M3P14 EXAMPLE SHEET 4

1. Find the continued fraction expansions of the following rational numbers: $\frac{40}{29}$, $\frac{144}{89}$, $\frac{414}{93}$.

2a. Define the Fibonacci numbers F_n by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $i \geq 2$. Describe, for all $n > 1$, the continued fraction expansion of $\frac{F_n}{F_{n-1}}$.

2b. Find the continued fraction expansion of $\frac{1+\sqrt{5}}{2}$.

2c. Show that the limit, as n goes to infinity, of $\frac{F_n}{F_{n-1}}$ is $\frac{1+\sqrt{5}}{2}$.

3a. Show that a positive integer n is expressible as $x^2 - xy + y^2$, with x and y integers if, and only if, for every prime p congruent to 2 mod 3, the exponent of p in the prime factorization of n is even. [Hint: use unique factorization in the Eisenstein integers.]

3b. Find x and y such that $x^2 - xy + y^2 = 91$.

4a. Find all solutions to the equation $x^2 - 5y^2 = 1$. Explicitly list all solutions with $x < 200$ and $x, y > 0$.

4b. Find all solutions to the equation $x^2 - 5y^2 = -1$.

5a. Find the value of the continued fraction $[1; 2, 2, 2, \dots]$.

5b. Find the values of the continued fraction $[1; 3, 5, 1, 3, 5, \dots]$.

6a. Show that, for n an integer, we have $\sqrt{n^2+1} = [n; 2n, 2n, 2n, \dots]$.

6b. Show that, for n an integer, we have $\sqrt{n^2+2} = [n; n, 2n, n, 2n, \dots]$.