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Function theory in multiply connected domains and applications to fluid dynamics.

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Abstract

In this thesis we shall be considering a variety of problems set in the complex plane whose common feature is that they involve domains of finite multiple connectivity. We choose to focus on a particular canonical class of domains, namely circular domains. We extend our results to more general domains using conformal mappings. Results are derived for these circular domains by using the theory of Schottky groups. Problems we consider include the construction of automorphic functions, Green's functions, and conformal mappings from circular domains to other commonly studied canonical domains. The abstract function-theoretic results we derive are applied to a number of physical problems of fluid dynamics.

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Contents

CHAPTER 1

Function theory of planar multiply connected domains.

In this first chapter we shall introduce the general ideas of function theory which we shall use in all subsequent chapters.

1.1 Canonical domains.

In this thesis we shall be considering a variety of problems set in the complex plane whose common feature is that they involve domains of finite multiple connectivity. A common approach when solving such problems is to first focus on some "canonical" domain characterized by simple geometries which simplify the problem to be solved, and then to make the extension to more general domains using conformal mapping theory. This of course relies on there existing a conformal map to the general domain from the canonical domain chosen. We shall say that two domains are of the same conformal *class* if there exists a conformal mapping which maps one onto the other in a "one-to-one" manner. We shall also refer to two such domains as being conformally equivalent.

Any domain of connectivity $N \geq 1$ is conformally equivalent only to domains of the same connectivity, [54]. In the case $N = 1$, the Riemann mapping theorem [1] states that in fact all simply connected domains are conformally equivalent to one another. Therefore, we may choose just one particular simply connected domain as a model for all others. A natural choice is the unit disc. However, for $N > 1$, not all domains of connectivity N are of the same conformal class [54], and thus in this case there is no single N-connected domain to which all other N-connected domains are conformally equivalent.

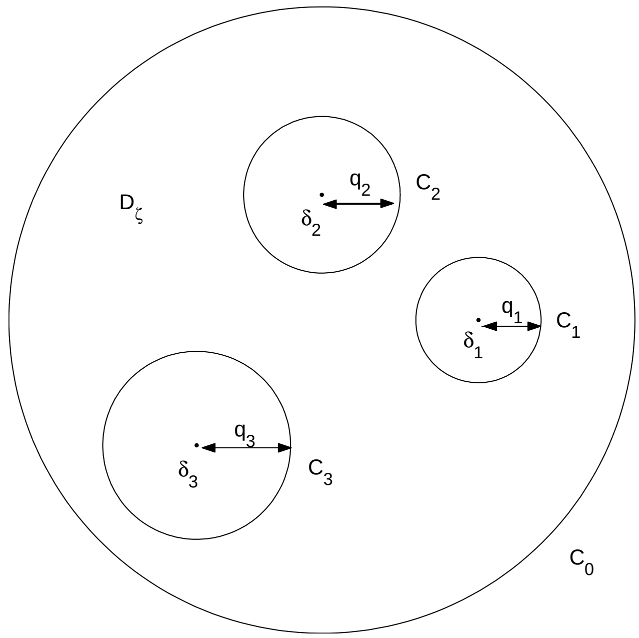

Fig. 1.1: Schematic of a multiply connected circular domain D_ζ consisting of the region bounded by the circle C_0 of unit radius centred on the origin, and circles $C_1,\ C_2,\ C_3$ in the interior of C_0.

1.1.1 Circular domains.

There are in fact a number of commonly studied varieties of canonical multiply connected domains, see [54], [62], [37] for example. Each has different characteristics which may make it more suitable than others in a particular context. The variety we shall focus on are ones whose boundaries are all circles. Such domains are referred to as **circular** domains. It is in fact true that for $N \geq 1$, every N-connected domain can be mapped with a one-to-one conformal map onto a circular domain. (This result was proved by Koebe and details of the proof can be found in [37].) In particular we may consider a $(N+1)$-connected circular domain D_ζ consisting of the region bounded by a circle C_0 of unit radius centred on the origin, and N other circles $C_1, ..., C_N$ in the interior of C_0. For $i = 1, ..., N$ we label the centre and radius of C_i as δ_i and q_i respectively. A schematic in the case $N = 3$ is shown in figure 1.1.

Many of the other common varieties of canonical domains have boundaries which are not Jordan curves but "slits", i.e. finite line segments. However, we choose to work primarily with circular domains firstly because of their greater geometrical simplicity, but secondly because associated with domains bounded by Jordan curves (of which these circular domains are of course a special case) there is a theory of functions developed by Schottky and Klein which we shall exploit to derive the main results of the thesis. This theory shall be described shortly. Note however that in chapter 4 we shall in fact discuss how to conformally map our chosen circular domains onto other commonly studied canonical domains.

Before proceeding, for clarity we shall briefly make the following remarks. Often we will want to study the effect of say, a transformation T on a set of points S perhaps representing a domain or a line. We denote by $T(S)$ the set of points which is the image of S under T. When considering compositions of transformations we write say T_1T_2 to denote the composition of first applying T_2 and then applying T_1. We shall sometimes refer to a composition of maps as a *product* and to the maps in the composition as *factors*. We shall also use terminology such as the *right-* and *left-hand ends* of a composition, where this is meant in the obvious sense. So for example, the right-most map in a composition is the map applied first, while the left-most map is the map applied last. We use exponent notation, writing for example T^{-1} for the inverse of T, and $T^2(\zeta)$ for the composition TT, etc. We also take the convention that T^0 is just the identity transformation.

1.2 Schottky groups constructed from circular domains.

Consider our $(N+1)$-connected circular domain D_ζ described in §1.1.1. We shall now describe an associated group of transformations of the extended complex plane which depend purely on the geometry of D_ζ. These transformations can be constructed as follows.

Consider the reflection of D_ζ in C_0, where we define the transformation of reflection in a circle in Appendix A. Reflection in C_0 is given by,

$$r_{C_0}(\zeta) = \frac{1}{\bar{\zeta}} \tag{1.1}$$

Denote this reflection of D_ζ as $\hat{D}_\zeta$. The boundaries of $\hat{D}_\zeta$ are of course the reflections of the circles $\{C_j | j = 1, ..., N\}$ in C_0. From §A.2, we see that under reflection in C_0, C_0 of course maps onto itself, while for $j \in \{1, ..., N\}$, the image of C_j is a circle in the exterior of C_0 which we denote as C'_j. A schematic for a case of $N = 3$ is shown in figure 1.2.

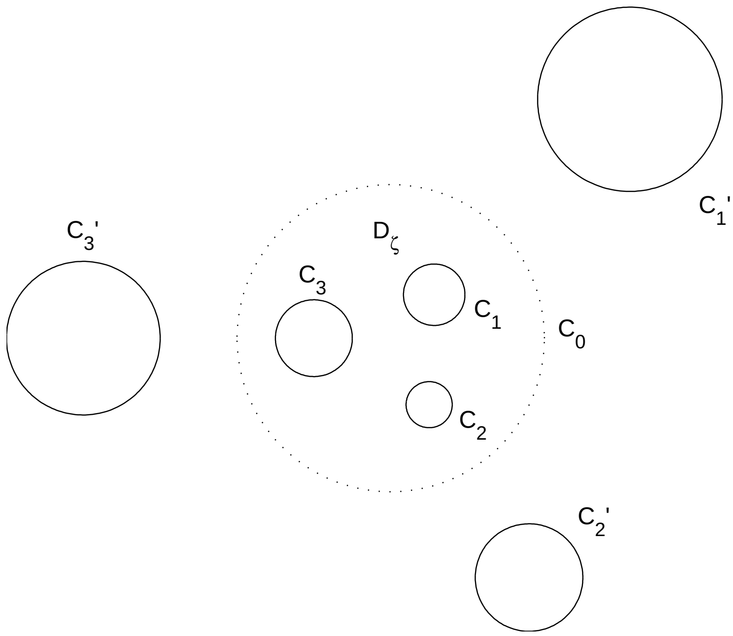

Fig. 1.2: Schematic of a typical circular domain D_ζ and its reflection in its boundary C_0.

Using the formulae (A.14) of §A.2, C_j' has centre δ_j' and radius q_j' given by

$$\delta_j' = \frac{\delta_j}{|\delta_j|^2 - q_j^2}, \qquad q_j' = \frac{q_j}{||\delta_j|^2 - q_j^2|}. \tag{1.2}$$

Consider now the map θ_j which is the composition of reflection in C_0 followed by reflection in C_j. Using the transformations $\phi_i(\zeta)$ introduced in Appendix A, where in particular

$$\phi_0(\zeta) = \frac{1}{\zeta}, \tag{1.3}$$

we have from (A.4),

$$\begin{aligned} \theta_j(\zeta) &= r_{C_j}(r_{C_0}(\zeta)) \\ &= \overline{\phi_j(\overline{\phi_0(\zeta)})} \\ &= \overline{\phi_j}(\phi_0(\zeta)) \\ &= \overline{\phi_j}(\zeta^{-1}) \end{aligned} \tag{1.4}$$

and thus from (A.5),

$$\theta_j(\zeta) = \delta_j + \frac{q_j^2 \zeta}{1 - \overline{\delta_j}\zeta}. \tag{1.5}$$

This may be written in the form

$$\boxed{\begin{aligned}&\theta_j(\zeta) = \frac{a_j\zeta + b_j}{c_j\zeta + d_j}\\ &\text{where}\\ &\qquad a_j = q_j^2 - |\delta_j|^2;\\ &\qquad b_j = \delta_j;\\ &\qquad c_j = -\overline{\delta_j};\\ &\qquad d_j = 1.\end{aligned}} \tag{1.6}$$

It is also easy to see that this may be written in the form,

$$\theta_j(\zeta) = \delta_j - \frac{q_j^2}{\overline{\delta_j}} + \frac{q_j^2}{\overline{\delta_j}(-\overline{\delta_j}\zeta + 1)}. \tag{1.7}$$

$\theta_j(\zeta)$ is a Möbius transformation of ζ. Möbius maps are a very important class of transformations of the extended complex plane. Some important facts about them which we shall call upon are given in the appendix B. Note that reflection in a circle is not a Möbius transformation of ζ as it involves conjugation of ζ. However, two successive reflections involves conjugating twice and thus leads to a Möbius transformation. Denoting the inverse of $\theta_j(\zeta)$ by $\theta_{-j}(\zeta)$ it is clear by geometrical arguments that this is given by the composition of reflection in C_j followed by reflection in C_0, i.e.

$$\theta_j^{-1}(\zeta) = r_{C_0}(r_{C_j}(\zeta)). \tag{1.8}$$

It is straightforward to show that

$$\boxed{\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)}.} \tag{1.9}$$

By geometrical arguments we make the following observations. Since reflection in C_0 maps C_j' onto C_j, and reflection in C_j leaves C_j itself invariant, then obviously $\theta_j(C_j') = C_j$. Furthermore, if ζ is in the exterior of C_j' then clearly $r_{C_0}(\zeta)$ is in the exterior of C_j, and so $r_{C_j}(r_{C_0}(\zeta))$ is in the interior of C_j. Thus θ_j maps C_j' onto C_j, with the exterior of C_j' mapping onto the interior of C_j. Note that θ_j must thus map the interior of C_j' onto the exterior of C_j.

There are N such maps $\{\theta_j | j = 1, ..., N\}$ where for $j = 1, ..., N$ $\theta_j(\zeta)$ "pairs" C_j and C_j' in the sense just described, i.e. θ_j maps C_j' onto C_j with the exterior of C_j'

mapping onto the interior of C_j. Consider the set Θ consisting of all compositions of these N maps and their inverses. This is obviously group. We say that this group is **generated** by the maps $\{\theta_j | i = 1, ..., N\}$ and refer to these maps as its **generators**. The **order** of the group is its number of generators N. In fact this is an example of a special type of group of transformations known as a **Schottky group**. For more on Schottky groups the reader is referred to [8], [52], [6], [9]. More general Schottky groups are generated from mutually disjoint pairs of Jordan curves [9]. The Schottky groups considered here where these Jordan curves are in fact all circles are examples of what are sometimes referred to as *classical* Schottky groups.

Thus we have shown how to construct a Schottky group from a N-connected circular domain D_ζ by reflecting it in one of its $N+1$ boundaries C_0 and then constructing the Möbius maps which pair each of the other N boundaries C_j with its reflection C'_j, $j \neq 0$.

But suppose now that we had chosen to initially reflect D_ζ in a different one of its boundaries; would we arrive at the same Schottky group? As shall now be demonstrated, the answer to this question is yes. For $i \in \{0, 1, ..., N\}$ consider the reflection of D_ζ in C_i. Denote this region $\hat{D}_\zeta^{(i)}$ (- note that the region $\hat{D}_\zeta^{(0)}$ is the region we have previously labelled $\hat{D}_\zeta$). The boundaries of $\hat{D}_\zeta^{(i)}$ are of course the reflections of the circles $\{C_j | j = 0, 1, ..., N\}$ in C_i. Under reflection in C_i, C_i of course maps onto itself, while for $j \in \{0, 1, ..., N\} \setminus \{i\}$, the image of C_j is also a circle. Denote this $C_j^{(i)}$ (- note that $C_j^{(0)}$ is the circle we have previously labelled C'_j). Consider the map $\theta_j^{(i)}$ which is the composition of reflection in C_i followed by reflection in C_j (- again we must make a point regarding our notation; here and in what follows, transformations and sets of transformations with the superscript '(0)' are those previously labelled without a superscript. Thus for example, $\theta_j^{(0)}$ is simply θ_j). We have

$$\begin{aligned}
\theta_j^{(i)}(\zeta) &= r_{C_j}(r_{C_i}(\zeta)) \\
&= \overline{\phi_j(\overline{\phi_i(\zeta)})} \\
&= \overline{\phi_j}(\phi_i(\zeta))
\end{aligned} \tag{1.10}$$

This is clearly a Möbius map. Denoting the inverse of $\theta_j^{(i)}(\zeta)$ by $\theta_{-j}^{(i)}(\zeta)$ it is clear by geometrical arguments that this is given by the composition of reflection in $C_j^{(i)}$

followed by reflection in C_i. In fact the maps $\{\theta_j^{(i)} | j = 0, 1, ..., N, j \neq i\}$ generate a Schottky group, $\Theta^{(i)}$ say.

Proposition 1.2.1 *For $i = 0, 1, ..., N$, $\Theta^{(i)}$ is the same group.*

Proof: For $i, j, k \in \{0, 1, ..., N\}$,

$$\begin{aligned}\theta_k^{(i)} &= r_{C_k}(r_{C_i}(\zeta)) \\ &= r_{C_k}(r_{C_j}(r_{C j}(r_{C_i}(\zeta)))) \\ &= \theta_k^{(j)}(\theta_{-i}^{(j)}(\zeta))\end{aligned} \tag{1.11}$$

where the first and third equalities follow by definition and the second equality follows from the fact that reflection in a circle is a self-inverse transformation. Now, suppose $k \neq i$. Then $\theta_k^{(i)}$ is one of the generators of $\Theta^{(i)}$. Now, also suppose $j \neq i$. Then obviously $\theta_{-i}^{(j)}$ is the inverse of one of the generators of $\Theta^{(j)}$. And $\theta_k^{(j)}$ is one of the generators of $\Theta^{(j)}$, unless $k = j$ in which case it is the identity. In either case, since $k \neq i$ then $\theta_{-i}^{(j)}$ is not the inverse of $\theta_k^{(j)}$. Thus $\theta_k^{(j)}(\theta_{-i}^{(j)})$ is a non-identity composition of the generators of $\Theta^{(j)}$. Hence by (1.11), each generator of $\Theta^{(i)}$ may be expressed as a composition of the generators of $\Theta^{(j)}$. Thus in fact, any map in $\Theta^{(i)}$ may be expressed as such a composition, and is thus also contained in $\Theta^{(j)}$, i.e., $\Theta^{(i)} \subseteq \Theta^{(j)}$. But by a similar argument we may deduce $\Theta^{(j)} \subseteq \Theta^{(i)}$. Hence in fact, $\Theta^{(i)} = \Theta^{(j)}$. This completes the proof.

Thus, starting with the $N + 1$ circles $\{C_j | j = 0, 1, ..., N\}$ bounding our circular domain D_ζ, no matter which one we choose to reflect the N others in, the resulting Schottky group is the *same* group Θ say.

1.3 Properties of Schottky groups.

Using the general theory of Schottky groups as described for example in [8], [52], [6] we can now state the following facts about our Schottky group Θ constructed from a circular domain D_ζ.

Note first that none of the generating maps $\{\theta_i | i = 1, ..., N\}$ can be expressed as a non-trivial composition of the other generating maps and their inverses. It follows that, the composition of each map in Θ in terms of the maps $\{\theta_i | i = 1, ..., N, -1, ..., -N\}$ is *unique*. It shall prove useful to categorize the maps in Θ in the following way: for $\ell > 0$ an integer, we say that a map $\theta \in \Theta$ is of **level** ℓ if its unique composition in terms of the generating maps and their inverses is a product of ℓ factors. The identity map is considered to be of level zero while the generating maps and their inverses are of level one. The level two maps are all possible compositions of any two of the level one maps which do not reduce to the identity. So, for example, the composition of one of the level one maps with its inverse is not a level two map. Similarly for maps of all higher levels.

It can be shown that every non-identity map in Θ is loxodromic (- see §B.4 for the definition of this term). Thus, in particular, for $j = 1, .., N$ the generator $\theta_j(\zeta)$ has two distinct fixed points, say A_j say and B_j, where these are respectively its source and sink, so that for any point ζ in the plane other than A_j or B_j,

$$\boxed{\theta_j^{-\infty}(\zeta) = A_j, \qquad \theta_j^{\infty}(\zeta) = B_j} \tag{1.12}$$

It can be shown that A_j is contained in the interior of C_j', while B_j is contained in the interior of C_j. Also, as explained in §B.4, an alternative representation for $\theta_j(\zeta)$ is

$$\boxed{\frac{\theta_j(\zeta) - B_j}{\theta_j(\zeta) - A_j} = \lambda_j \frac{\zeta - B_j}{\zeta - A_j}} \tag{1.13}$$

where λ_j is some complex constant, $\lambda \neq 0, \infty$, where $|\lambda_j| < 1$.

Now note that the connected region $D_\zeta \cup \hat{D}_\zeta$ exterior to the $2N$ circles $\{C_j, C_j' | j = 0, 1, ..., N\}$ has the following special significance. Label this region F. Consider the image of F under maps in Θ. It can be shown that the images of F under two different maps in Θ are mutually disjoint. Also, for $\theta \in \Theta$, as the level of θ tends to infinity, so the image of F under θ shrinks to a point. In fact it can be shown that the union of the regions in set $\{\theta(F) | \theta \in \Theta\}$ covers the whole of the extended complex plane. F is referred to as a **fundamental region** of Θ. Every point in the plane can be reached as the image of some point in F under a unique map in Θ. We now make the following distinction. Points in the plane which may be reached as the image of a point in F under a map of *finite* level are referred to as the **ordinary** points of Θ. Points in the plane which may only be reached as the image of a point in F under a map of *infinite* level are referred to as the **singular** points of Θ. A Schottky group has infinitely many singular points; each is the image of F under a different infinite level map of Θ. Each of these singular points lies outside of F. These singular points occur in clusters, i.e. as close as we desire to any singular point there are infinitely many others. In fact, it can be shown that the singular points of Θ are precisely the fixed points of all the non-identity maps in Θ.

It should be noted that, F is not unique in having the property that its images under all maps in Θ cover the whole of the extended plane without overlapping. For example, it is clear that for any $\theta \in \Theta$ (where θ is of finite level so that the region $\theta(F)$ is not a single point), $\theta(F)$ also has this property and so is a fundamental region of Θ. Note, furthermore for our particular Schottky groups, for $j = 1, ..., N$ the region $D_\zeta \cup \hat{D}_\zeta^{(j)}$ which we shall denote as $F^{(j)}$ is a fundamental region of $\Theta^{(j)}$. However as shown above in Proposition 1.2.1, $\Theta^{(j)} = \Theta$, and thus $F^{(j)}$ must be a fundamental region of Θ. Hence in fact we can state

Proposition 1.3.1 *For our particular Schottky group Θ, we can take as a fundamental region the region consisting of D_ζ and the reflection of D_ζ in any one of its boundaries $C_0, C_1, ..., C_N$.*

1.3.1 Additional properties of our particular Schottky groups.

Note that the properties of Schottky groups described so far in §1.3 hold for general Schottky groups. The particular Schottky groups which we have constructed from circular domains have additional properties which Schottky groups in general do not possess. These are due to the fact that the associated fundamental region F can be regarded as consisting of two "halves", namely D_ζ and $\hat{D}_\zeta$, where the transformation $r_{C_0}(\zeta)$ of reflection in C_0 provides a one-to-one map of each of these halves onto the other, leaving C_0 invariant. We shall now present some of the additional special properties of these Schottky groups.

The first of these is that for the generator $\theta_j(\zeta)$, $j \in \{1, ..., N\}$ its fixed points A_j and B_j are in fact reflections of one another in the unit circle C_0, i.e.

$$A_j = \overline{B_j}^{-1} \tag{1.14}$$

This result is stated and proved in an appendix as Proposition C.0.1. Furthermore, these fixed points in fact lie on the same ray as the centre δ_j of the circle C_j (- see Proposition (C.0.2)) Also, the constant λ_j in the representation of θ_j in the form (1.13) is in fact real. This result is stated and proved in an appendix as Proposition C.0.3. Thus $\theta_j(\zeta)$ is in fact hyperbolic (- see §B.4 for the definition of this term).

