We have $\beta = 2n + \frac{1}{\beta}$. So $\beta^2 - 2n\beta - 1 = 0$. Thus $\beta = n \pm \sqrt{n^2+1}$. Since $\beta > n$, we must have $\beta = n + \sqrt{n^2+1}$, and the result follows.

6b. Show that, for n a positive integer, we have $\sqrt{n^2+2} = [n; n, 2n, n, 2n, \dots]$.

Let $\beta = [2n; n, 2n, n, 2n, n, \dots]$. Then

$$\beta = 2n + \frac{1}{n + \frac{1}{\beta}} = 2n + \frac{\beta}{n\beta+1}.$$

We then have $n\beta^2 + \beta = 2n^2\beta + 2n + \beta$, so $\beta^2 - 2n\beta - 2 = 0$. Thus $\beta = n \pm \sqrt{n^2+2}$; since $\beta > 2n$ we must have $\beta = n + \sqrt{n^2+2}$ and the result follows.