Next we point out that for the inverse $\theta_{-j}(\zeta)$ of $\theta_j(\zeta)$ we have

$$\begin{aligned} \theta_{-j}(\zeta) &= r_{C_0}(r_{C_j}(\zeta)) \\ &= r_{C_0}(r_{C_j}(r_{C_0}(r_{C_0}(\zeta)))) \\ &= r_{C_0}(\theta_j(r_{C_0}(\zeta))) \\ &= \frac{1}{\overline{\theta_j\left(\frac{1}{\overline{\zeta}}\right)}} \end{aligned} \tag{1.15}$$

where note that the second equality follows from the fact that reflection in a circle is self-inverse. Obviously it follows from (1.15) that,

$$\theta_j(\zeta) = \frac{1}{\overline{\theta_{-j}\left(\frac{1}{\overline{\zeta}}\right)}} \tag{1.16}$$

In fact, consider a general transformation $\theta \in \Theta$ expressed as a unique composition of the generators of Θ and their inverses

$$\theta(\zeta) = \theta_{i_\ell}\theta_{i_{\ell-1}}...\theta_{i_2}\theta_{i_1}(\zeta) \tag{1.17}$$

where $i_1,\ i_2, ..., i_{\ell-1},\ i_\ell\ \in\ \{1, ..., N, -1, ..., -N\}$. Then on repeated use of (1.15) and (1.16) we have

$$\begin{aligned} \theta^{-1}(\zeta) &= \theta_{-i_1}\theta_{-i_2}...\theta_{-i_{\ell-1}}\theta_{-i_\ell}(\zeta) \\ &= \left(\overline{\theta_{i_1}\theta_{i_2}...\theta_{i_{\ell-1}}\theta_{i_\ell}\left(\overline{\zeta}^{-1}\right)}\right)^{-1} \end{aligned} \tag{1.18}$$

We now introduce the notation ${}_r\theta$ to denote the map whose composition as a product of the generators of Θ and their inverses is the reverse of that of θ. Thus given $\theta(\zeta)$ of the form (1.17) then

$${}_r\theta(\zeta) = \theta_{i_1}\theta_{i_2}...\theta_{i_{\ell-1}}\theta_{i_\ell}(\zeta) \tag{1.19}$$

We shall refer to ${}_r\theta$ as the *reverse* of θ. Obviously, for $\theta \in \Theta$, ${}_r\theta$ is also contained in Θ. Thus from (1.18) it follows that

$$\theta^{-1}(\zeta) = \frac{1}{\overline{\left({}_r\theta(\frac{1}{\overline{\zeta}})\right)}} \tag{1.20}$$

1.4 Functions associated with Schottky groups.

We shall now describe some special functions associated with a general Schottky group Θ. The first we mention is known as the **Schottky-Klein prime function** [6]. We shall henceforth refer to this simply as the prime function. This is a function defined with respect to two variables $\zeta,\ \gamma \in \mathbb{C} \cup \{\infty\}$. We shall denote it $\omega(\zeta, \gamma)$. It has the following properties:

- For any ζ and γ it is single-valued.
- It has the property
$$\omega(\zeta, \gamma) = -\omega(\gamma, \zeta) \tag{1.21}$$
- As a function of ζ it has the following properties. It is analytic everywhere in the extended ζ-plane except for at (i) the point $\zeta = \infty$, unless $\gamma = \infty$ in which case this is not a singularity; (ii) the singular points of Θ. The singularities at the singular points are *not* poles as the singular points occur in clusters, i.e. as close as we wish to any singular point is another singular point. However, if the point $\zeta = \infty$ is not a singular point of Θ and if $\gamma \neq \infty$, then this singularity *is* a simple pole. The prime function vanishes only at the points $\{\theta(\gamma)|\theta \in \Theta\}$, i.e. at γ and all images of γ under non-identity maps in Θ. These zeros are simple zeros.

$\omega(\zeta, \gamma)$ can be thought of as generalising the irreducible factor $(\zeta - \gamma)$.

We shall now present an explicit representation of the prime function in terms of the elements of the associated Schottky group Θ. To do this we first introduce the following notation. For a Schottky group Θ we shall need to consider the following subgroups. The notation ${}_i\Theta_j$ is used to denote all transformations of the full group which do not have a power of θ_i or θ_i^{-1} on the left hand end or a power of θ_j or θ_j^{-1} on the right hand end. As a special case of this, the notation Θ_j simply means all substitutions of the group which do not have any positive or negative power of θ_j at the right hand end (but with no stipulation about what appears on the left hand end). Similarly, ${}_j\Theta$ means all substitutions which do not have any positive or negative power of θ_j at the left hand end (but with no stipulation about what appears on the right hand end). In addition, the single prime notation will be used to denote a subset where the identity is excluded from the set; thus Θ_1' denotes all substitutions, excluding the identity and all transformations with a positive or negative power of θ_1 at the right hand end. The double prime notation will be used to denote a subset where the identity *and* all inverse substitutions are excluded from the set. This means, for example, that if $\theta_1\theta_2$ is included in the set, the transformation $\theta_2^{-1}\theta_1^{-1}$ must be excluded. Thus, Θ'' means all substitutions of the group excluding the identity and all inverses. Similarly the notation ${}_1\Theta_2''$ denotes all substitutions of the group, excluding inverses and the identity, which do not have any power of θ_1 or θ_1^{-1} on the left hand end or any power of θ_2 or θ_2^{-1} on the right hand end. In the same way, Θ_j'' denotes all substitutions of the group, excluding the identity and all inverses, which do not have any positive or negative power of θ_j at the right hand end.

It can be shown (see [6]) that for a Schottky group Θ for which the following infinite product converges, it represents the associated prime function $\omega(\zeta,\gamma)$

$$\boxed{\omega(\zeta,\gamma) = (\zeta-\gamma)\prod_{\theta\in\Theta''}\frac{(\zeta-\theta(\gamma))(\gamma-\theta(\zeta))}{(\zeta-\theta(\zeta))(\gamma-\theta(\gamma))}} \tag{1.22}$$

An immediate question to ask is for what Schottky groups does the infinite product (1.22) converge. On this issue we make the following remarks. Associated with a Schottky group Θ one may construct special infinite series known as Poincaré theta series which are discussed in [10],[6],[9],[8] for example, and also in appendix D of this thesis. The representation (1.22) of the prime function derives directly from a particular Poincaré theta series [6], and under conditions where this series converges this representation (1.22) of the prime function is also well-defined. Some criteria are known for which this series converges. For example, it is known to converge under certain conditions on the circles $\{C_i|i=1,...,N\}$ associated with Θ. These are described for example in [10], [6], [9]. They state that convergence is guaranteed if, roughly speaking, the circles $\{C_i|i=1,...,N\}$ are far enough apart. However, *precise* criteria for which convergence is assured and hence the representation (1.22) of the prime function is valid remain to be determined. Furthermore, it would be valuable to know alternative explicit representations for the prime function which are well-defined under conditions where the representation (1.22) does not converge. These are issues for further investigation. However it is emphasized that in all explicit computations made using the representation (1.22) of the prime function presented later in the thesis, no major convergence problems were encountered. Many of these computations were checked using completely independent methods as shall be shown. It appears that for a wide range of distributions of circles $\{C_i | i = 1, ..., N\}$ the convergence of the infinite product in (1.22) is adequate for practical purposes.

We shall proceed under the assumption that the infinite product (1.22) converges. We point out that to actually compute the expression (1.22) for the prime function, one must of course truncate the infinite product to a finite product. This may be done in a very natural way by including all maps in Θ of up to some chosen level and truncating the contribution to the product from all higher-level maps.

Recall as stated above that $\omega(\zeta, \gamma)$ can be thought of as generalising the irreducible factor $(\zeta - \gamma)$. Suppose we consider the trivial case of $N = 0$ so that D_ζ is simply the unit disc and F is the entire extended complex plane. Then the associated Schottky group contains just the identity, and indeed (1.22) reduces to $(\zeta - \gamma)$.

Also, note that an expression of the form

$$\frac{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)} \tag{1.23}$$

is called a cross-ratio of the points $\zeta_1, ..., \zeta_4$ ([1]). We introduce the following notation to denote such a cross-ratio,

$$\frac{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}. \tag{1.24}$$

Notice that in (1.22) each term in the product is a cross-ratio. Thus, using the notation (1.24), (1.22) may also be written as

$$\omega(\zeta, \gamma) = (\zeta - \gamma) \prod_{\theta \in \Theta''} \{\zeta, \theta(\zeta), \gamma, \theta(\gamma)\}. \tag{1.25}$$

Now note that it is known ([1]) that the cross-ratio of four points is invariant under a Möbius transformation. That is, for a Möbius map $T(\zeta)$, denoting $\zeta_i' = T(\zeta_i)$, $i = 1, ..., 4$, we have

$$\{\zeta_1', \zeta_2', \zeta_3', \zeta_4'\} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} \tag{1.26}$$

Note that in the product in (1.22), for each non-identity map $\theta \in \Theta$ we may include in Θ'' either θ or θ^{-1} - the value of the product does not depend on the choice we make. This can be seen as follows

$$\begin{aligned}\omega(\zeta,\gamma) &= (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\zeta,\theta(\zeta),\gamma,\theta(\gamma)\}\\ &= (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\theta^{-1}(\zeta),\zeta,\theta^{-1}(\gamma),\gamma\}\\ &= (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\zeta,\theta^{-1}(\zeta),\gamma,\theta^{-1}(\gamma)\}\end{aligned} \tag{1.27}$$

where the second equality follows from the fact that the cross-ratio is invariant if we apply the Möbius transformation θ^{-1} to each of the four arguments, and the third equality follows simply from (1.24).

Note that the representation (1.22) of the prime function can be rearranged as follows. Consider a general map θ in Θ'' of the form

$$\theta(\zeta) = \frac{a\zeta+b}{c\zeta+d}, \tag{1.28}$$

so

$$\theta^{-1}(\zeta) = \frac{d\zeta-b}{-c\zeta+a}. \tag{1.29}$$

By straightforward rearrangements it is possible to show that

$$\zeta-\theta^{-1}(\gamma) = \left(\frac{c\zeta+d}{c\gamma-a}\right)(\gamma-\theta(\zeta)) \tag{1.30}$$

Also, the two fixed points of θ, A_θ and B_θ say, are the solutions of

$$\zeta-\theta(\zeta)=0. \tag{1.31}$$

It is straightforward to show that

$$(\zeta-A_\theta)(\zeta-B_\theta) = \left(\frac{c\zeta+d}{c}\right)(\zeta-\theta(\zeta)) \tag{1.32}$$

Thus from (1.30) and (1.32) it is clear that the representation (1.22) of the prime function can be rearranged into the form

$$\boxed{\omega(\zeta,\gamma) = K(\zeta-\gamma)\prod_{\theta\in\Theta''}\frac{(\zeta-\theta(\gamma))(\zeta-\theta^{-1}(\gamma))}{(\zeta-A_\theta)(\zeta-B_\theta)}} \tag{1.33}$$

where K is a factor independent of ζ.

Finally, let us introduce the following notation which will be useful later in the thesis. We may write

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma) \tag{1.34}$$

where we define the function $\omega'(\zeta,\gamma)$ as

$$\begin{aligned}\omega'(\zeta,\gamma) &= \prod_{\theta\in\Theta''} \frac{(\zeta-\theta(\gamma))(\gamma-\theta(\zeta))}{(\zeta-\theta(\zeta))(\gamma-\theta(\gamma))} \\ &= \prod_{\theta\in\Theta''} \{\zeta,\theta(\zeta),\gamma,\theta(\gamma)\}.\end{aligned} \tag{1.35}$$

Note that $\omega'(\zeta,\gamma)$ does not have a zero at $\zeta=\gamma$. As a special case when $\gamma=\infty$ (1.35) becomes

$$\omega'(\zeta,\infty) = \prod_{\theta\in\Theta''} \frac{(\theta(\infty)-\zeta)}{(\theta(\zeta)-\zeta)} \tag{1.36}$$

This has simple zeros at the images of ∞ under all non-identity maps of Θ, but not at ∞ itself. It also has singularities at the singular points of Θ. Indeed,

$$\begin{aligned}\zeta-\theta(\zeta) &= \left(\zeta+\frac{d}{c}\right)^{-1}(\zeta-A_\theta)(\zeta-B_\theta) \\ &= (\zeta-\theta^{-1}(\infty))^{-1}(\zeta-A_\theta)(\zeta-B_\theta)\end{aligned} \tag{1.37}$$

where the first equality follows from (1.32) and the second equality from (1.29). Thus it is straightforward to write (1.36) as

$$\omega'(\zeta,\infty) = \prod_{\theta\in\Theta''} \frac{(\zeta-\theta(\infty))(\zeta-\theta^{-1}(\infty))}{(\zeta-A_\theta)(\zeta-B_\theta)} \tag{1.38}$$

Also, we mention that later in the thesis we shall be considering products of prime functions and it shall prove convenient to introduce the notation $\omega_n(\zeta,\gamma)$ as

$$\omega_n(\zeta,\gamma) \equiv \prod_{k=0}^{n-1} \omega(\zeta, e^{2\pi i k/n}\gamma). \tag{1.39}$$

Let us now describe some further properties of the prime function derived from its representation as (1.22).

First we ask how the value of $\omega(\zeta,\gamma)$ changes if we transform ζ by one of the transformations of Θ. Following [6] let us introduce for $i=1,...,N$, the function $\mathcal{V}_i(\zeta)$ which has simple zeros at the points $\{\theta(B_i)|\theta\in\Theta\}$ and simple poles at the points $\{\theta(A_i)|\theta\in\Theta\}$. We may write

$$\mathcal{V}_i(\zeta) = \prod_{\theta\in\Theta}\left(\frac{\zeta-\theta(B_i)}{\zeta-\theta(A_i)}\right) \tag{1.40}$$

Note that obviously since A_i and B_i are fixed points of θ_i (1.40) is the same as

$$\mathcal{V}_i(\zeta) = \prod_{\theta \in \Theta_i} \left(\frac{\zeta - \theta(B_i)}{\zeta - \theta(A_i)} \right) \tag{1.41}$$

Also introduce

$$\mathcal{V}_i(\zeta_1, \zeta_2) = \frac{\mathcal{V}_i(\zeta_1)}{\mathcal{V}_i(\zeta_2)} \tag{1.42}$$

which from (1.41) may be written as

$$\begin{aligned} \mathcal{V}_i(\zeta_1, \zeta_2) &= \prod_{\theta \in \Theta_i} \left(\frac{(\zeta_1 - \theta(B_i))(\zeta_2 - \theta(A_i))}{(\zeta_1 - \theta(A_i))(\zeta_2 - \theta(B_i))} \right) \\ &= \prod_{\theta \in \Theta_i} \left(\{\zeta_1, \theta(A_i), \zeta_2, \theta(B_i)\} \right). \end{aligned} \tag{1.43}$$

Notice

$$\mathcal{V}_i(\zeta_2, \zeta_1) = (\mathcal{V}_i(\zeta_1, \zeta_2))^{-1} \tag{1.44}$$

It can be shown ([6]) that for $i, j = 1, ..., N$,

$$\mathcal{V}_i(\theta_j(\zeta), \zeta) = \mathcal{T}_{i,j} \tag{1.45}$$

where, for $i, j = 1, ..., N$, we may write

$$\begin{aligned} \mathcal{T}_{i,j} &= \prod_{\theta \in \ _j\Theta_i} \{A_j, \theta(B_i), B_j, \theta(A_i)\}, \qquad i \neq j \\ \mathcal{T}_{i,i} &= \lambda_i \left(\prod_{\theta \in \ _i\Theta_i''} \{A_i, \theta(B_i), B_i, \theta(A_i)\} \right)^2 \end{aligned} \tag{1.46}$$

where λ_i is the constant in the representation (1.13) of $\theta_i(\zeta)$. Note in particular that for $i, j = 1, ..., N$, $\mathcal{T}_{i,j}$ and thus $\mathcal{V}_i(\theta_j(\zeta), \zeta)$ are *independent* of ζ.

We also point out a property that will be useful later which is

$$\mathcal{T}_{i,j} = \mathcal{T}_{j,i} \tag{1.47}$$

A proof of this is proved in [6].

We can now state the following result whose proof is given in [6].

Proposition 1.4.1 *If θ_i is one of the generators of Θ then*

$$\boxed{\omega(\theta_i(\zeta), \gamma) = -\left(\lambda_i \mathcal{T}_{i,i}^{-1}\right)^{\frac{1}{2}} \mathcal{V}_i(\gamma, \zeta) \left(\frac{\theta_i(\zeta) - A_i}{\zeta - A_i} \right) \omega(\zeta, \gamma)} \tag{1.48}$$

Note that the transformation of the prime function for any $\theta \in \Theta$ obviously follows by considering θ as its composition in terms of the generators of Θ and their inverses.

In fact, where we use the prime function later in the thesis, it most commonly appears in products of ratios of the form $\omega(\zeta, \gamma_1)/\omega(\zeta, \gamma_2)$, where $\gamma_1 \neq \gamma_2$. It follows easily from (1.48) and (1.42) that for $\gamma_1 \neq \gamma_2$,

$$\boxed{\frac{\omega(\theta_i(\zeta), \gamma_1)}{\omega(\theta_i(\zeta), \gamma_2)} = \mathcal{V}_i(\gamma_1, \gamma_2)\frac{\omega(\zeta, \gamma_1)}{\omega(\zeta, \gamma_2)}} \tag{1.49}$$

We shall use the property (1.49) of the prime function frequently throughout the thesis.

We point out the special case of (1.49) when either γ_1 or γ_2 is the point at infinity. Writing the left-hand side of (1.48) as

$$\frac{(\theta_i(\zeta) - \gamma)\omega'(\theta_i(\zeta), \gamma)}{(\zeta - \gamma)\omega'(\zeta, \gamma)} \tag{1.50}$$

we see that as $\gamma \to \infty$ so this tends to

$$\frac{\omega'(\theta_i(\zeta), \gamma)}{\omega'(\zeta, \gamma)}. \tag{1.51}$$

On the other hand, as $\gamma \to \infty$ so

$$\mathcal{V}_i(\gamma) \to 1. \tag{1.52}$$

Thus, from (1.48) it follows that

$$\frac{\omega'(\theta_i(\zeta), \infty)}{\omega'(\zeta, \infty)} = -\left(\lambda_i \mathcal{T}_{i,i}^{-1}\right)^{\frac{1}{2}} \mathcal{V}_i(\zeta)^{-1} \left(\frac{\theta_i(\zeta) - A_i}{\zeta - A_i}\right) \tag{1.53}$$

Hence, from (1.48) and (1.53) we have

$$\frac{\omega(\theta_i(\zeta), \gamma)}{\omega'(\theta_i(\zeta), \infty)} = \mathcal{V}_i(\gamma)\frac{\omega(\zeta, \gamma)}{\omega'(\zeta, \infty)} \tag{1.54}$$

Another property that will be useful later is the following.

Proposition 1.4.2 *Consider $\omega(\theta(\zeta), \theta(\gamma))$ where $\theta \in \Theta$. Writing $\theta(\zeta)$ in the form*

$$\theta(\zeta) = \frac{a\zeta + b}{c\zeta + d} \tag{1.55}$$

it can be shown that

$$\omega(\theta(\zeta), \theta(\gamma)) = \frac{(ad - bc)}{(c\zeta + d)(c\gamma + d)}\omega(\zeta, \gamma) \tag{1.56}$$

Proof: From (1.25) we have

$$\omega(\theta(\zeta),\theta(\gamma)) = (\theta(\zeta)-\theta(\gamma)) \prod_{\varphi\in\Theta''} \{\theta(\zeta),\varphi(\theta(\zeta)),\theta(\gamma),\varphi(\theta(\gamma))\}. \tag{1.57}$$

Then, from the invariance of cross-ratios under Möbius transformations stated by (1.26), we have

$$\omega(\theta(\zeta),\theta(\gamma)) = (\theta(\zeta)-\theta(\gamma)) \prod_{\varphi\in\Theta''} \{\zeta,\theta^{-1}(\varphi(\theta(\zeta))),\gamma,\theta^{-1}(\varphi(\theta(\gamma)))\}. \tag{1.58}$$

But since $\theta \in \Theta$ then it is straightforward that

$$\{\theta^{-1}\varphi\theta | \varphi \in \Theta\} = \Theta \tag{1.59}$$

and thus it follows from (1.58) that

$$\omega(\theta(\zeta),\theta(\gamma)) = \left(\frac{\theta(\zeta)-\theta(\gamma)}{\zeta-\gamma}\right)\omega(\zeta,\gamma) \tag{1.60}$$

But now note that from the form (1.55) of θ it is straightforward to show that

$$\begin{aligned} \theta(\zeta)-\theta(\gamma) &= \frac{a\zeta+b}{c\zeta+d} - \frac{a\gamma+b}{c\gamma+d} \\ &= \frac{(ad-bc)(\zeta-\gamma)}{(c\zeta+d)(c\gamma+d)} \end{aligned} \tag{1.61}$$

Thus the result follows from (1.60) and (1.61). This completes the proof.

1.4.1 Additional properties of functions associated with our particular Schottky groups.

The functions and their properties described so far in §1.4 are for general Schottky groups. However, for the particular Schottky groups we have constructed from circular domains these functions possess some additional special properties which we shall now present. Note that these arise from the special properties described in §1.3.1 of these particular Schottky groups which result from the reflectional symmetry of its associated fundamental region.

A very important additional property of the prime function in this case is stated as follows.

Proposition 1.4.3

$$\boxed{\omega(\overline{\zeta}^{-1},\overline{\gamma}^{-1}) = -\frac{1}{\overline{\zeta\gamma}}\overline{\omega(\zeta,\gamma)}} \tag{1.62}$$

Proof: From (1.25) we have

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\zeta,\theta(\zeta),\gamma,\theta(\gamma)\}. \tag{1.63}$$

Recall from (1.27) that in writing the prime function in the form (1.25), for any given non-identity map in Θ it does not matter whether we choose this map or its inverse in Θ''. Thus we may write

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\zeta,\theta^{-1}(\zeta),\gamma,\theta^{-1}(\gamma)\}. \tag{1.64}$$

And so,

$$\omega(\overline{\zeta}^{-1},\overline{\gamma}^{-1}) = (\overline{\zeta}^{-1}-\overline{\gamma}^{-1})\prod_{\theta\in\Theta''}\{\overline{\zeta}^{-1},\theta^{-1}(\overline{\zeta}^{-1}),\overline{\gamma}^{-1},\theta^{-1}(\overline{\gamma}^{-1})\}. \tag{1.65}$$

Now consider a general map $\theta\in\Theta''$. From (1.20) we have

$$\theta^{-1}(\zeta) = \frac{1}{\overline{\left({}_r\theta(\frac{1}{\overline{\zeta}})\right)}} \tag{1.66}$$

where ${}_r\theta$ is the reverse of θ defined as the map whose composition as a product of the generators of Θ and their inverses is the reverse of that of θ. Hence from (1.65) and (1.66) we have

$$\omega(\overline{\zeta}^{-1},\overline{\gamma}^{-1}) = (\overline{\zeta}^{-1}-\overline{\gamma}^{-1})\prod_{\theta\in\Theta''}\left\{\frac{1}{\overline{\zeta}},\frac{1}{\overline{({}_r\theta(\zeta))}},\frac{1}{\overline{\gamma}},\frac{1}{\overline{({}_r\theta(\gamma))}}\right\}. \tag{1.67}$$

Now, using the invariance of cross-ratios to Möbius transformations we have

$$\omega(\overline{\zeta}^{-1},\overline{\gamma}^{-1}) = (\overline{\zeta}^{-1}-\overline{\gamma}^{-1})\prod_{\theta\in\Theta''}\{\overline{\zeta},\overline{({}_r\theta(\zeta))},\overline{\gamma},\overline{({}_r\theta(\gamma))}\}. \tag{1.68}$$

Let us now argue that we may choose Θ'' such that for any $\theta\in\Theta''$, ${}_r\theta$ is also contained in Θ''. Thus, suppose $\theta\in\Theta''$ and we suppose we choose to include in Θ'' ${}_r\theta$ rather than its inverse $({}_r\theta)^{-1}$. Then we must check that the reverse of $({}_r\theta)^{-1}$ is certainly not in Θ'' as otherwise this would contradict our required property of Θ''. But it is of course clear that, for any $\theta\in\Theta$

$$({}_r\theta)^{-1} \;=\; {}_r(\theta^{-1}) \tag{1.69}$$

i.e., the inverse of the reverse of θ is the same as the reverse of the inverse of θ. Thus the reverse of $({}_r\theta)^{-1}$ is θ^{-1} and this is indeed not in Θ''. Thus in defining the prime function, we may indeed choose Θ'' such that for any $\theta \in \Theta''$, ${}_r\theta$ is also contained in Θ''.

Thus from (1.68) it follows that

$$\begin{aligned} \omega(\overline{\zeta}^{-1}, \overline{\gamma}^{-1}) &= (\overline{\zeta}^{-1} - \overline{\gamma}^{-1}) \prod_{\theta\in\Theta''} \{\overline{\zeta}, \overline{(\theta(\zeta))}, \overline{\gamma}, \overline{(\theta(\gamma))}\} \\ &= \frac{(\overline{\zeta}^{-1} - \overline{\gamma}^{-1})}{\overline{\zeta} - \overline{\gamma}} \overline{\omega(\zeta, \gamma)} \\ &= -\frac{1}{\overline{\zeta\gamma}} \overline{\omega(\zeta, \gamma)} \end{aligned} \tag{1.70}$$

This completes the proof of (1.62).

From the explicit formulae (1.6) for the generating maps, it can be shown that if the distribution of the circles $\{C_i | i = 1, .., N\}$ in the interior of C_0 possesses certain symmetries then the associated prime function exhibits these too as illustrated by the following results.

Proposition 1.4.4 *Suppose the distribution of the circles $\{C_i | i = 1, .., N\}$ in the interior of C_0 is reflectionally symmetric in the real axis, i.e., for each $i \in \{1, ..., N\}$, the reflection of C_i in the real axis is C_j for some $j \in \{1, ..., N\}$, possibly $j = i$. Then*

$$\overline{\omega}(\zeta, \gamma) = \omega(\zeta, \gamma). \tag{1.71}$$

Proof: Suppose that for $i, j \in \{1, ..., N\}$ (possibly $i = j$) the reflection of C_i in the real axis is C_j. Then, clearly

$$\overline{\delta_i} = \delta_j \quad \text{and} \quad q_i = q_j. \tag{1.72}$$

Thus from (1.6) it follows that

$$\overline{\theta_i}(\zeta) = \theta_j(\zeta) \tag{1.73}$$

Hence, if the distribution of the circles $\{C_i | i = 1, .., N\}$ is reflectionally symmetric in the real axis then it is clear that

$$\{\overline{\theta}(\zeta) | \theta(\zeta) \in \Theta\} = \Theta, \tag{1.74}$$

and so

$$\begin{aligned}\overline{\omega}(\zeta,\gamma) &= \overline{\omega(\overline{\zeta},\overline{\gamma})} \\ &= (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\zeta,\overline{\theta}(\zeta),\gamma,\overline{\theta}(\gamma)\} \\ &= \omega(\zeta,\gamma).\end{aligned} \tag{1.75}$$

where the third equality follows from (1.74). This completes the proof.

Proposition 1.4.5 *Suppose the distribution of the circles $\{C_i|i = 1,..,N\}$ in the interior of C_0 is reflectionally symmetric in the imaginary axis, i.e., for each $i \in \{1,...,N\}$, the reflection of C_i in the imaginary axis is C_j for some $j \in \{1,...,N\}$, possibly $j = i$. Then*

$$\overline{\omega}(-\zeta,-\gamma) = -\omega(\zeta,\gamma). \tag{1.76}$$

Proof: Suppose that for $i, j \in \{1,...,N\}$ (possibly $i = j$) the reflection of C_i in the imaginary axis is C_j. Then, clearly

$$-\overline{\delta_i} = \delta_j \quad \text{and} \quad q_i = q_j. \tag{1.77}$$

Thus from (1.6) it follows that

$$-\overline{\theta_i}(-\zeta) = \theta_j(\zeta) \tag{1.78}$$

Hence, if the distribution of the circles $\{C_i|i = 1,..,N\}$ is reflectionally symmetric in the imaginary axis then it is clear that

$$\{-\overline{\theta}(-\zeta)|\theta(\zeta)\in\Theta\} = \Theta, \tag{1.79}$$

and so

$$\begin{aligned}-\overline{\omega}(-\zeta,-\gamma) &= -\overline{\omega(\overline{-\zeta},\overline{-\gamma})} \\ &= (\zeta-\gamma)\prod_{\theta\in\Theta''}\{\zeta,-\overline{\theta}(-\zeta),\gamma,-\overline{\theta}(-\gamma)\} \\ &= \omega(\zeta,\gamma).\end{aligned} \tag{1.80}$$

where the third equality follows from (1.79). This completes the proof.

From propositions 1.4.4 and 1.4.5 we may deduce

Proposition 1.4.6 *Suppose the distribution of the circles $\{C_i | i = 1, .., N\}$ in the interior of C_0 is reflectionally symmetric in both the real and imaginary axes. Then*

$$\omega(-\zeta, -\gamma) = -\omega(\zeta, \gamma) \tag{1.81}$$

1.5 A special case.

It can be shown ([1]) that any doubly-connected circular domain is actually conformally equivalent to a concentric annulus bounded by circles of radius 1 and $q_1 < 1$ both centred on the origin. It follows that, any doubly connected domain is conformally equivalent to a concentric annulus. In fact, for $N > 0$, any domain of connectivity $N + 1$, is conformally equivalent to a circular domain which is the interior of an outer circle C_0 and exterior of N inner circles, where one of these inner circles C_1 say is concentric with the outer circle C_0.

So suppose our circular domain D_ζ is bounded by the unit circle C_0 and N other circles in the interior of C_0 where one of these say C_1 is also centred on the origin of radius $q_1 < 1$. In this case by (1.6) the generating map θ_1 is given by

$$\theta_1(\zeta) = q_1^2 \zeta \tag{1.82}$$

It is clear that the fixed points of this map $\theta_1(\zeta)$ are

$$B_1 = 0, \qquad A_1 = \infty \tag{1.83}$$

where B_1 is the sink and A_1 the source. Also, it is clear that,

$$\frac{(\theta_1(\zeta) - B_1)(\zeta - A_1)}{(\theta_1(\zeta) - A_1)(\zeta - B_1)} = q_1^2 \tag{1.84}$$

Thus writing $\theta_1(\zeta)$ in the form (1.13) we have

$$\lambda_1 = q_1^2 \tag{1.85}$$

Furthermore, note that we must slightly modify our expression (1.41) of the function $\mathcal{V}_1(\zeta)$ in this case. $\mathcal{V}_1(\zeta)$ must have simple zeros at all images of B_1 under maps in Θ and simple poles at all images of A_1 under maps in Θ. In this case $B_1 = 0$ and $A_1 = \infty$. So in this case we should replace the term in (1.41) corresponding to the identity map of Θ simply by ζ which of course has a simple zero at $\zeta = 0$ and a simple pole at $\zeta = \infty$ as required. So in this case our expression for $\mathcal{V}_1(\zeta)$ is

$$\mathcal{V}_1(\zeta) = \zeta \prod_{\theta \in \Theta_1'} \left(\frac{\zeta - \theta(0)}{\zeta - \theta(\infty)} \right) \tag{1.86}$$

Thus, (1.43) reduces to

$$\begin{aligned} \mathcal{V}_1(\zeta_1, \zeta_2) &= \frac{\mathcal{V}_1(\zeta_1)}{\mathcal{V}_1(\zeta_2)} \\ &= \frac{\zeta_1}{\zeta_2} \prod_{\theta \in \Theta_1'} \{\zeta_1, \theta(\infty), \zeta_2, \theta(0)\} \end{aligned} \tag{1.87}$$

where the second equality follows from (1.86). Also from (1.46),

$$\begin{aligned} \mathcal{T}_{i,1} &= \prod_{\theta \in {}_1\Theta_i} \left(\frac{\theta(B_i)}{\theta(A_i)} \right), \quad i \neq 1 \\ \mathcal{T}_{1,1} &= q_1^2 \left(\prod_{\theta \in {}_1\Theta_1''} \left(\frac{\theta(0)}{\theta(\infty)} \right) \right)^2 \end{aligned} \tag{1.88}$$

where we have made use of (1.85).

1.5.1 Doubly connected case

Suppose D_ζ is in fact doubly connected and C_0 and C_1 are its only boundaries. To simplify the notation, we now denote the radius of C_1 as q rather than q_1. In this case the above theory simplifies greatly. $\theta_1(\zeta)$ is the only generating map of the Schottky group Θ. (1.22) thus simplifies to

$$\omega(\zeta, \gamma) = -\frac{\gamma}{C^2} P(\zeta\gamma^{-1}, q) \tag{1.89}$$

where we define

$$\boxed{P(\zeta, q) = (1 - \zeta) \prod_{k=1}^{\infty} (1 - q^{2k}\zeta)(1 - q^{2k}\zeta^{-1})} \tag{1.90}$$

and the constant C is given by

$$C = \prod_{k=1}^{\infty} (1 - q^{2k}) \tag{1.91}$$

We may also write

$$P(\zeta, q) = (1 - \zeta) P'(\zeta, q) \tag{1.92}$$

where we define

$$P'(\zeta, q) = \prod_{k=1}^{\infty}(1 - q^{2k}\zeta)(1 - q^{2k}\zeta^{-1}) \tag{1.93}$$

Let us now show how some of the properties of $\omega(\zeta, \gamma)$ stated earlier simplify in this doubly connected case. First consider (1.48). In this case, the left-hand side of (1.48) is

$$\frac{\omega(\theta_1(\zeta), \gamma)}{\omega(\zeta, \gamma)} = \frac{P(q^2\zeta\gamma^{-1}, q)}{P(\zeta\gamma^{-1}, q)} \tag{1.94}$$

and since Θ_1 consists only of the identity map, the right hand-side of (1.48) simply reduces to

$$-\gamma\zeta^{-1} \tag{1.95}$$

Thus (1.48) becomes

$$P(q^2\zeta\gamma^{-1}, q) = -\gamma\zeta^{-1}P(\zeta\gamma^{-1}, q) \tag{1.96}$$

equivalently,

$$\boxed{P(q^2\zeta, q) = -\zeta^{-1}P(\zeta, q)} \tag{1.97}$$

This may also be easily checked directly from the expression (1.90) for $P(\zeta, q)$.

Note that in this case $\overline{\theta_1}(\zeta) = \theta_1(\zeta)$ and so from (1.75) we have $\overline{\omega}(\zeta, \gamma) = \omega(\zeta, \gamma)$. Obviously from (1.90) we can see that $\overline{P}(\zeta, q) = P(\zeta, q)$.

Now, consider the property (1.62). Simply using (1.89) it is clear that

$$\overline{\omega}(\zeta, \gamma) = \omega(\zeta, \gamma) = -\frac{\gamma}{C^2}P(\zeta\gamma^{-1}, q) \tag{1.98}$$

and

$$-\zeta\gamma\omega(\zeta^{-1}, \gamma^{-1}) = \frac{\zeta}{C^2}P(\gamma\zeta^{-1}, q) \tag{1.99}$$

Hence, by (1.62), it follows that

$$P(\gamma\zeta^{-1}, q) = -\frac{\gamma}{\zeta}P(\zeta\gamma^{-1}, q) \tag{1.100}$$

equivalently,

$$\boxed{P(\zeta^{-1}, q) = -\zeta^{-1}P(\zeta, q)} \tag{1.101}$$

This may also be easily checked directly from the expression (1.90) for $P(\zeta, q)$.

From (1.97) and (1.101) it of course also follows that

$$\boxed{P(q^2\zeta, q) = P(\zeta^{-1}, q)} \tag{1.102}$$

This may also be easily checked directly from the expression (1.90) for $P(\zeta, q)$.

Furthermore, in this case (1.86) reduces to

$$\mathcal{V}_1(\zeta) = \zeta \tag{1.103}$$

and (1.43) reduces to

$$\mathcal{V}_1(\zeta_1, \zeta_2) = \frac{\mathcal{V}_1(\zeta_1)}{\mathcal{V}_1(\zeta_2)} = \frac{\zeta_1}{\zeta_2} \tag{1.104}$$

where the second equality follows from (1.103). And (1.45) reduces to

$$\mathcal{T}_{1,1} = \mathcal{V}_1(\theta_1(\zeta), \zeta)) = q_1^2 \tag{1.105}$$

where the second equality follows from (1.104).

The function $P(\zeta, q)$ is in fact related to the first Jacobi theta function ([69]) which is given by the expression:

$$\vartheta_1(\tau, q) = 2C\sin\tau \prod_{k=1}^{\infty}(1 - q^{2k}e^{2i\tau})(1 - q^{2k}e^{-2i\tau}) \tag{1.106}$$

Indeed if we define

$$\tau = -\log\zeta \tag{1.107}$$

then it is straightforward to show that

$$P(\zeta, q) = -\frac{i}{Cq^{1/4}}e^{-i\tau/2}\vartheta_1(i\tau/2, q) \tag{1.108}$$

Further properties of the function $P(\zeta, q)$, including its relations with elliptic functions are discussed for example in Chapter 2 of [42].

CHAPTER 2

Automorphic functions.

2.1 Introduction.

We begin this chapter by stating the following definition.

Definition 2.1.1 *Automorphic function*

> If a function is invariant under every element of a group of transformations, we say it is **automorphic** with respect to the group.

Automorphic functions are thus generalizations of periodic functions. We shall in this chapter present explicit representations for functions automorphic with respect to a Schottky group of transformations.

2.2 Representations of automorphic functions.

Consider a general Schottky group Θ of order N with associated fundamental region F. Suppose a function $f(\zeta)$ is automorphic with respect to Θ. Every ordinary point in the plane may be reached as the image $\theta(\zeta)$ of a point ζ in F under a map $\theta \in \Theta$. By the automorphicity of f, $f(\theta(\zeta)) = f(\zeta)$. Thus, if the behaviour of $f(\zeta)$ is known in F, then so is its behaviour at every ordinary point in the plane. Suppose also that f is meromorphic in F. An explicit representation for $f(\zeta)$ is known (see [6] and [11] for example), and will now be described.

Before writing down such a representation, one should be aware of the following facts stated as Propositions 2.2.1 and 2.2.2 which may be deduced about $f(\zeta)$ and which are proved in for example [10].

Proposition 2.2.1 *$f(\zeta)$ takes every value in $\mathbb{C} \cup \{\infty\}$ the same number of times.*

So, in particular, $f(\zeta)$ has the same number of poles as zeros in F, say M. Label these as $\{\alpha_k | k = 1, ..., M\}$ and $\{\beta_k | k = 1, ..., M\}$ respectively.

Proposition 2.2.2 *There are N relations between the poles and zeros in F of $f(\zeta)$. These may be written as*

$$\boxed{\frac{\prod_{k=1}^{M} \mathcal{V}_i(\beta_k)}{\prod_{k=1}^{M} \mathcal{V}_i(\alpha_k)} = \prod_{j=1}^{N} \mathcal{T}_{i,j}^{m_j}, \qquad i = 1, ..., N} \tag{2.1}$$

where $\{m_k | k = 1, ..., N\}$ are integers, and where $\mathcal{V}_i(\zeta_1, \zeta_2)$ and $\mathcal{T}_{i,j}$ are given by (1.41) and (1.46) respectively.

The N conditions (2.1) will be referred to henceforth as the **automorphicity conditions** of the function $f(\zeta)$.

Now following for example [11] and [6], let us show how $f(\zeta)$ may be represented in terms of the associated prime function $\omega(\zeta, \gamma)$. Consider the following expression.

$$\boxed{f(\zeta) = r(\zeta) \frac{\prod_{k=1}^{M} \omega(\zeta, \beta_k)}{\prod_{k=1}^{M} \omega(\zeta, \alpha_k)}} \tag{2.2}$$

where the pre-factor $r(\zeta)$ is given by

$$\boxed{r(\zeta) = R \prod_{i=1}^{N} (\mathcal{V}_i(\zeta))^{-m_i}} \tag{2.3}$$

where R is a multiplicative constant and $\{m_k | k = 1, ..., N\}$ are integers and $\mathcal{V}_i(\zeta)$ is given by (1.41).

By the properties of $\omega(\zeta, \gamma)$ as described in §1.4, it is straightforward to check that $f(\zeta)$ as defined by the expression (2.2) has the following properties. First, notice (2.2) is clearly single-valued at every point ζ. Also, it is analytic in F except at the points $\{\alpha_k | k = 1, ..., M\}$ where it has simple poles. Also it has simple zeros at the points $\{\beta_k | k = 1, ..., M\}$. Also, by the properties of $\mathcal{V}_i(\zeta)$ the factor $r(\zeta)$ has poles at $\{A_i | i = 1, ..., N\}$ and all images of these points under maps in Θ, and zeros at $\{B_i | i = 1, ..., N\}$ and all images of these points under maps in Θ. However, these are all singular points of Θ and thus lie *outside* F. Thus $f(\zeta)$ is single-valued at every point in F and also meromorphic in F.

Now suppose that the poles and zeros of f satisfy the conditions (2.1). Let us check whether $f(\zeta)$ is automorphic. Take one of the N generators of Θ, say θ_n. Then,

$$\begin{aligned}
f(\theta_n(\zeta)) &= \left(R\prod_{i=1}^{N}(\mathcal{V}_i(\theta_n(\zeta)))^{-m_i}\right) \frac{\prod_{k=1}^{M}\omega(\theta_n(\zeta),\beta_k)}{\prod_{k=1}^{M}\omega(\theta_n(\zeta),\alpha_k)} \\
&= \left(R\prod_{i=1}^{N}(\mathcal{V}_i(\zeta)\mathcal{T}_{i,n})^{-m_i}\right)\left(\prod_{k=1}^{M}\mathcal{V}_n(\beta_k,\alpha_k)\right) \frac{\prod_{k=1}^{M}\omega(\zeta,\beta_k)}{\prod_{k=1}^{M}\omega(\zeta,\alpha_k)} \\
&= \left(\prod_{i=1}^{N}(\mathcal{T}_{i,n})^{-m_i}\right)\left(\prod_{k=1}^{M}\mathcal{V}_n(\beta_k,\alpha_k)\right)\left(R\prod_{i=1}^{N}(\mathcal{V}_i(\zeta))^{-m_i}\right) \frac{\prod_{k=1}^{M}\omega(\zeta,\beta_k)}{\prod_{k=1}^{M}\omega(\zeta,\alpha_k)} \\
&= f(\zeta)
\end{aligned} \tag{2.4}$$

where the second equality follows by the transformation properties (1.45) and (1.49), and the final equality follows from (1.42), (1.47) and (2.1).

So the expression (2.2) where $\{\alpha_k|k = 1, ..., M\}$ and $\{\beta_k|k = 1, ..., M\}$ satisfy the conditions (2.1) represents an automorphic function.

Note that in the special case where one of the poles or zeros of $f(\zeta)$ is at the point at infinity we must modify our representation (2.2) for $f(\zeta)$ by replacing the corresponding factor $\omega(\zeta,\infty)$ by $\omega'(\zeta,\infty)$. For example, if $\alpha_1 = \infty$ and the remaining poles $\{\alpha_k|k = 2, ..., M\}$ and the zeros $\{\beta_k|k = 1, ..., M\}$ are all at finite points in F then we may write

$$f(\zeta) = r(\zeta)\frac{\prod_{k=1}^{M}\omega(\zeta,\beta_k)}{\omega'(\zeta,\infty)\prod_{k=2}^{M}\omega(\zeta,\alpha_k)} \tag{2.5}$$

where the pre-factor $r(\zeta)$ is still given by (2.3). Recalling (1.36) notice that as $\zeta \to \infty$ so $\omega'(\zeta,\infty) \to 1$. Also, recalling the properties of the prime function stated in §1.4, $\omega(\zeta,\gamma)$ has a simple pole at $\zeta = \infty$ (-provided this is not a singular point of Θ). Hence the expression (2.5) has the required simple pole at $\zeta = \infty$. Furthermore, recalling (1.41) we see that $\mathcal{V}_i(\gamma) \to 1$ as $\gamma \to \infty$, and thus the N conditions (2.1) satisfied by the poles $\{\alpha_k|k = 2, ..., M\}$ and the zeros $\{\beta_k|k = 1, ..., M\}$ are

$$\frac{\prod_{k=1}^{M}\mathcal{V}_i(\beta_k)}{\prod_{k=2}^{M}\mathcal{V}_i(\alpha_k)} = \prod_{j=1}^{N}\mathcal{T}_{i,j}^{m_j}, \qquad i = 1, ..., N. \tag{2.6}$$

It is straightforward to check following almost the same steps as in (2.4) together with the transformation property (1.54) that the expression (2.5) where $\{\alpha_k|k = 2, ..., M\}$ and $\{\beta_k|k = 1, ..., M\}$ satisfy the conditions (2.6) is automorphic with respect to Θ.

Similarly, if $\beta_1 = \infty$ and $\{\alpha_k | k = 1, ..., M\}$ and $\{\beta_k | k = 2, ..., M\}$ are finite then we may write

$$f(\zeta) = r(\zeta) \frac{\omega'(\zeta, \infty) \prod_{k=2}^{M} \omega(\zeta, \beta_k)}{\prod_{k=1}^{M} \omega(\zeta, \alpha_k)} \tag{2.7}$$

where $\{\alpha_k | k = 1, ..., M\}$ and $\{\beta_k | k = 2, ..., M\}$ satisfy the conditions

$$\frac{\prod_{k=2}^{M} \mathcal{V}_i(\beta_k)}{\prod_{k=1}^{M} \mathcal{V}_i(\alpha_k)} = \prod_{j=1}^{N} \mathcal{T}_{i,j}^{m_j}, \qquad i = 1, ..., N. \tag{2.8}$$

It is straightforward to check these representations are correct in the case $N = 0$. In this case, the Schottky group consists only of the identity transformation, and the associated fundamental region F is the whole extended complex plane. In this case, clearly *any* function $f(\zeta)$ which is single-valued at every point in the whole extended complex plane is invariant under the identity map and thus automorphic with respect to this trivial Schottky group. If furthermore, $f(\zeta)$ is meromorphic everywhere in the extended complex plane then in fact $f(\zeta)$ is a **rational** function. Thus we can represent f as a constant multiple of a ratio of products of factors of the form $(\zeta - \gamma)$. And indeed, in this case the prime function $\omega(\zeta, \gamma)$ of course reduces to $(\zeta - \gamma)$ and the factor $r(\zeta)$ given by (2.3) reduces simply to a multiplicative constant R. Thus, (2.2) indeed reduces to a rational function. Also, from the fact that $\omega'(\zeta, \gamma)$ reduces to 1, (2.5) and (2.7) also reduce to rational functions with respectively a simple pole and simple zero at the point at infinity. Note that in this simply connected case there are *no* conditions on the zeros and poles of the function $f(\zeta)$: there exist rational functions with *any* possible choices of poles and zeros.

Example 2.2.1

For general N, taking the integers $\{m_k | k = 1, ..., N\}$ in the pre-factor $r(\zeta)$ (2.3) all to be zero, the function given by

$$f(\zeta) = R \frac{\prod_{k=1}^{M} \omega(\zeta, \beta_k)}{\prod_{k=1}^{M} \omega(\zeta, \alpha_k)} \tag{2.9}$$

represents an automorphic function with poles $\{\alpha_k | k = 1, ..., N\}$ and zeros $\{\beta_k | k = 1, ..., N\}$ in F where for $i = 1, ..., N$

$$\prod_{k=1}^{M} \mathcal{V}_i(\beta_k, \alpha_k) = 1 \tag{2.10}$$

Example 2.2.2

Alternatively, for general N, taking $m_1 = -1$ and $\{m_k | k = 2, ..., N\}$ all to be zero, so that

$$\begin{aligned} r(\zeta) &= R\mathcal{V}_1(\zeta) \\ &= R \prod_{\theta\in\Theta_1} \left(\frac{\zeta - \theta(B_1)}{\zeta - \theta(A_1)}\right) \end{aligned} \tag{2.11}$$

then

$$f(\zeta) = \left(R \prod_{\theta\in\Theta_1} \frac{\zeta - \theta(B_1)}{\zeta - \theta(A_1)}\right) \frac{\prod_{k=1}^{M} \omega(\zeta, \beta_k)}{\prod_{k=1}^{M} \omega(\zeta, \alpha_k)} \tag{2.12}$$

represents an automorphic function with poles $\{\alpha_k | k = 1, ..., N\}$ and zeros $\{\beta_k | k = 1, ..., N\}$ in F where for $i = 1, ..., N$

$$\prod_{k=1}^{M} \mathcal{V}_i(\beta_k, \alpha_k) = \mathcal{T}_{i,1}^{-1} \tag{2.13}$$

Example 2.2.3

Let us consider the representation of functions automorphic with respect to our Schottky groups constructed from circular domains. In particular, consider those of the form considered in §1.5 where one of the boundaries of the circular domain, C_1 say, is concentric to the outer boundary C_0. In this case, by the expression (1.86), (2.11) becomes

$$\begin{aligned} r(\zeta) &= R\mathcal{V}_1(\zeta) \\ &= R\zeta \prod_{\theta\in\Theta_1'} \frac{\zeta - \theta(0)}{\zeta - \theta(\infty)} \end{aligned} \tag{2.14}$$

so that (2.12) becomes

$$f(\zeta) = \left(R\zeta \prod_{\theta\in\Theta_1'} \frac{\zeta - \theta(0)}{\zeta - \theta(\infty)}\right) \frac{\prod_{k=1}^{M} \omega(\zeta, \beta_k)}{\prod_{k=1}^{M} \omega(\zeta, \alpha_k)} \tag{2.15}$$

and from (1.87) and (1.88) the N conditions (2.13) become

$$\prod_{k=1}^{M} \frac{\beta_k}{\alpha_k} \prod_{\theta\in\Theta_1'} \{\beta_k, \theta(\infty), \alpha_k, \theta(0)\} = q_1^{-2} \left(\prod_{\theta\in\,{}_1\Theta_1''} \frac{\theta(\infty)}{\theta(0)}\right)^2 \tag{2.16}$$

plus for $i = 2, ..., N$,

$$\prod_{k=1}^{M} \mathcal{V}_i(\beta_k, \alpha_k) = \prod_{\theta \in {}_1\Theta_i} \left(\frac{\theta(A_i)}{\theta(B_i)} \right) \tag{2.17}$$

Furthermore, if C_1 is the only inner boundary of D_ζ then from (1.103), (2.9) reduces to

$$f(\zeta) = R \frac{\prod_{k=1}^{M} P(\zeta/\beta_k, q_1)}{\prod_{k=1}^{M} P(\zeta/\alpha_k, q_1)} \tag{2.18}$$

and from (1.104) the relations (2.10) reduce to

$$\prod_{k=1}^{M} \beta_k = \prod_{k=1}^{M} \alpha_k, \qquad i = 1, ..., N \tag{2.19}$$

Also, from (1.103), we have

$$r(\zeta) = R\mathcal{V}_1(\zeta) = R\zeta \tag{2.20}$$

and so (2.12) reduces to the form

$$f(\zeta) = R\zeta \frac{\prod_{k=1}^{M} P(\zeta/\beta_k, q_1)}{\prod_{k=1}^{M} P(\zeta/\alpha_k, q_1)} \tag{2.21}$$

and from (1.104) and (1.105), the relations (2.13) reduce to

$$\prod_{k=1}^{N} \alpha_k = q_1^2 \prod_{k=1}^{N} \beta_k, \qquad i = 1, ..., N \tag{2.22}$$

Crowdy [20] has explicitly constructed automorphic functions using this last representation.

2.2.1 An alternative representation: Poincaré theta series.

A function f automorphic with respect to a Schottky group Θ may be represented in forms other than those presented above which are written in terms of the associated prime function $\omega(\zeta, \gamma)$. One alternative representation ([10], [6], [8]) is in terms of Poincaré theta series. This is described in appendix E. Note that as described in §1.4, Poincaré theta series are in fact related to the prime function. We prefer to use the representation of in terms of the prime function for a number of reasons. Firstly, we find it conceptually more attractive since as described above it is the natural generalization of the trivial case of rational functions written in the forms described, with the prime function $\omega(\zeta,\gamma)$ generalizing the factor $(\zeta-\gamma)$. Furthermore, we find it the easier of the two representations to use in practice, as described in appendix D.

2.3 Summary

Given a circular domain D_ζ of our chosen form with associated Schottky group Θ and associated fundamental region F, we have in this chapter presented an explicit representation for a function $f(\zeta)$ with the properties that it is meromorphic in F and furthermore automorphic with respect to Θ. This representation is in terms of the prime function $\omega(\zeta,\gamma)$ associated with Θ. This representation is precisely of the form stated for example in [6] and [11].

CHAPTER 3

Green's functions for Laplace's equation.

3.1 Introduction.

In this chapter we turn our attention to Green's functions for Laplace's equation in planar domains of finite connectivity. These functions are of great importance in applied mathematics and the physical sciences since so many situations in the "real world" can be mathematically formulated in terms of Laplace's equation. It is desirable to have explicit formulae for these functions not only from a purely mathematical point of view but also for the purpose of their computation in practical use. Formulae are known for the Green's function of any simply connected domain. However, for domains of connectivity $N > 1$ explicit representations of the Green's function are known only in certain cases. In this chapter we shall derive formulae for Green's functions of general domains of arbitrary finite connectivity up to conformal mapping from the canonical class of circular domains. These formulae are also presented in [26] and [30].

3.2 Definition of Green's function.

Throughout the chapter we shall make use of well-known results on Green's functions discussed for example in [54] and [62].

For an arbitrary $(N+1)$-connected planar domain D in a z-plane, bounded by the $N+1$ mutually disjoint curves $\{\partial D_j | j = 0, 1, ..., N\}$ it can be shown that there exists a unique function $G(z; \alpha)$ defined with respect to points z and α in D with the properties that it is real-valued and:

(i)

$$G(z;\alpha) + \log r_0 \tag{3.1}$$

is harmonic with respect to z throughout D including at the point α, where r_0 is

$$r_0 = |z - \alpha|; \tag{3.2}$$

(ii) for $z \in D_j$, $j = 0, 1, ..., N$.

$$G(z;\alpha) = 0. \tag{3.3}$$

$G(z;\alpha)$ is known as the **Green's function** of D. This is in fact sometimes referred to as the *first-kind* Green's function of D to distinguish it from what are known as its *modified* Green's functions which we shall now define.

3.3 Modified Green's functions.

For $j = 0, 1, ..., N$, it can be shown that there exists a unique function $G_j(z;\alpha)$ defined with respect to points z and α in D with the properties that it is real-valued and:

(i)

$$G_j(z;\alpha) + \log r_0 \tag{3.4}$$

is harmonic with respect to z throughout D including at the point α, where r_0 is given by (3.2);

(ii) for $z \in \partial D_j$

$$G_j(z;\alpha) = 0, \tag{3.5}$$

while for $z \in \partial D_k,\ k = 0, 1, ..., N,\ k \neq j$,

$$G_j(z;\alpha) = \gamma_{jk}(\alpha), \tag{3.6}$$

for some real $\gamma_{jk}(\alpha)$ which is independent of z but not necessarily α, and

$$\oint_{\partial D_k} \frac{\partial G_j}{\partial n} ds = 0, \tag{3.7}$$

where n and s denote directions normal and tangential to a curve.

$G_j(z,\alpha)$ is known as a **modified Green's function** of D. Clearly there are $N+1$ different modified Green's functions associated with the $(N+1)$-connected domain D. For $j = 0, 1, ..., N$, the boundary ∂D_j has a special significance with respect to the function $G_j(z;\alpha)$, namely that $G_j(z;\alpha)$ vanishes along it. Note that the conditions (3.7) force the values of the quantities $\{\gamma_{jk}(\alpha) | k = 0, 1, ..., N, \ k \neq j\}$ and these will generally be non-zero. Similarly, the constraint (3.5) forces the value of

$$\oint_{\partial D_j} \frac{\partial G_j}{\partial n} ds \tag{3.8}$$

which will generally be non-zero.

For a general domain D it was in fact shown by Koebe [49] that its first kind Green's function $G(z,\alpha)$ can be constructed from its modified Green's functions $\{G_j(z,\alpha) | j = 0, 1, ..., N\}$. Thus, if explicit formulae are known for the modified Green's functions then one can derive an explicit formula for the first kind Green's function. We shall now describe this construction of $G(z,\alpha)$ from $\{G_j(z,\alpha) | j = 0, 1, ..., N\}$.

3.4 A construction of Green's functions in terms of modified Green's functions.

In this section we describe the construction due to Koebe [49] of a first-kind Green's function of a general domain D in terms of its modified Green's functions.

For $i, j \in \{0, 1, ..., N\}$, $i \neq j$ consider the function given by the difference of two modified Green's functions,

$$G_i(z;\alpha) - G_j(z;\alpha). \tag{3.9}$$

From the properties of the modified Green's functions described in §3.3, clearly the function given by (3.9) is a real-valued function of z and α which is harmonic *everywhere* in D including at $z = \alpha$, and furthermore on each boundary component of D its value is independent of z. In fact, adding any function of α to (3.9) the resulting function still has these properties. In particular, for $j = 1, ..., N$ consider

$$\hat{\sigma}_j(z;\alpha) = G_0(z;\alpha) - G_j(z;\alpha) + \gamma_{j,0}(\alpha) \tag{3.10}$$

where $\gamma_{j,0}(\alpha)$ is as in (3.6), i.e. the value of $G_j(z;\alpha)$ for $z \in \partial D_0$. $\hat{\sigma}_j(z;\alpha)$ has the properties that it is a real-valued function of z and α which is harmonic everywhere in D including at $z = \alpha$, and

$$\hat{\sigma}_j(z;\alpha) = \begin{cases} 0 & \text{for } z \in \partial D_0 \\ b_{k,j}(\alpha) & \text{for } z \in \partial D_k, \ k = 1, ..., N \end{cases} \tag{3.11}$$

where we introduce for $j, k \in \{1, ..., N\}$,

$$b_{k,j}(\alpha) = \gamma_{0,k}(\alpha) - \gamma_{j,k}(\alpha) + \gamma_{j,0}(\alpha) \tag{3.12}$$

Now consider a linear combination of the N functions $\hat{\sigma}_j(z;\alpha)$, $j = 1, ..., N$ of the form

$$\sigma_l(z;\alpha) = \sum_{j=1}^{N} a_{j,l}(\alpha)\hat{\sigma}_j(z;\alpha) \tag{3.13}$$

where $\{a_{j,l}(\alpha)|j = 1, ..., N\}$ are real-valued. $\sigma_l(z;\alpha)$ also has the properties that it is a real-valued function of z and α which is harmonic everywhere in D including at $z = \alpha$.

From (3.11) it is clear that $\sigma_l(z;\alpha)$ vanishes on ∂D_0. But we shall now show that it is in fact possible to pick $\{a_{j,l}(\alpha)|l, j = 1, ..., N\}$ such that for $l = 1, ..., N$

$$\sigma_l(z;\alpha) = \begin{cases} 1 & \text{for } z \in \partial D_l \\ 0 & \text{for } z \in \partial D_k, \ k = 1, ..., N, \ k \neq l \end{cases} \tag{3.14}$$

To see this, first note that from (3.13) and (3.11) it follows that for $k = 1, ..., N$, for $z \in \partial D_k$,

$$\sigma_l(z;\alpha) = \sum_{j=1}^{N} a_{j,l}(\alpha)b_{k,j}(\alpha) \tag{3.15}$$

Letting A and B denote the NxN matrices with entries in the j-th row and k-th column of $a_{j,k}$ and $b_{j,k}$ respectively, it is clear that the right-hand side of (3.15) is the entry in the k-th row and l-th column of the matrix BA. Thus if B is invertible and we pick $\{a_{j,l}(\alpha)|l, j = 1, ..., N\}$ such that A is the inverse of B, then it follows that BA equals the NxN identity matrix I. It is straightforward to see from (3.15) that this is equivalent to (3.14).

So it remains to show that B is invertible. First note that this is equivalent to showing that,

$$Bx = 0 \tag{3.16}$$

does not possess a non-trivial solution for an Nx1 column vector x. Suppose that such a solution does exist. Writing $x = (x_1, x_2, ..., x_N)^{\mathrm{T}}$, where the superscript T denotes transpose, then it is straightforward to show that this implies that the function

$$\sigma^*(z;\alpha) = \sum_{j=1}^{N} x_j \hat{\sigma}_j(z;\alpha) \tag{3.17}$$

is zero on the N boundaries ∂D_k, $k = 1, ..., N$. But also, from (3.11) it follows that $\sigma^*(z;\alpha)$ is zero on ∂D_0 too. Hence $\sigma^*(z;\alpha)$ vanishes everywhere on the boundary of D. But the function $\sigma^*(z;\alpha)$ given by (3.17) is clearly harmonic everywhere in D. Hence, it follows from the maximum and minimum principles of harmonic functions [53] that $\sigma^*(z;\alpha)$ must be identically zero in D. Thus the analytic extension $\tilde{\sigma}^*(z;\alpha)$ of $\sigma^*(z;\alpha)$ must be a constant. But from (3.17) we have

$$\tilde{\sigma}^*(z;\alpha) = \sum_{j=1}^{N} x_j \tilde{\hat{\sigma}}_j(z;\alpha) \tag{3.18}$$

where for $j = 1, ..., N$, $\tilde{\hat{\sigma}}_j(z;\alpha)$ is the analytic extension of $\hat{\sigma}_j(z;\alpha)$, which from (3.10) is given by

$$\tilde{\hat{\sigma}}_j(z;\alpha) = \tilde{G}_0(z;\alpha) - \tilde{G}_j(z;\alpha) + \gamma_{j,0}(\alpha) \tag{3.19}$$

where for $j = 0, 1, ..., N$, $\tilde{G}_j(z;\alpha)$ is the analytic extension of the modified Green's function $G_j(z;\alpha)$. For $j = 0, 1, ..., N$ we shall write

$$\tilde{G}_j(z;\alpha) = G_j(z;\alpha) + iH_j(z;\alpha) \tag{3.20}$$

where $H_j(z;\alpha)$ is the harmonic conjugate of $G_j(z;\alpha)$. Now note that it is clear that for $j, k = 0, 1, ..., N$,

$$\begin{aligned} \oint_{\partial D_k} d[\tilde{G}_j] &= \oint_{\partial D_k} \frac{\partial \tilde{G}_j}{\partial s} ds \\ &= \oint_{\partial D_k} \left(\frac{\partial G_j}{\partial s} + i \frac{\partial H_j}{\partial s} \right) ds \\ &= i \oint_{\partial D_k} \frac{\partial H_j}{\partial s} ds \\ &= i \oint_{\partial D_k} \frac{\partial G_j}{\partial n} ds \end{aligned} \tag{3.21}$$

where the third equality follows from the fact that G_j is constant on ∂D_k, and the fourth equality follows from the Cauchy-Riemann relations satisfied by G_j and H_j. Note that, the integral on the left-hand side of (3.21) is the change in $\tilde{G}_j$ on traversing ∂D_k. From the properties of the modified Green's functions described at the start of §3.3 it follows that for $j, k = 0, 1, ..., N$, the change in $\tilde{G}_j$ on traversing ∂D_k is zero for all k except $k = j$. It follows from (3.19) that for $j, k = 1, ..., N$, on traversing ∂D_k the change in $\tilde{\tilde{\sigma}}_j(z; \alpha)$ is zero except if $k = j$. Hence it follows from (3.18) that since $\tilde{\sigma}^*(z; \alpha)$ is a constant, then for $k = 1, ..., N$, on traversing ∂D_k $\tilde{\sigma}^*(z; \alpha)$ must not change and so x_k must be zero. So the vector x must be zero. But this contradicts our original assumption. Hence B is indeed invertible.

So we have demonstrated that in terms of linear combinations of the modified Green's functions of a general $(N + 1)$-connected domain D one can construct functions $\sigma_j(z; \alpha)$, $j = 1, ..., N$ which are real-valued and harmonic everywhere in D, and furthermore have the property that

$$\sigma_j(z; \alpha) = \begin{cases} 1 & \text{for } z \in \partial D_j \\ 0 & \text{for } z \in \partial D_k, \ k = 0, 1, ..., N, \ k \neq j \end{cases} \tag{3.22}$$

Furthermore, it is straightforward to see that one may define

$$\sigma_0(z : \alpha) = 1 - \sum_{j=1}^{N} \sigma_j(z; \alpha) \tag{3.23}$$

where this clearly has the properties that it is real-valued and harmonic everywhere in D, and furthermore by (3.22), has the property

$$\sigma_0(z; \alpha) = \begin{cases} 1 & \text{for } z \in \partial D_0 \\ 0 & \text{for } z \in \partial D_k, \ k = 1, ..., N, \end{cases} \tag{3.24}$$

The functions $\sigma_j(z; \alpha)$, $j = 0, 1, ..., N$ associated with D are known as its **harmonic measures**.

Now notice that we can express the first kind Green's function $G(z, \alpha)$ of D in terms of the modified Green's functions $\{G_j(z, \alpha) | j = 1, ..., N\}$ and the harmonic measures $\{\sigma_j(z) | j = 1, ..., N\}$ as follows. For any $j \in \{0, 1, ..., N\}$,

$$G(z, \alpha) = G_0(z, \alpha) - \sum_{j=1}^{N} \gamma_{0j} \sigma_j(z) \tag{3.25}$$

where the constants $\{\gamma_{0j} | j = 1, ..., N\}$ are as in (3.6). This expression follows easily from the properties described above for the modified Green's functions and the harmonic measures. It is clear that the function on the right-hand side of (3.25) is harmonic everywhere in D except for the same logarithmic singularity at $z = \alpha$ as the modified Green's function $G_j(z;\alpha)$. Also, for $z \in \partial D_0$, this function vanishes since $G_0(z;\alpha)$ and the harmonic measures $\{\sigma_j(z)|j = 1,...,N\}$ vanish there. But furthermore, this function vanishes on each of the N other boundaries $\{\partial D_j|j = 1,...,N\}$ since for $j = 1,...,N$, for $z \in \partial D_j$ $G_0(z;\alpha) = \gamma_{0j}$ while $-\gamma_{0j}\sigma_j(z) = -\gamma_{0j}$.

So we have described how for a general domain D, in terms of its modified Green's functions one can construct its harmonic measures and then its first kind Green's function. Thus, it remains to find explicit formulae for the modified Green's functions of D.

3.5 Conformal invariance of Green's functions.

But now suppose that $z(\zeta)$ is a one-to-one conformal map of a domain D_ζ in a ζ-plane onto a domain D in a z-plane, where the boundary of D_ζ maps onto the boundary of D. Suppose that $G^{(\zeta)}(\zeta,\alpha)$ is the first kind Green's function of D_ζ with a singularity at $\alpha \in D_\zeta$. Then it is straightforward to show that the first kind Green's function $G(z, z_\alpha)$ of D with a singularity at $z_\alpha = z(\alpha)$ is given by

$$G(z, z_\alpha) = G^{(\zeta)}(\zeta(z);\alpha) \tag{3.26}$$

where $\zeta = \zeta(z)$ is the inverse of the one-to-one conformal mapping of D_ζ onto D. Thus, if we can construct Green's functions for some canonical class of domains, formulae for Green's functions for general multiply connected domains follow up to conformal mappings. We shall construct Green's functions for multiply connected circular domains. We shall do this by deriving explicit formulae for their modified Green's functions and then using the construction described in §3.4.

3.6 Green's functions of circular domains.

Let D_ζ be a circular domain of our chosen form bounded by the unit circle C_0 and N other circles $\{C_j|j = 1,...,N\}$ in the interior of C_0. As usual, label the associated Schottky group generated by N Möbius maps of the form (1.6) as Θ. And recall that for $j = 0, 1, ..., N$, denoting the reflection of D_ζ in C_j by $\hat{D}_\zeta^{(j)}$, the region $F^{(j)}$ consisting of D_ζ and $\hat{D}_\zeta^{(j)}$ is a fundamental region of Θ. Finally, recall we may construct the prime function $\omega(\zeta, \gamma)$.

3.6.1 Modified Green's functions of a circular domain.

It will now be argued that for $j = 0, 1, ..., N$ an explicit expression for the modified Green's function $G_j(\zeta; \alpha)$ is

$$\boxed{G_j(\zeta;\alpha) = -\frac{1}{2}\log\left|\frac{\omega(\zeta,\alpha)\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\omega(\zeta,\overline{\phi}_j(\overline{\alpha}))\overline{\omega}(\phi_j(\zeta),\overline{\alpha})}\right|.} \tag{3.27}$$

where recall the function $\phi_j(\zeta)$ is defined as in (A.5), so that $\overline{\phi_j}(\bar{\zeta})$ is the reflection of ζ in C_j. It follows that the analytic extension $\tilde{G}_j(\zeta; \alpha)$ of this modified Green's function is given by

$$\tilde{G}_j(\zeta;\alpha) = -\frac{1}{2}\log\left(\frac{\omega(\zeta,\alpha)\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}\right). \tag{3.28}$$

Let us prove the that the representation (3.27) for the modified Green's function $G_j(\zeta; \alpha)$ is correct. First define for $j = 0, 1, ..., N$ the functions

$$\hat{R}_j(\zeta;\alpha) = \frac{\omega(\zeta,\alpha)}{\omega(\zeta,\overline{\phi_j}(\bar{\alpha}))} \tag{3.29}$$

It is clear from the properties of the prime function $\omega(\zeta, \gamma)$ that for $j = 0, 1, ..., N$, $\hat{R}_j(\zeta; \alpha)$ has the following properties. First, it is analytic everywhere in D_ζ with a single isolated simple zero at $\zeta = \alpha$. It has a simple pole at $\zeta = \overline{\phi_j}(\bar{\zeta})$ but since α lies in D_ζ, then $\overline{\phi_j}(\bar{\zeta})$ lies outside D_ζ, in fact $\overline{\phi_j}(\bar{\zeta})$ lies in $\hat{D}_\zeta^{(j)}$. For θ a non-identity map in Θ, $\hat{R}_j(\zeta; \alpha)$ also has a simple zero at $\zeta = \theta(\alpha)$ and a simple pole at $\zeta = \theta(\overline{\phi_j}(\bar{\zeta}))$. However, these points lie in the image $\theta(F^{(j)})$ of the fundamental region $F^{(j)}$ of Θ, and hence in particular lie outside D_ζ. Thus, the zero at $\zeta = \alpha$ and the pole at $\zeta = \overline{\phi_j}(\bar{\zeta})$ are the only zeros and poles of $\hat{R}_j(\zeta; \alpha)$ in the fundamental region $F^{(j)}$ of Θ. It is clear that if $\hat{R}_j(\zeta; \alpha)$ is multiplied by a factor independent of ζ (but not necessarily α) then the resulting function still has these properties. It can in fact be shown that for $j = 0, 1..., N$, the function

$$\frac{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\overline{\omega}(\phi_j(\zeta),\overline{\alpha})} \tag{3.30}$$

is simply $\hat{R}_j(\zeta;\alpha)$ multiplied by a factor independent of ζ. A proof of this is given in §F.1. Thus,

$$R_j(\zeta;\alpha) = \left(\frac{\omega(\zeta,\alpha)\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}\right)^{1/2} \tag{3.31}$$

is also $\hat{R}_j(\zeta;\alpha)$ multiplied by a factor independent of ζ.

So for $j = 0, 1..., N$,

$$-\log|R_j(\zeta;\alpha)| \tag{3.32}$$

is harmonic in D_ζ except for a logarithmic singularity at $\zeta = \alpha$. And in the neighbourhood of this point we have

$$-\log|R_j(\zeta;\alpha)| = -\log|\zeta - \alpha| + \mathcal{O}(1), \tag{3.33}$$

In fact, as we shall now show, $R_j(\zeta;\alpha)$ is the correct multiple of $\hat{R}_j(\zeta;\alpha)$ such that for $j = 0, 1, ..., N$, (3.32) satisfies the boundary conditions (3.5), (3.7), (3.7) required of the modified Green's functions of D_ζ. First, it is shown in §F.2 that for $k = 0, 1, ..., N$, for ζ on the circle C_k,

$$|R_j(\zeta;\alpha)| = \left|\frac{\mathcal{V}_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\mathcal{V}_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right|^{1/2}. \tag{3.34}$$

where recall $\mathcal{V}_j(\gamma_1,\gamma_2)$ is given by (1.43) and we adopt the convention that $\mathcal{V}_0(\zeta,\alpha) \equiv 1$. It is immediate that for ζ on C_j, $|R_j(\zeta;\alpha)| = 1$ and so $-\log|R_j(\zeta;\alpha)| = 0$ as required. Furthermore, for ζ on C_k, $k \neq j$, we have

$$-\log|R_j(\zeta;\alpha)| = -\frac{1}{2}\log\left|\frac{\mathcal{V}_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\mathcal{V}_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right| \tag{3.35}$$

which is independent of ζ. Finally note that from (3.21) it follows that for $k = 0, 1, ..., N$,

$$\oint_{\partial C_k} \frac{\partial G_j}{\partial n} ds = \mathrm{Im}[[\tilde{G}_j]_{C_k}] \tag{3.36}$$

where $[\tilde{G}_j]_{C_k}$ denotes the change in $\tilde{G}_j$ on traversing the circle C_k. Thus it is required that

$$\mathrm{Im}[[\tilde{G}_j]_{C_k}] = 0 \quad \text{for } k = 0, 1, ..., N,\ k \neq j. \tag{3.37}$$

Consider now $R_j(\zeta;\alpha)$ given by (3.31). In the fundamental region $F^{(j)}$ associated with the Schottky group Θ, $R_j(\zeta;\alpha)$ has precisely two logarithmic singularities of equal and opposite strength: one at $\zeta = \alpha$ in D_ζ, the other at $\zeta = \overline{\phi}_j(\bar{\alpha})$ in $\hat{D}^{(j)}$. A natural way to define a branch of $R_j(\zeta;\alpha)$ is therefore to join by a branch cut this pair of logarithmic singularities in the fundamental region, and furthermore for θ a non-identity map in Θ, to join $\theta(\alpha)$ and $\theta(\overline{\phi_j}(\bar{\alpha}))$. It follows that for $k = 0, 1, ..., N$ the change in $R_j(\zeta;\alpha)$ on traversing C_k is zero for all k except $k = j$. Hence, $R_j(\zeta;\alpha)$ satisfies the remaining condition (3.37).

Thus we have demonstrated that for $j = 0, 1, ..., N$, the function (3.32) satisfies all the defining properties of the modified Green's functions of D_ζ. It is straightforward to show that any function satisfying these properties is unique. Indeed, if there are two functions with these properties then their difference is harmonic everywhere in D_ζ including at α, and zero on the whole boundary of D_ζ. Hence the difference is identically zero in D_ζ by the minimum and maximum principles for harmonic functions ([53]).

Note finally that for $j, k = 0, 1, ..., N$, the values $\gamma_{j,k}(\alpha)$ of (3.6) for the modified Green's functions of the circular domain D_ζ are given by (3.35), i.e.

$$\gamma_{j,k}(\alpha) = -\frac{1}{2}\log\left|\frac{\mathcal{V}_j(\overline{\phi_j}(\bar{\alpha}), \alpha)}{\mathcal{V}_k(\overline{\phi_j}(\bar{\alpha}), \alpha)}\right| \tag{3.38}$$

Note that in the special case of $j = 0$, the circle C_0 is the circle $|\zeta| = 1$ so that $\phi_0(\zeta) = \zeta^{-1}$ and so the function $G_0(\zeta, \alpha)$ takes the form

$$G_0(\zeta;\alpha) = -\frac{1}{2}\log\left|\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\bar{\alpha})}\right| \tag{3.39}$$

which, on use of (1.62), reduces to

$$G_0(\zeta;\alpha) = -\log\left|\frac{\omega(\zeta,\alpha)}{\alpha\omega(\zeta,\bar{\alpha}^{-1})}\right|. \tag{3.40}$$

Its analytic extension is then simply

$$\tilde{G}_0(\zeta;\alpha) = -\log\left(\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\bar{\alpha}^{-1})}\right). \tag{3.41}$$

3.6.2 First kind Green's function of a circular domain.

Having derived explicit formulae for the modified Green's functions of a circular domain we now use these to construct an explicit formula for its first kind Green's function as described in §3.4.

First we define for $j = 1, ..., N$,

$$\hat{\sigma}_j(\zeta;\alpha) = G_0(\zeta;\alpha) - G_j(\zeta;\alpha) + \gamma_{j,0}(\alpha) \tag{3.42}$$

where now the modified Green's functions $\{G_j(\zeta;\alpha)|j = 0, 1, ..., N\}$ are given explicitly by (3.27) and $\{\gamma_{j,0}(\alpha)|j = 1, ..., N\}$ are given by (3.38). Then the harmonic measures $\{\sigma_l(z;\alpha)|l = 1, ..., N\}$ are given by

$$\sigma_l(z;\alpha) = \sum_{j=1}^{N} a_{j,l}(\alpha)\hat{\sigma}_j(\zeta;\alpha) \tag{3.43}$$

where for $j, l = 1, ..., N$, $a_{j,l}(\alpha)$ is the entry in the j-th row and l-th column of the inverse of the NxN matrix whose entry in the k-th row and j-th column for $k, j = 1, ..., N$ is given by

$$b_{k,j}(\alpha) = \gamma_{0,k}(\alpha) - \gamma_{j,k}(\alpha) + \gamma_{j,0}(\alpha) \tag{3.44}$$

Note that from (3.38) it follows that

$$\begin{aligned} b_{k,j}(\alpha) &= \frac{1}{2}\log\left|\frac{\mathcal{V}_k(\overline{\phi_0}(\bar{\alpha}),\alpha)}{\mathcal{V}_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right| \\ &= \frac{1}{2}\log|\mathcal{V}_k(\bar{\alpha}^{-1},\overline{\phi_j}(\bar{\alpha}))| \\ &= \frac{1}{2}\log|\mathcal{V}_k(\bar{\alpha}^{-1},\theta_j(\bar{\alpha}^{-1}))| \\ &= \frac{1}{2}\log|\mathcal{T}_{k,j}| \end{aligned} \tag{3.45}$$

where the third equality follows from (1.4) and the fourth equality follows from (1.45). $\mathcal{T}_{k,j}$ is given by (1.46). Note that $\mathcal{T}_{k,j}$ is in fact *independent* of α and depends *only* on the parameters $\{q_j, \delta_j|j = 1, .., N\}$ defining the circular domain D_ζ.

Note incidentally that the harmonic measure $\sigma_0(z;\alpha)$ of D_ζ is given by (3.23) in terms of $\sigma_l(z;\alpha)$, $l = 1, ..., N$.

Finally, the explicit formula for the first kind Green's function of D_ζ is given by (3.25). Note that all the geometrical information on the shape of the circular domain D_ζ is encoded in this formula by the prime function $\omega(\zeta,\gamma)$.

3.6.3 The simply connected case.

It is instructive to see how the general theory just presented reduces to perhaps more familiar formulas in the simply connected case. In this case the circular domain D_ζ is just the disc bounded by the unit circle C_0 centred on the origin. In this case D_ζ has just one boundary component and so it has just one modified Green's function which is also its first kind Green's function. From (3.40) this is given by

$$\begin{aligned} G(\zeta;\alpha) &= G_0(\zeta;\alpha) \\ &= -\log\left|\frac{\omega(\zeta,\alpha)}{\alpha\omega(\zeta,\bar{\alpha}^{-1})}\right| \\ &= -\log\left|\frac{\zeta-\alpha}{\alpha(\zeta-\bar{\alpha}^{-1})}\right| \end{aligned} \tag{3.46}$$

where the last equality follows from the fact that in this case the prime function reduces simply to $\omega(\zeta,\gamma)=\zeta-\gamma$. It is straightforward to check that for $|\zeta|=1$,

$$\left|\frac{\zeta-\alpha}{\alpha(\zeta-\bar{\alpha}^{-1})}\right| = 1 \tag{3.47}$$

so that the function given by (3.46) vanishes on the boundary of D_ζ as required.

3.6.4 The doubly connected case.

We now present details of a doubly connected case. Consider the circular domain which is the annulus bounded by the circles $|\zeta|=1$ and $|\zeta|=q$ for some $q<1$, denoted respectively C_0 and C_1. The Schottky group we construct from such a circular domain is as in §1.5.1. In particular, recall that the associated prime function $\omega(\zeta,\gamma)$ can be written in terms of the function $P(\zeta,q)$ which is defined by (1.90). It follows that in this case

$$G_0(\zeta,\alpha) = -\frac{1}{2}\log\left|\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\bar{\alpha})}\right| = -\frac{1}{2}\log\left|\frac{P(\zeta\alpha^{-1},q)P(\alpha\zeta^{-1},q)}{P(\zeta\bar{\alpha},q)P(\zeta^{-1}\bar{\alpha}^{-1},q)}\right|,$$

$$G_1(\zeta,\alpha) = -\frac{1}{2}\log\left|\frac{\omega(\zeta,\alpha)\overline{\omega}(q^2\zeta^{-1},q^2\alpha^{-1})}{\omega(\zeta,q^2\bar{\alpha}^{-1})\overline{\omega}(q^2\zeta^{-1},\bar{\alpha})}\right| = -\frac{1}{2}\log\left|\frac{P(\zeta\alpha^{-1},q)P(\alpha\zeta^{-1},q)}{P(\zeta\bar{\alpha}q^{-2},q)P(q^2\zeta^{-1}\bar{\alpha}^{-1},q)}\right|. \tag{3.48}$$

Also, it can be shown that the harmonic measure $\sigma_1(\zeta)$ of D_ζ which takes the value 1 on C_1 and 0 on C_0 is given by

$$\sigma_1(\zeta) = \frac{\log|\zeta|}{\log q}. \tag{3.49}$$

Finally, the first-kind Green's function of D_ζ is given by

$$G(\zeta,\alpha) = -\frac{1}{2\pi}\left(\log\left|\frac{\alpha P(\zeta\alpha^{-1},q)}{P(\zeta\bar{\alpha},q)}\right| - \log|\alpha|\frac{\log|\zeta|}{\log q}\right) \tag{3.50}$$

3.7 Examples.

In this section we present evidence of numerical computation of the above formulae for Green's functions of multiply connected domains. Figures 3.1 and 3.2 show the contours of the Green's functions of circular domains of the form of D_ζ. Figures 3.3–3.8 show the contours of the Green's functions of other multiply connected domains constructed as the image under a conformal map of such circular domains. The conformal maps to the domains in figures 3.3 and 3.4 are simply Möbius maps. However, the conformal maps to the slit domains in figures 3.5–3.8 are more complicated. These maps are described in §G.2.

Note that in the left of all these figures are shown the critical "separatrix" contours which separate the domain into qualitatively distinct contour types. Points at which these critical contours intersect themselves are referred to as *critical points* of the Green's function. Note that in all these examples the domain is triply connected and the Green's function exhibits two critical points. This is consistent with the general result ([55]) that the Green's function of an $(N+1)$-connected domain has precisely N critical points in the domain.

3.8 A numerical check.

It is important to give some quantitative validation of our new construction of Green's functions. Note that Shen, Strang and Wathen [66] have studied the Green's functions of domains of the type shown in figures 3.5–3.8 bounded by slits in a line. At points in the intervals between the slits, the Green's function may be expressed in terms of formulae first stated by Widom [70] which were derived from Schwarz-Christoffel mappings. Based on this formulation, Shen *et al* [66] have developed a means for determining the positions of the critical points of the Green's function of such slit domains. This method is now briefly described. Then as an independent

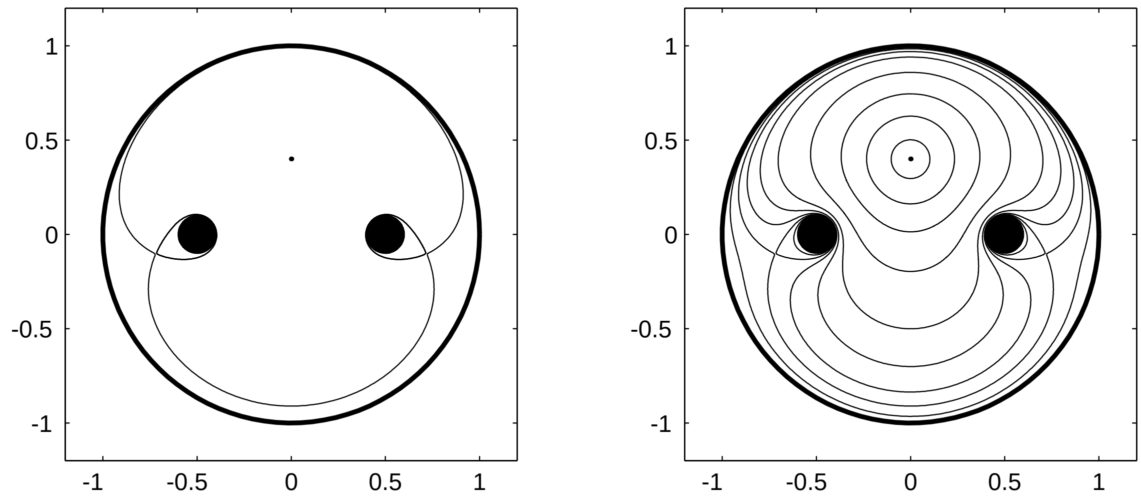

Fig. 3.1: Domain bounded by the unit circle and circles of radius 0.1 centred at ± 0.5. The singularity is at $0.4i$.

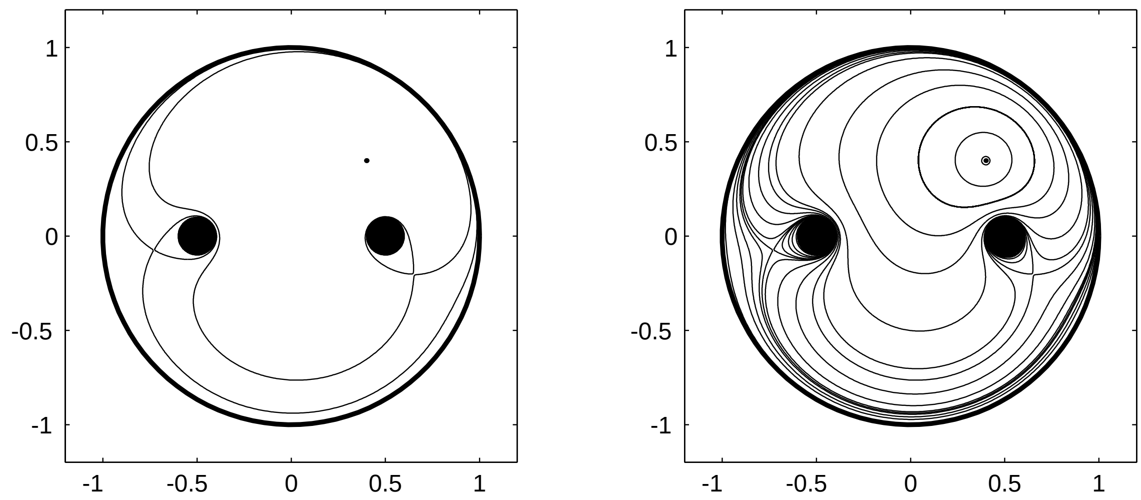

Fig. 3.2: Domain bounded by the unit circle and circles of radius 0.1 centred at ± 0.5. The singularity is at $0.4(1+i)$.

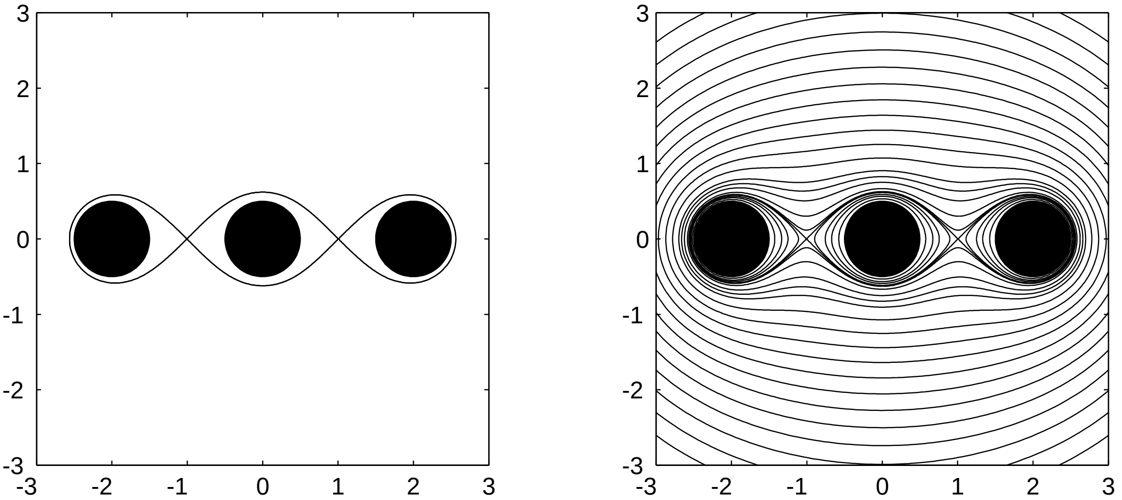

Fig. 3.3: Domain exterior to three circles each of radius 0.5 centred at ± 2 and 0. The singularity is at infinity.

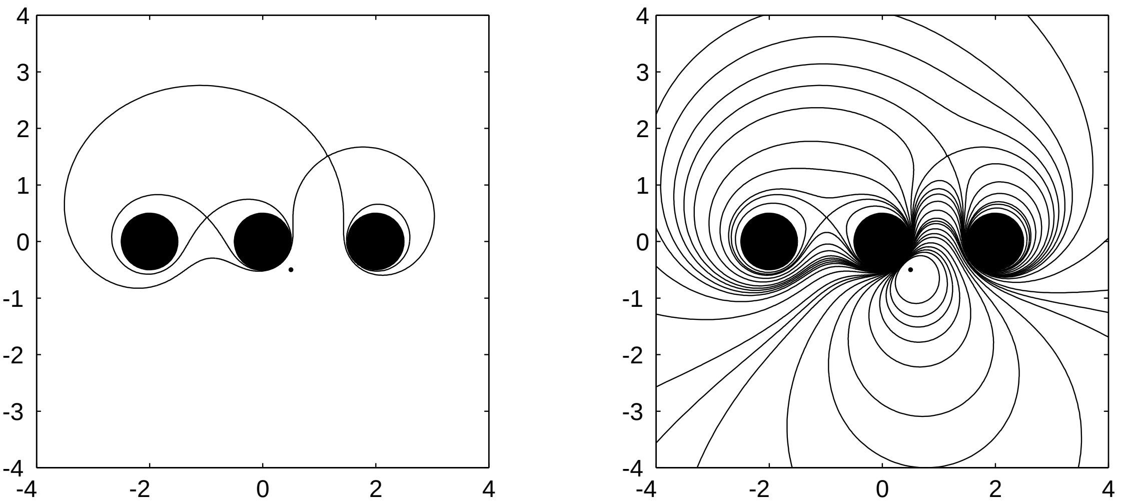

Fig. 3.4: Domain exterior to three circles each of radius 0.5 centred at ± 2 and 0. The singularity is at $0.5(1 - i)$.

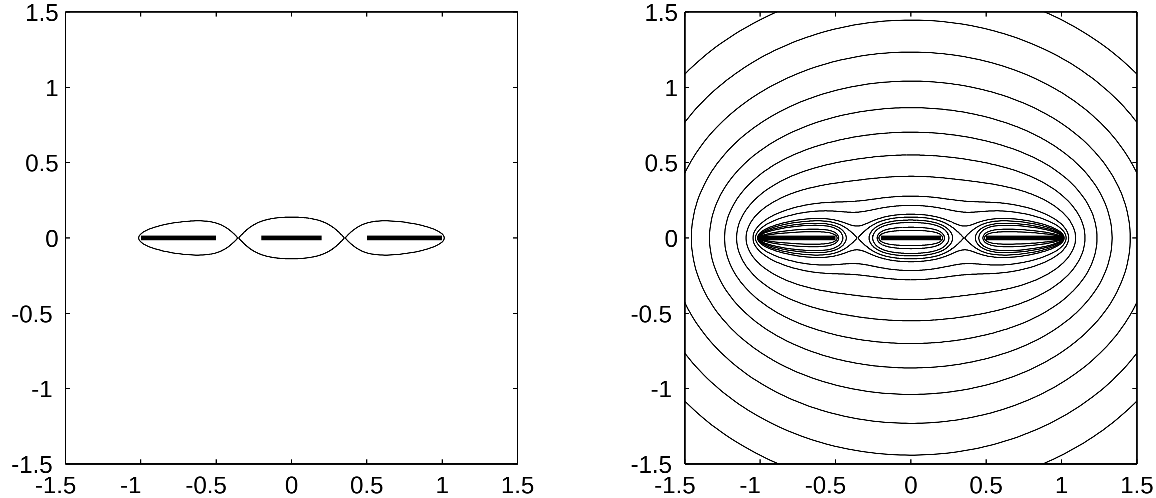

Fig. 3.5: Domain exterior to three slits between $[-1,-0.5]$,$[-0.2,0.2]$ and $[0.5,1]$. The singularity is at infinity.

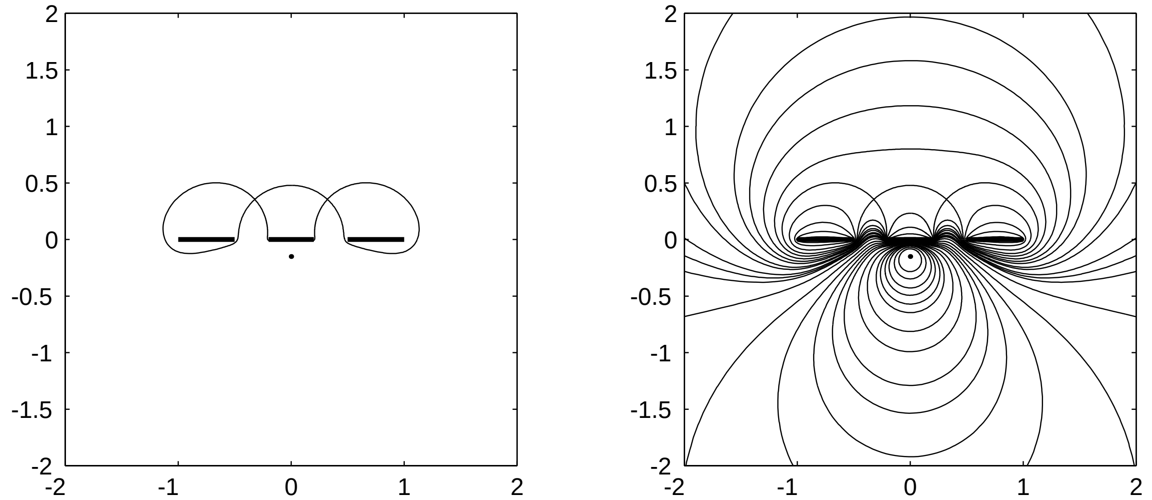

Fig. 3.6: Domain exterior to three slits between $[-1,-0.5]$,$[-0.2,0.2]$ and $[0.5,1]$. The singularity is at $-0.1513i$ (whose pre-image in the ζ-plane is $\zeta = 0.5i$).

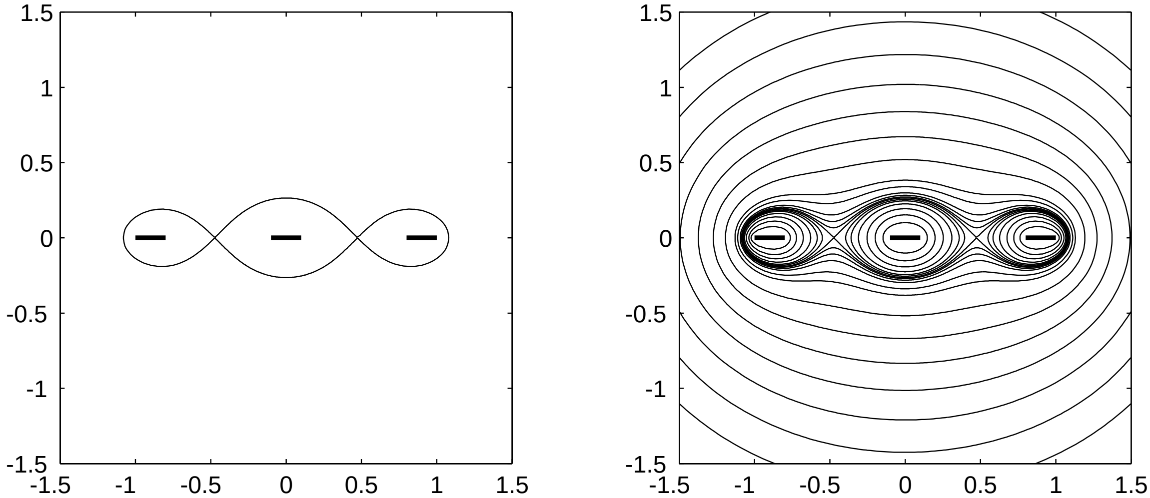

Fig. 3.7: Domain exterior to three slits between $[-1, -0.8]$,$[-0.1, 0.1]$ and $[0.8, 1]$. The singularity is at infinity.

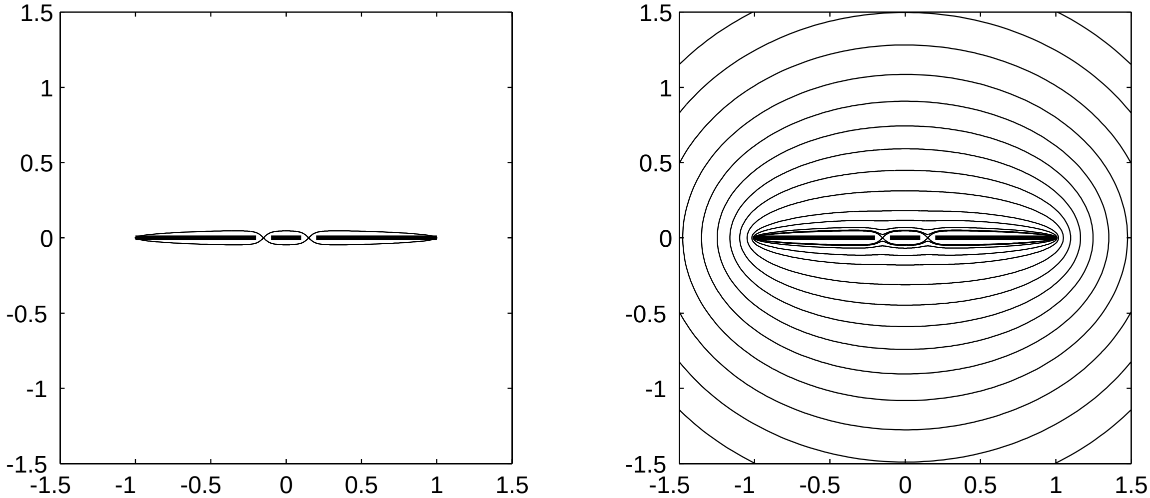

Fig. 3.8: Domain exterior to three slits between $[-1, -0.2]$,$[-0.1, 0.1]$ and $[0.2, 1]$. The singularity is at infinity.

check on our formulae for Green's functions stated above, we shall compare the values for the critical points of the Green's functions of such slit domains computed using both our formulae and this alternative method.

Consider a set of $N+1$ disjoint slits $[-1, a_1], [b_1, a_2], ..., [b_{N-1}, a_N], [b_N, 1]$ on the real axis. The region exterior to these slits is a $(N+1)$-connected domain. Recall as stated above the general result on Green's functions in multiply connected domains which says that the Green's function of an $(N+1)$-connected domain D has precisely N critical points in D (see [55]). Let $\{\varsigma_k | k = 1, ..., N\}$ be the critical points of the Green's function associated with the domain exterior to these slits. Define polynomials

$$\begin{aligned} Q(z) &= (z^2 - 1) \prod_{k=1}^{N} (z - a_k)(z - b_k), \\ P(z) &= \prod_{k=1}^{N} (z - \varsigma_k) \equiv z^N - c_1 z^{N-1} - c^2 z^{N-2} - ... - c_N \end{aligned} \tag{3.51}$$

Let $\{I_k | k = 1, ..., N\}$ denote the N gaps between the slits, so for example I_1 denotes the interval of the real axis given by $[a_1, b_1]$. Then, according to Theorem 3 of [66], if we define an NxN matrix M and a Nx1 column vector b by

$$M_{jk} = \int_{I_j} \frac{t^{N-k} dt}{\sqrt{Q(t)}}, \quad \mathbf{b}_j = \int_{I_j} \frac{t^N dt}{\sqrt{Q(t)}} \tag{3.52}$$

then the vector $\mathbf{c} \equiv (c_1, c_2, ..., c_N)^T$ solves the linear system $M\mathbf{c} = \mathbf{b}$. Having solved for $\mathbf{c}$, the values $\{\varsigma_k | k = 1, ..., N\}$ can then be determined.

The following table gives data for a symmetric distribution of three slits $[-1, -b]$, $[-a, a]$, $[b, 1]$ along the real axis together with the position, ς_1, of one of the two critical points in this case (by symmetry, the other critical point is at $-\varsigma_1$). The values of the second column of the table are computed using our new Green's function formulas presented above and Newton's method to find the zero of the derivative of this Green's function in the gaps between the slits. The values of the third column are computed using the formulas of Shen *et al* [66]. In all cases, agreement to several decimal places of accuracy is found.

Geometry (a, b)	New method σ_1	Shen *et al* σ_1,
(0.1, 0.2)	0.150077	0.150067
(0.1, 0.3)	0.200123	0.200119
(0.1, 0.4)	0.250416	0.250414
(0.1, 0.5)	0.301591	0.301591
(0.1, 0.6)	0.473644	0.473644
(0.1, 0.7)	0.550757	0.550757

Table 2. Comparison of critical point positions of differing right-left symmetric distributions of three equipotential slits on the real axis.

Table 3 gives the data for a symmetric distribution of five slits $[-1, -d]$, $[-c, -b]$, $[-a, a]$, $[b, c]$ and $[d, 1]$ along the real axis and the positions σ_1 (between a and b) and σ_2 (between c and d) of the critical points in the gap-intervals as calculated by our formulae above and by the formulae of Shen *et al* [66]. Again, there is good agreement to several decimal places.

Geometry (a, b, c, d)	New method (σ_1, σ_2)	Shen *et al* (σ_1, σ_2)
(0.05, 0.15, 0.25, 0.35)	(0.100161, 0.299993)	(0.100071, 0.299885)
(0.1,0.35,0.55, 0.8)	(0.226884, 0.683378)	(0.226896, 0.683390)
(0.18, 0.23, 0.59, 0.64)	(0.205137, 0.613791)	(0.205067, 0.615306)

Table 3. Comparison of critical point positions of differing right-left symmetric distributions of five equipotential slits on the real axis.

3.9 Summary

We have constructed explicit formulae for the modified Green's functions, harmonic measures and first kind Green's function of a multiply connected circular domain. Formulae for the Green's function of more general multiply connected domains follow using conformal mapping. Numerical checks against alternative independent formulations validate our formulae.

CHAPTER 4

Mappings to canonical domains.

4.1 Introduction.

The theory of Green's functions has underlying connections with the theory of conformal mappings to multiply-connected domains of certain canonical classes. These connections were first made by Koebe [49] and are described for example in [54] and [62]. However, explicit formulae for these conformal maps do not seem to appear in the existing literature. In this chapter we shall use the explicit representations for the Green's functions presented in the previous chapter to construct such formulae. Note that these formulae are also presented in [29].

As referred to in §1.1 there are a number of canonical classes of multiply connected domains other than circular domains. Each is characterized by different geometries which make it more suited to particular situations than others. The most common of these are found listed in standard texts on function theory and conformal mapping such as [54] and [62]. The five most important are, for domains of connectivity $N+1$, $N \geq 0$: (a) the region bounded by a circle with N enclosed circular arc slits all centred on the centre of the circle; (b) the region bounded by a concentric circular ring with $N-1$ concentric circular arc slits between the two circumferences of the ring; (c) the circular arc domain bounded by $N+1$ circular arc slits all centred on the origin; (d) the parallel slit domain bounded by $N+1$ finite-length slits aligned at the same angle to the real axis; (e) the radial slit domain bounded by $N+1$ finite-length slits lying on rays emanating from the origin.

It is very useful to have explicit formulae for the conformal maps from an arbitrary domain onto the conformally equivalent domain of each of these classes. Or, which is almost as desirable, is to have explicit formulae for the conformal maps between conformally equivalent domains of each of these classes so that given an arbitrary domain if one can determine, possibly by numerical means, a conformal map to a domain of one of these classes then the conformal maps to domains of each of the other classes follow easily. Such formulae do not however appear to be documented. However, as referred to above, it turns out that these maps may in fact be constructed using modified Green's functions [54],[62]. We shall in fact explicitly construct conformal maps from a general circular domain to domains of the classes (a)–(e) just described using the modified Green's functions associated with the circular domain.

4.2 Mathematical formulation.

Let D_ζ be the usual circular domain bounded by the circle $|\zeta| = 1$ which we label C_0 and N other circles $\{C_j | j = 1, ..., N\}$ in the interior of C_0. As usual we can construct an associated Schottky group Θ generated by N Möbius maps of the form (1.6). And recall that for $j = 0, 1, ..., N$, denoting the reflection of D_ζ in C_j by $\hat{D}_\zeta^{(j)}$, the region $F^{(j)}$ consisting of D_ζ and $\hat{D}_\zeta^{(j)}$ is a fundamental region of Θ. Finally we may construct the prime function $\omega(\zeta, \gamma)$.

Recall, we may express each of the $N + 1$ modified Green's functions $G_j(\zeta; \alpha)$ of D_ζ in the form (3.27), $j = 0, 1, ..., N$. The analytic extension of $G_j(\zeta; \alpha)$ is $\tilde{G}_j(\zeta; \alpha)$ which is given by (3.28).

4.2.1 The circle with concentric circular slits.

For $j = 0, 1, ..., N$, let $R_j(\zeta; \alpha)$ denote the conformal map from D_ζ to the region bounded by an outer circle and M inner concentric circular slits, where C_j maps to the outer circle and the point $\zeta = \alpha$ maps to the origin. It can be shown ([54],[62]) that in terms of $\tilde{G}_j(\zeta; \alpha)$ we have

$$R_j(\zeta; \alpha) = \exp(-\tilde{G}_j(\zeta; \alpha)), \tag{4.1}$$

so that from (3.28) we have

$$R_j(\zeta; \alpha) = \left(\frac{\omega(\zeta, \alpha)\overline{\omega}(\phi_j(\zeta), \phi_j(\alpha))}{\omega(\zeta, \overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta), \bar{\alpha})} \right)^{1/2}. \tag{4.2}$$

The mappings (4.2) have been normalized to take the boundary circle C_j to the unit circle in the image domain.

Note that in the special case $j = 0$, (4.2) simplifies to

$$R_0(\zeta;\alpha) = \frac{1}{|\alpha|}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})}. \tag{4.3}$$

4.2.2 The circular ring with concentric circular slits

Let $S_{ij}(\zeta)$ denote a conformal mapping which maps C_i and C_j respectively onto the outer and inner circumferences of a circular ring domain while the remaining $N-1$ circles bounding D are mapped to circular arc slits between these two bounding circumferences. It can be shown ([54]) that in terms of the modified Green's functions of D_ζ

$$S_{ij}(\zeta) = \exp(\tilde{G}_j(\zeta;\alpha) - \tilde{G}_i(\zeta;\alpha)) \tag{4.4}$$

4.2.3 The circular slit domain.

Let $P(\zeta;\alpha,\beta)$ be the conformal mapping from D_ζ to a circular slit domain where the point $\zeta = \alpha$ maps to the origin and the point $\zeta = \beta$ maps to infinity. Following Schiffer [62], the required mapping is given by

$$P(\zeta;\alpha,\beta) = \frac{\exp(-\tilde{G}_j(\zeta;\alpha))}{\exp(-\tilde{G}_j(\zeta;\beta))} \tag{4.5}$$

Any of the $N+1$ modified Green's functions $\{\tilde{G}_j(\zeta;\alpha)|j = 0, 1..., N\}$ can be used in these formulas but let us use $\tilde{G}_0(\zeta;\alpha)$ which is given by the simple form (3.41). An explicit formula for the required mapping is thus

$$P(\zeta;\alpha,\beta) = \frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\beta}^{-1})}{\omega(\zeta,\beta)\omega(\zeta,\bar{\alpha}^{-1})} \tag{4.6}$$

4.2.4 The parallel slit domain

Consider the conformal mapping from D_ζ to a parallel slit domain where each slit makes an angle χ to the real axis in the image domain. Each boundary circle of D_ζ maps to a different slit. Suppose the point $\zeta = \alpha$ is the point mapping to infinity. Let this conformal mapping be $\phi_\chi(\zeta;\alpha)$. It is shown in Schiffer [62] that such a mapping is given by

$$\phi_\chi(\zeta;\alpha) = e^{i\chi}\left[\cos\chi\phi(\zeta;\alpha) - i\sin\chi\psi(\zeta;\alpha)\right] \tag{4.7}$$

where

$$\phi(\zeta;\alpha) = \frac{1}{i}\frac{\partial}{\partial y_0}\tilde{G}_j(\zeta;\alpha), \quad \psi(\zeta;\alpha) = \frac{\partial}{\partial x_0}\tilde{G}_j(\zeta;\alpha) \tag{4.8}$$

with $\alpha = x_0 + iy_0$. Note that *any* of the $N+1$ modified Green's functions $\{\tilde{G}_j(\zeta;\alpha)|j = 0, 1..., N\}$ can be used in these formulas but let us use $\tilde{G}_0(\zeta;\alpha)$ which is given by the simple form (3.41). By a simple change of variables, (4.8) can also be written

$$\phi(\zeta;\alpha) = -\left(\frac{\partial}{\partial\bar{\alpha}} - \frac{\partial}{\partial\alpha}\right)\tilde{G}_0(\zeta;\alpha), \quad \psi(\zeta;\alpha) = \left(\frac{\partial}{\partial\bar{\alpha}} + \frac{\partial}{\partial\alpha}\right)\tilde{G}_0(\zeta;\alpha) \tag{4.9}$$

so that (4.7) takes the form

$$\phi_\chi(\zeta;\alpha) = \left[\frac{\partial}{\partial\alpha} - e^{2i\chi}\frac{\partial}{\partial\bar{\alpha}}\right]\tilde{G}_0(\zeta;\alpha). \tag{4.10}$$

From (3.41) it follows that

$$\begin{aligned}\frac{\partial\tilde{G}_0(\zeta;\alpha)}{\partial\alpha} &= \frac{1}{2\alpha} - \frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)},\\ \frac{\partial\tilde{G}_0(\zeta;\alpha)}{\partial\bar{\alpha}} &= \frac{1}{2\bar{\alpha}} - \frac{1}{\bar{\alpha}^2}\frac{\omega_\alpha(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})}\end{aligned} \tag{4.11}$$

where the notation $\omega_\alpha(\zeta,\alpha)$ denotes the derivative of $\omega(\zeta,\alpha)$ with respect to its second argument. Combining all this leads to the final formula

$$\phi_\chi(\zeta;\alpha) = -e^{2i\chi}\left(\frac{1}{2\bar{\alpha}} - \frac{1}{\bar{\alpha}^2}\frac{\omega_\alpha(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})}\right) + \frac{1}{2\alpha} - \frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)}. \tag{4.12}$$

4.2.5 The radial slit domain.

Let $Q(\zeta;\alpha,\beta)$ be the conformal mapping from D_ζ to a radial slit domain where the point $\zeta = \alpha$ maps to the origin and the point $\zeta = \beta$ maps to infinity. Neither Nehari

[54] nor Schiffer [62] give a simple way to construct this mapping from the modified Green's functions, but it turns out to be given by

$$Q(\zeta;\alpha,\beta) = \frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\beta)\omega(\zeta,\bar{\beta}^{-1})}. \tag{4.13}$$

To establish this, the formulae (1.62), (1.9) and (1.49) can be used to demonstrate that, on each boundary circle $\{C_j | j = 0, 1, ..., M\}$, the argument of $Q(\zeta;\alpha,\beta)$ is constant (this is easily done by computing $\overline{Q(\zeta;\alpha,\beta)}$ on each circle and showing that it is proportional to $Q(\zeta;\alpha,\beta)$). This shows that the images of each circle are rays emanating from the origin. Then, standard arguments based on the argument principle can be used to show the univalency of the mapping on use of the fact that it has just one zero and one pole inside the original circular domain.

4.3 Examples.

As illustrative examples of the above mapping formulas in action, Figure 4.1 shows the case of an arbitrarily chosen triply connected circular domain and its images under the five mappings of §4.2.1–4.2.5. Figure 4.2 shows the case of a quadruply connected domain.

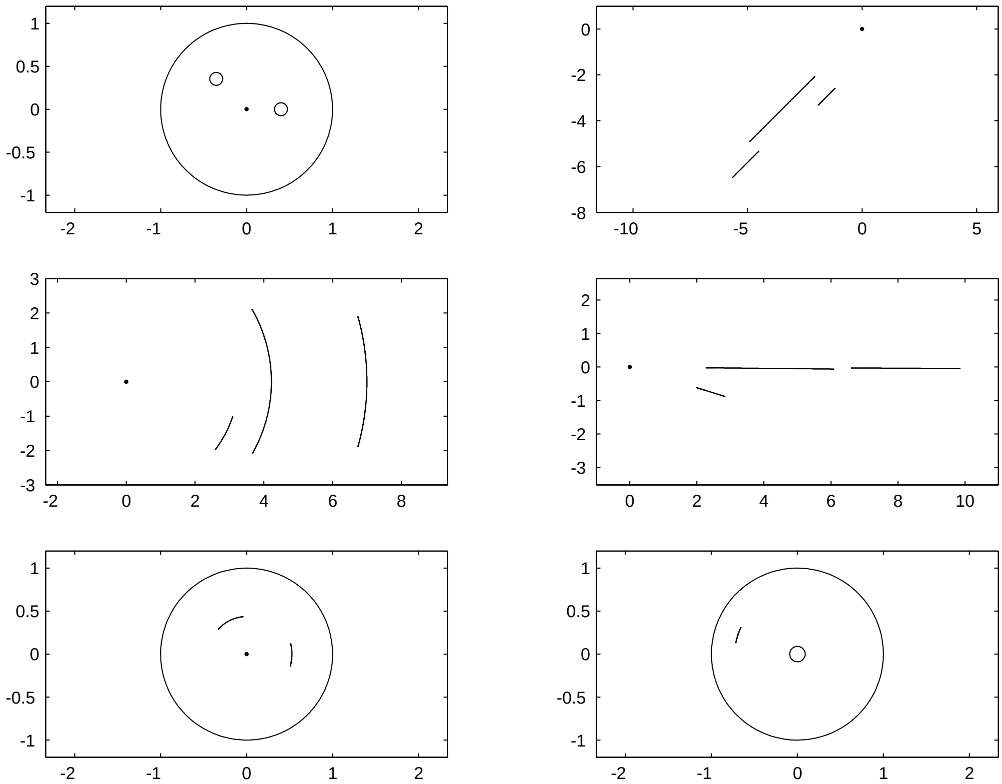

Fig. 4.1: Conformal mappings from a triply connected circular domain with parameters $\delta_1 = 0.4, q_1 = 0.075, \delta_2 = 0.5e^{3\pi i/4}, q_2 = 0.075$ with parameters $\chi = \pi/4$, $\alpha = -0.15$ and $\beta = 0.1$ to five types of conformally equivalent slit domains.

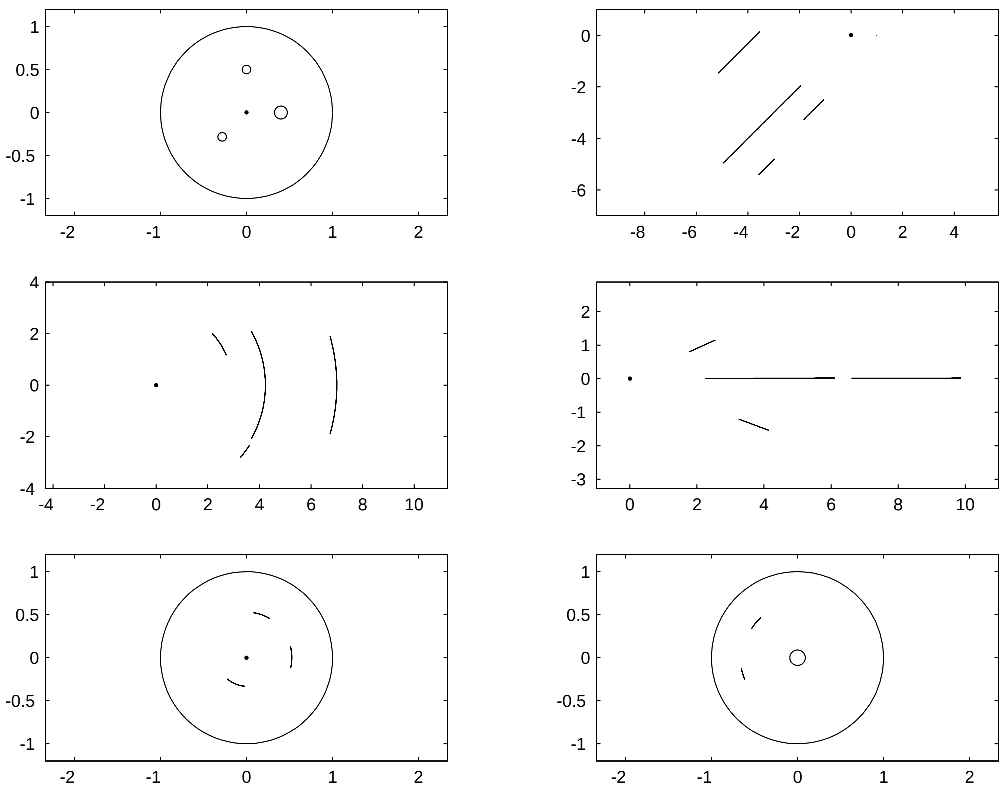

Fig. 4.2: Conformal mappings from a quadruply connected circular domain with parameters $\delta_1 = 0.4, q_1 = 0.075, \delta_2 = 0.5i, q_2 = 0.05, \delta_3 = 0.4e^{5\pi i/4}, q_4 = 0.05$ with parameters $\chi = \pi/4$, $\alpha = -0.15$ and $\beta = 0.1$ to the five types of conformally equivalent slit domains.

CHAPTER 5

Applications to fluid dynamics

In this chapter we present some applications to fluid dynamics of the theory presented in the previous chapters .

5.1 Multiply connected quadrature domains.

In this section we consider a particular class of domains known as quadrature domains. An excellent introduction to quadrature domains with an extensive bibliography on the the subject is given by [41].

The simplest example of a quadrature domain is a disc. Recall the well known result referred to as the "mean value theorem" [1] which states that for a disc D in a z-plane with centre $z = a$ and radius r, for any function $h(z)$ integrable and analytic in D

$$\int\int_D h(z)dxdy = \pi r^2 h(a), \tag{5.1}$$

where $z = x + iy$. Thus the two-dimensional integral on the left-hand side of (5.1) reduces simply to the evaluation of the integrand at a single point. General quadrature domains are characterized by special integration properties such as (5.1) which are referred to as quadrature identities. A *classical* quadrature domain D is a special type of quadrature domain whose quadrature identity is of the form that for any function $h(z)$ integrable and analytic in D,

$$\int\int_D h(z)dxdy = \sum_{k=1}^{M}\sum_{j=0}^{n_k-1} c_{k,j}h^{(j)}(z_k) \tag{5.2}$$

where $\{z_k|k = 1, ..., M\}$ is some set of points in D, $h^{(j)}$ denotes the j-th derivative of h, and $c_{k,j} \in \mathbb{C}$, and $M, n_k \in \mathbb{Z}$. That is, the two-dimensional integral on the left-hand side of (5.2) reduces simply to the evaluation of the integrand and derivatives of the integrand at a finite set of points in D. Thus for example, a disc is a classical quadrature domain.

Quadrature domains are interesting in their own right, but they also happen to be relevant to the mathematical study of a wide range of physical problems, particularly in fluid dynamics. An account of such applications is given by [21]. These include Hele-Shaw flows, free surface Stokes flows, free surface Euler flows, viscous sintering and vortical flows. Indeed, later in this chapter we shall present a new class of equilibrium vorticity distributions of planar flows of ideal inviscid fluids obtained using quadrature domain theory.

All this motivates our interest in quadrature domains. In particular, we consider the problem of finding explicit formulae describing quadrature domains of arbitrary multiple (finite) connectivity N. Most of the existing literature consists only of the cases $N = 1$ and $N = 2$. One way of parameterizing any domain is in terms of a conformal map from a domain of some canonical class of the same connectivity. We choose the canonical class of circular domains which we have so far been studying. It turns out that the conformal map from such a circular domain D_ζ to a quadrature domain D of the same connectivity $N \geq 1$ is a function which is automorphic with respect to the Schottky group Θ constructed from D_ζ as in chapter 1. In chapter 2 we describe how to represent such functions explicitly for any finite N. Thus we may use these formulae to parameterize quadrature domains of *any* finite multiple connectivity. A range of explicit examples are presented to demonstrate the efficacy of this method.

But parameterization in terms of conformal mappings is not the only means of constructing quadrature domains. It turns out that the boundary of a quadrature domain is an algebraic curve [4] and based on this fact Crowdy [16] has presented yet another alternative method. For purposes of comparison we have constructed quadrature domains using conformal maps as well as this algebraic curve method.

5.1.1 Properties of a conformal mapping from a circular domain to a quadrature domain.

Any domain is the image under a conformal map of a circular domain. In this section we shall describe the properties of a conformal map from a circular domain to a quadrature domain. In particular, we shall show that given a circular domain

D_ζ of our chosen form described in chapter 1, with associated Schottky group Θ and a fundamental region F, the image D of D_ζ under a conformal map $z(\zeta)$ which is automorphic with respect to Θ and meromorphic in F, is a quadrature domain.

To begin, consider a general domain D (*not* assumed to be a quadrature domain) of connectivity $(N+1)$ in the z-plane bounded by analytic curves $\{\partial D_i | i = 0, 1, ..., N\}$. Denote the complete boundary of D given by $\cup_{i=0}^{N} \partial D_i$ as ∂D. For $i = 0, 1, ..., N$ label as D_i the region bounded by ∂D_i disjoint from D. We sometimes refer to these regions as the "holes" of D. Now consider D as the image under a conformal map $z(\zeta)$ from our $(N+1)$-connected circular domain D_ζ in a ζ-plane. Suppose that for $i = 0, 1, ..., N$, C_i maps to ∂D_i.

Now consider an integral over D of the form appearing on the left-hand side of (5.2). That is

$$\int\int_D h(z) dx dy, \tag{5.3}$$

where $h(z)$ is integrable and analytic in D. Using Green's theorem, this is equal to

$$\frac{1}{2i} \oint_{\partial D} h(z) \overline{z} dz. \tag{5.4}$$

And we may write this as

$$\frac{1}{2i} \sum_{j=0}^{N} \oint_{\partial D_j} h(z) \overline{z} dz. \tag{5.5}$$

Now note that for any analytic curve C there exists a function $S(z)$ known as its **Schwarz function** which satisfies

$$S(z) = \overline{z}, \tag{5.6}$$

for any point z on C, and which is analytic in a neighbourhood of C ([32]).

For $j = 0, 1, ..., N$ let $S_j(z)$ denote the Schwarz function of ∂D_j. Thus we can write (5.5) as

$$\frac{1}{2i} \sum_{j=0}^{N} \oint_{\partial D_j} h(z) S_j(z) dz. \tag{5.7}$$

Now parameterizing in terms of ζ we have,

$$\overline{z} = \overline{z(\zeta)} = \overline{z}(\overline{\zeta}) \tag{5.8}$$

where we define

$$\overline{z}(\zeta) = \overline{z(\overline{\zeta})} \tag{5.9}$$

Now, for ζ on C_0,

$$\overline{\zeta} = \zeta^{-1}, \tag{5.10}$$

and for ζ on C_j, from (A.7) and (1.4)

$$\overline{\zeta} = \phi_j(\zeta) = \overline{\theta_j}(\zeta^{-1}). \tag{5.11}$$

So, on ∂D_0,

$$\overline{z} = \overline{z}(\zeta^{-1}), \quad \zeta \in C_0 \tag{5.12}$$

And on ∂D_j,

$$\overline{z} = \overline{z}(\overline{\theta}(\zeta^{-1})), \quad \zeta \in C_j \tag{5.13}$$

So we deduce,

$$\begin{aligned} S_0(z) &= \overline{z}(\zeta^{-1}) \\ S_j(z) &= \overline{z}(\overline{\theta_j}(\zeta^{-1})), \qquad j = 1, ..., N \end{aligned} \tag{5.14}$$

where note that for ζ in D_ζ or on its boundary, $\zeta = \zeta(z)$ defines some function of z, which is analytic for z in and on the boundary of D, so that the expressions on the right-hand sides of (5.14) are indeed functions of z.

Now suppose that $z(\zeta)$ is automorphic with respect to the Schottky group Θ. Then for $j = 1, ..., N$,

$$z(\zeta) = z(\theta_j(\zeta)), \tag{5.15}$$

and

$$\begin{aligned} S_j(z) &= \overline{z}(\overline{\theta_j}(\zeta^{-1})) \\ &= \overline{z(\theta_j(\overline{\zeta}^{-1}))} \\ &= \overline{z(\overline{\zeta}^{-1})} \\ &= \overline{z}(\zeta^{-1}) \\ &= S_0(z) \end{aligned} \tag{5.16}$$

So we deduce

$$S_0(z) = S_1(z) = ... = S_N(z) = S(z) \qquad \text{say,} \tag{5.17}$$

where we define

$$S(z) = \overline{z}(\zeta^{-1}). \tag{5.18}$$

So then we can write (5.7) as

$$\frac{1}{2i}\oint_{\partial D} h(z)S(z)dz. \tag{5.19}$$

where ∂D denotes the whole boundary of D.

Now note that $z(\zeta)$ is conformal in D_ζ and thus meromorphic in this domain. But furthermore suppose that $z(\zeta)$ is *also* meromorphic in $\hat{D}_\zeta$, and thus everywhere in the fundamental region F. Then clearly $z(\overline{\zeta}^{-1})$ is meromorphic for $\zeta \in D_\zeta$, since for $\zeta \in D_\zeta$, $\overline{\zeta}^{-1}$ is contained in $\hat{D}_\zeta$. Thus $\overline{z}(\zeta^{-1})$ is meromorphic for $\zeta \in D_\zeta$, and hence $S(z)$ is meromorphic for $z \in D$. But then the integrand $h(z)S(z)$ in (5.19) is analytic in D except for the singularities of $S(z)$ which are just poles. Hence we can simply apply the residue theorem [1] and so (5.19) and hence (5.3) simply reduce to a discrete sum of the form on the right-hand side of (5.2), and so D is a (classical) quadrature domain.

In particular, suppose the poles of $z(\zeta)$ in F are at $\{\alpha_k | k = 1, ..., M\}$. If D is finite then these must all be in $\hat{D}_\zeta$. Otherwise, if D is infinite, all of these except one must be in $\hat{D}_\zeta$. (Note that since $z(\zeta)$ is a conformal map of D_ζ onto D, it must map D_ζ onto D in a one-to-one manner, and thus no more than one pole of $z(\zeta)$ can be contained in D.) Suppose α_k is a pole of $z(\zeta)$ in $\hat{D}_\zeta$. Then $S(z(\zeta))$ has a pole at $\overline{\alpha_k}^{-1}$. So let

$$z_k = z(\overline{\alpha_k}^{-1}) \tag{5.20}$$

Obviously $z_k \in D$. And $S(z)$ has a pole at z_k. Suppose α_k is a pole of $z(\zeta)$ of order n_k (n_k a finite integer ≥ 1). Then it follows that $S(z)$ has a pole of order n_k at the point z_k. In the neighbourhood of the point z_k we have

$$S(z) = \sum_{j=0}^{n_k - 1} s_{k,j}(z - z_k)^{-(j+1)} + \mathcal{O}(1) \tag{5.21}$$

for some $s_{k,j}$. Then the corresponding quadrature identity of D is of the form (5.2) where $c_{k,j} = \pi s_{k,j}$, where $s_{k,j}$ is defined by (5.21).

So to summarize the above, given a circular domain D_ζ of our chosen form, under a conformal map $z(\zeta)$ which is automorphic with respect to the associated Schottky group Θ constructed as in §1.2 and meromorphic in the associated fundamental region F, the image D of D_ζ is a (classical) quadrature domain.

But recall we presented explicit representations for such functions in chapter 2. Thus we may use these to construct quadrature domains of any finite connectivity. Note that in the simply-connected case our circular domain D_ζ reduces simply to the unit disc, and as described in (2.2), these functions are simply rational functions. This agrees with the well-known ([4]) fact that the conformal map from a disc to a simply connected quadrature domain is a rational function.

5.1.2 Examples

We shall now construct some explicit examples of quadrature domains using the method described above. These examples have previously been presented in [23]. Most of them have been considered by Richardson [60] in a physical context, namely singularity-induced Hele-Shaw flows. Singularity-driven flows in a Hele-Shaw cell are flows induced by injecting fluid in or out of the cell. In this context these quadrature domains represent connected regions of fluid formed from circular "blobs" of fluid which have coalesced. These quadrature domains are generalizations of the simple case of a single circular disc. Note that as a numerical check on our work, computations were checked using constructions of the conformal maps in terms of both the prime function and Poincaré theta series as described in §2.2.1 and also using a constructive method proposed by Crowdy [16] which is based on the fact that the boundary of a quadrature domain is an algebraic curve. Good numerical agreement was found in all cases.

Example 5.1.1

The quintuply-connected quadrature domain D in Figure 5.1 is the image under a conformal map $z(\zeta)$ of a circular domain D_ζ in the ζ-plane bounded by the unit circle labelled C_0, and four circles $C_1, .., C_4$ in the interior of C_0 where for $k = 1, ..., 4$ C_k is of radius q centred at $\delta e^{i(\pi/4+(k-1)\pi/2)}$, where δ is real. The conformal map is an automorphic function which written in terms of the prime function is of the form

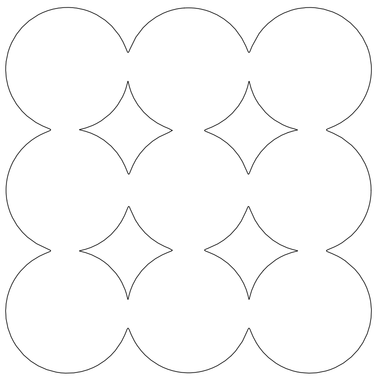

Fig. 5.1: A quintuply connected quadrature domain constructed using the conformal map (5.22) constructed in terms of the prime function.

(2.5). Specifically,

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega_4(\zeta,\beta_1)\omega_4(\zeta,\beta_2)}{\omega'(\zeta,\infty)\omega_4(\zeta,\alpha_1)\omega_4(\zeta,\alpha_2)}. \qquad (5.22)$$

where α_1, β_1 are real, and α_2, β_2 lie on the ray $\arg[\zeta] = \pi/4$. The associated automorphicity conditions on the zeros and poles of the map are of the form (2.6). Note that the associated quadrature identity is of the form (5.2) with $M = 9$ and $n_k = 1$ for $k = 1, ..., M$.

Example 5.1.2

The septuply-connected quadrature domain D in Figure 5.2 is the image under a conformal map $z(\zeta)$ of a circular domain D_ζ in the ζ-plane bounded by the unit circle labelled C_0, and six circles $C_1, .., C_6$ in the interior of C_0 where for $k = 1, ..., 6$ C_k is of radius q centred at $\delta e^{i(\pi/6+(k-1)\pi/3)}$, where δ is real. The conformal map is an automorphic function which in terms of the prime function is of the form (2.5). Specifically,

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega_6(\zeta,\beta_1)}{\omega'(\zeta,\infty)\omega_6(\zeta,\alpha_1)}, \qquad (5.23)$$

where R is a multiplicative constant. The associated automorphicity conditions on the zeros and poles are of the form (2.6). This automorphic function may also be represented in terms of Poincaré theta series. Indeed, Figure 5.2 shows the images of the conformal map represented in the form (5.23) but also in terms of Poincaré theta series. As can be seen, these images are virtually indistinguishable.

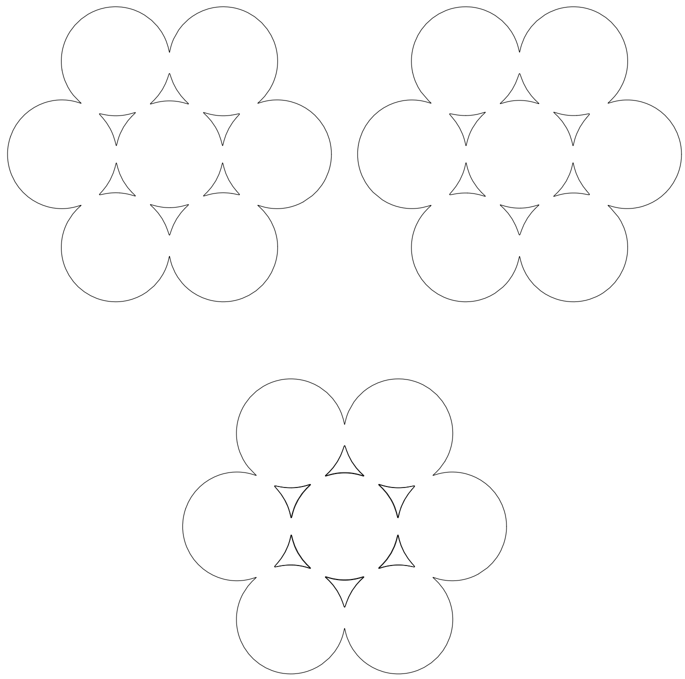

Fig. 5.2: Septuply connected domain constructed using the prime function (top left) and Poincaré theta series (top right) at level 2, together with their superposition (lower).

Example 5.1.3

The quintuply-connected quadrature domain D in Figure 5.3 is the image under a conformal map $z(\zeta)$ of a circular domain D_ζ in the ζ-plane bounded by the unit circle $|\zeta| = 1$ labelled C_0, and four circles $C_1, .., C_4$ in the interior of C_0 where C_1 is also centred on the origin and is of radius q_1, while C_2 is centred at some point on the ray $\arg[\zeta] = \frac{\pi}{3}$ and C_3 and C_4 are the rotations of C_2 through $2\pi/3$ and $4\pi/3$ respectively.

The conformal map $z(\zeta)$ is an automorphic function which in terms of the prime function is of the form (2.15) and whose poles and zeros satisfy relations of the form (2.16) and (2.17). Specifically

$$z(\zeta) = \left(R\zeta \prod_{\theta \in \Theta_1'} \frac{\zeta - \theta(0)}{\zeta - \theta(\infty)} \right) \frac{\omega_3(\zeta, \beta_1)\omega_3(\zeta, \beta_2)}{\omega_3(\zeta, \alpha_1)\omega_3(\zeta, \alpha_2)} \tag{5.24}$$

where

$$\theta_1(\zeta) = q_1^2 \zeta \tag{5.25}$$

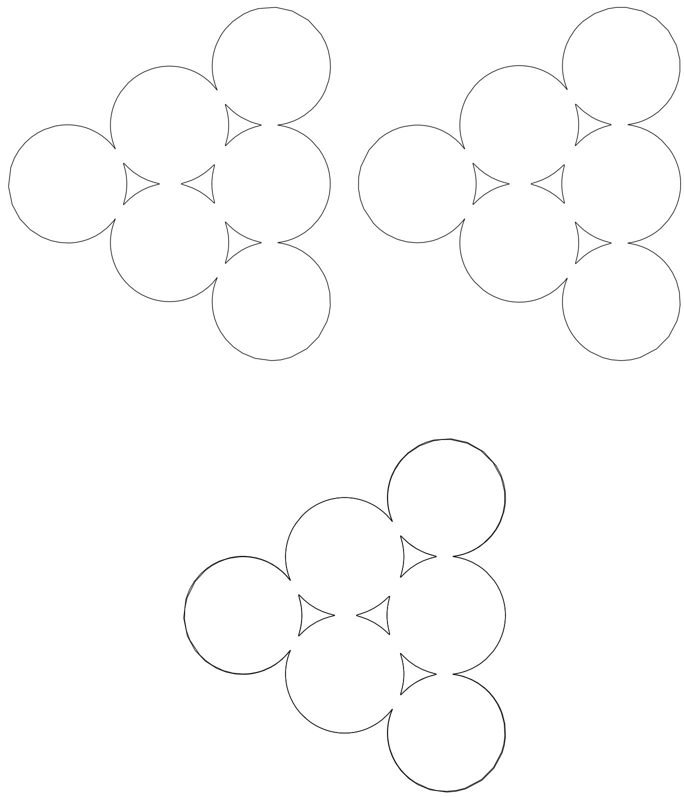

Fig. 5.3: Quintuply connected domain constructed using prime function (top left) and Poincaré theta series (top right) at level 2, with superposition (lower).

is the Möbius map associated with the inner circle C_1 centred on the origin, and R is a multiplicative constant. Also, α_1, β_1 are real, and α_2, β_2 lie on the ray $\arg[\zeta] = \pi/3$.

The image of the conformal map constructed using both conformal mapping methods is shown in Figure 5.3 along with their superposition. Again, the plots are virtually indistinguishable. Note that the quadrature identity associated with such a domain is of the form (5.2) with $M = 6$ and $n_k = 1$ for $k = 1, ..., M$.

Example 5.1.4

The quintuply connected quadrature domain D in Figure 5.4 is the image under a conformal map $z(\zeta)$ of a circular domain D_ζ in the ζ-plane bounded by the unit circle labelled C_0, and four circles $C_1, .., C_4$ in the interior of C_0 where for $k = 1, ..., 4$ C_k is of radius q centred at $\delta e^{i(\pi/4+(k-1)\pi/2)}$, where δ is real. The conformal map is

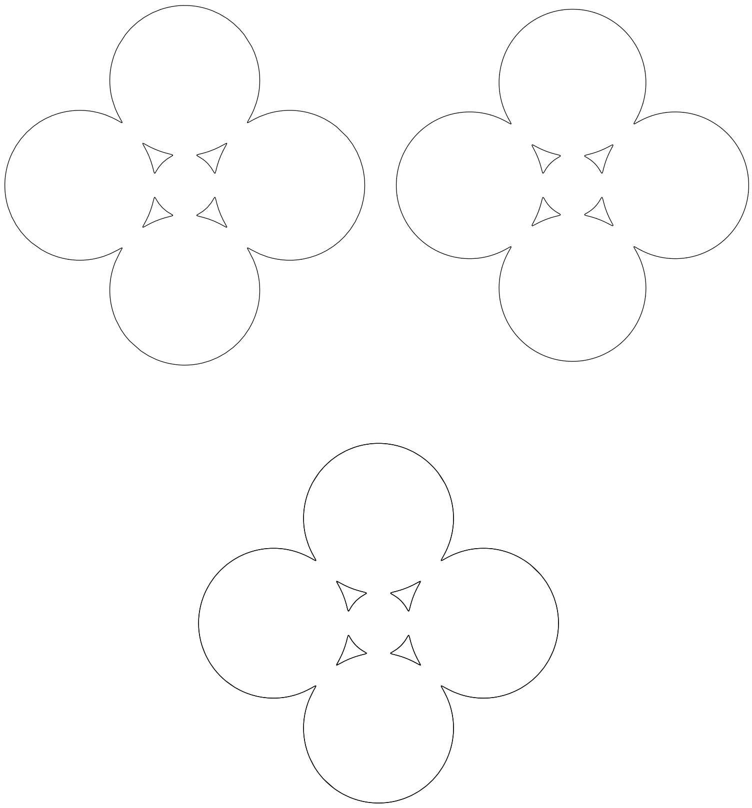

Fig. 5.4: A quintuply connected quadrature domain constructed using conformal maps in terms of the prime function (top left) and using algebraic curves (top right). The superposition is shown in the lower diagram.

an automorphic function which in terms of the prime function is of the form (2.5). Specifically,

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega_4(\zeta,\beta_1)}{\omega'(\zeta,\infty)\omega_4(\zeta,\alpha_1)}. \tag{5.26}$$

where α_1 and β_1 are real. The associated automorphicity conditions are of the form (2.6).

Figure 5.4 shows the quadrature domain constructed as the image under the conformal map (5.26) of the circular domain D, and also the same quadrature domain constructed using the algebraic curve method presented by Crowdy [16]. Again, the plots are virtually indistinguishable.

To summarize, so far in this section §5.1 we have demonstrated that given a circular domain D_ζ of our chosen form, under a conformal map $z(\zeta)$ which is automorphic with respect to an associated Schottky group Θ and meromorphic in an associated fundamental region F, the image of D_ζ is a quadrature domain. In chapter 2 we give explicit representations for such functions, and thus using these we are able to explicitly construct quadrature domains of any finite connectivity.

5.1.3 Vortical solutions of the Euler equations.

One of the problems of fluid dynamics which can be interpreted in terms of quadrature domain theory is the study of equilibrium vortical solutions of the Euler equations for two-dimensional flows of ideal inviscid fluids in a completely unbounded plane. We shall now present explicit formulae describing a new class of such solutions using the results on quadrature domains presented above. These results also appear in [24] and [25].

The simplest vorticity distributions are those consisting purely of point vortices. These are describable in terms of explicit mathematical formulae. The next simplest vorticity distributions are vortex patches by which we mean connected finite-area regions of constant uniform vorticity surrounded by irrotational flow. Few exact vortex patch equilibrium solutions are known. Perhaps the most famous vortex patch solution is the Kirchoff elliptical vortex patch ([50]). More complicated vorticity distributions usually require numerical treatment and cannot be described in terms of explicit mathematical formulae. While computational methods for vortex dynamics are highly sophisticated exact solutions are still important. The new equilibrium vortical solutions we shall present here consist of point vortices and vortex patches and are described by explicit formulae. Essentially, the unbounded region of irrotational flow exterior to the vortex patches is a quadrature domain, and thus may be parameterized in terms of a conformal map as described in §5.1. We point out that these new solutions appear to be the first examples of exact solutions of the Euler equations involving multiple interacting vortex patches.

We briefly remark that such vortical solutions have applications to physical problems such as modelling atmospheric flows (see [51] and [7] for example).

5.1.4 Mathematical formulation.

We consider two-dimensional flows of ideal inviscid fluids. These are of course incompressible and may be described in terms of a streamfunction $\psi(z, \overline{z})$, which is a real-valued function. Suppose at an initial instant we have $N+1$ mutually disjoint patches $\{D_i | i = 0, 1, ..., N\}$ in *pure solid body rotation* with the same vorticity ω, with the flow in the $(N+1)-$connected infinite region D exterior to the patches purely irrotational except possibly for point vortex singularities. Label the boundary of D_i as ∂D_i and the complete boundary of D given by $\cup_{i=0}^{N} \partial D_i$ as ∂D.

At this initial instant the patches are rotating about the origin with angular velocity $\omega/2$. Do there exist such vortical configurations of vortex patches and possibly point vortices which rotate about the origin with constant angular velocity $\omega/2$ for all subsequent time without change of form? Suppose we use a frame of reference which rotates about the origin with this constant angular velocity $\omega/2$. Then in this frame the configuration of patches and any point vortices must be stationary, and furthermore, the velocity field at points inside the patches must be zero.

Suppose that in fact D is initially a quadrature domain. Then for $i = 0, 1, ..., N$ denoting the Schwarz function of ∂D_i by $S_i(z)$ we have

$$S_0(z) \equiv S_1(z) \equiv ... \equiv S_N(z) \equiv S(z) \tag{5.27}$$

for some $S(z)$, where $S(z)$ is meromorphic in D. Suppose that at this initial instant, the streamfunction for this flow in this rotating frame of reference, is given by:

$$\psi(z, \overline{z}) = \begin{cases} 0 & z \in D_i, \quad i = 0, 1, ..., M \\ \frac{\omega}{4}\left(z\overline{z} - \int^{z} S(z')dz' - \int^{\overline{z}} \overline{S}(z')dz'\right) & z \in D \end{cases} \tag{5.28}$$

The associated velocity field is then given by

$$u - iv \equiv 2i\psi_z(z, \overline{z}) = \begin{cases} 0 & z \in D_i, i = 0, 1, ..., N \\ \frac{i\omega}{2}(\overline{z} - S(z)) & z \in D \end{cases} \tag{5.29}$$

Of course the velocity field must be continuous everywhere in the flow, including at points along the boundaries of the vortex patches where there is actually a "jump" in the vorticity field. This means that in this rotating frame of reference, since the flow inside the patches must actually be stagnant, the velocity field must be zero everywhere along the boundaries of the patches. Note that this automatically guarantees that the boundaries of the patches are streamlines of the flow, and so the patches do not change shape or position in this rotating frame. But this requirement is indeed satisfied since recall $S(z) = \overline{z}$ everywhere on the boundary of D.

Furthermore, it is clear from (5.29) that any singularities of $S(z)$ in D correspond to flow singularities in D. Since the only flow singularities in D are the point vortices, then $S(z)$ must be analytic in D except for $j = 1, ..., M$, for z near z_j we must have

$$u - iv = -\frac{i\Gamma_j}{2\pi}\frac{1}{z - z_j} + \text{terms analytic at } z_j \tag{5.30}$$

All that remains is to ensure that in the rotating frame of reference, for $j = 1, ..., M$ the non-self-induced velocity of the point vortex at z_j is zero so that it is stationary. This leads to M conditions to be satisfied. We shall refer to these conditions as the **stationarity conditions** on the point vortices. These stationarity conditions are stated explicitly in appendix G.

Then in the rotating frame of reference the arrangement of vortex patches and point vortices will remain fixed. Thus, in the "laboratory" frame of reference, the arrangement will rotate about the origin with constant angular velocity $\omega/2$ without change of form. Thus these are equilibrium solutions of the Euler equation, and we can write down *explicit* formulae describing them. Formulae for the circulations of the vortex patches and point vortices can also be written down. These are given in Appendix H.

Briefly, let us now reinterpret the Rankine vortex solution. This solution consists of a single vortex patch D_1 which is a disc, say centre the origin and radius a, which is in pure solid body rotation with vorticity ω, with the flow in the region D exterior to D_1 completely irrotational and no point vortices. This is known to be a equilibrium solution, with D_1 rotating about the origin with constant angular velocity $\omega/2$. Now note that it is well-known that the exterior of a circle (in fact any ellipse) is a simply-connected quadrature domain. The Schwarz function of the circle is $S(z) = a^2 z^{-1}$. Notice this has no singularities in the region D. Thus the Rankine vortex is of precisely this class of vortical solutions just described.

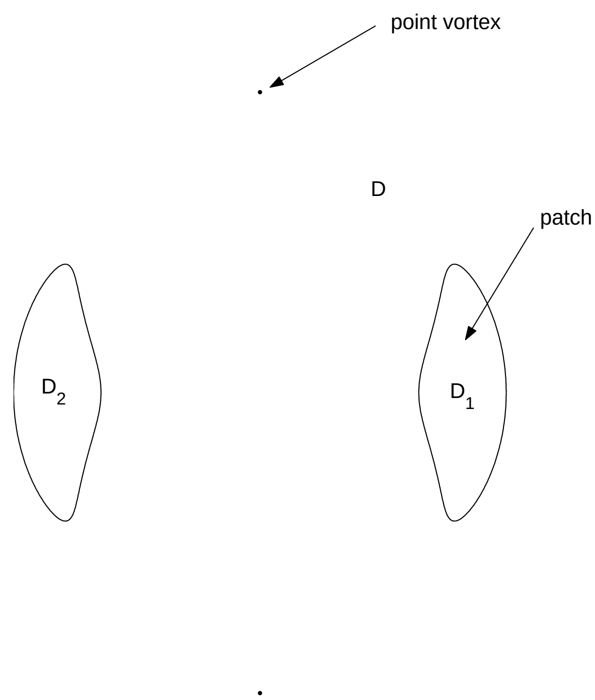

Fig. 5.5: Schematic illustrating the vortex configuration of interest consisting of two point vortices at $\pm i$ and two rotationally-symmetric uniform vortex patches.

Note that Crowdy [19] has already constructed solutions of this form with a single vortex patch so that the infinite region D is simply connected, but unlike the Rankine vortex, this patch is surrounded by a distribution of satellite point vortices.

Let us now present some explicit examples of solutions of the form just described.

5.1.5 Solutions with two patches.

The explicit formulae we present in this section for a class of solutions of the form described above are also presented in [24], and the reader is referred here for further discussion of these.

Consider at an initial instant, a rotationally symmetric configuration as shown schematically in Figure 5.5 consisting of two point vortices at $z_1 = i, z_2 = -i$, together with two vortex patches D_1 and D_2 whose centroids lie on the real axis. In this case D is doubly connected so we take our circular domain D_ζ to be an annulus $\rho < |\zeta| < 1$, with $|\zeta| = 1$ mapping onto ∂D_1 and $|\zeta| = \rho$ mapping onto ∂D_2. The form for $z(\zeta)$ is

$$z(\zeta) = R\frac{P(-\zeta\sqrt{\rho}^{-1}, \rho)P(-\zeta\sqrt{\rho}, \rho)P(\zeta\sqrt{\rho}, \rho)}{P(\zeta\sqrt{\rho}^{-1}, \rho)P(\zeta\sqrt{\rho}e^{i\phi}, \rho)P(\zeta\sqrt{\rho}e^{-i\phi}, \rho)} \tag{5.31}$$

where ϕ and R are parameters. A derivation of this map is given in appendix G. Observe that (5.31) is an automorphic function of the form (2.18). The associated automorphicity condition is of the form (2.19), and indeed it can be easily checked that the poles and zeros of (5.31) satisfy this, in fact for any choice of R,ρ and ϕ. Thus for *any* choice of R,ρ and ϕ, (5.31) represents a function automorphic with respect to Θ. The only conditions on the parameters R, ρ and ϕ are that in the rotating frame of reference the point vortices are at $\pm i$ and are stationary. For a fixed ρ these two conditions force the values of ϕ and R. We have found that solutions exist for ρ in the interval

$$\rho \in [0, \rho_{crit} = 0.735] \tag{5.32}$$

Hence there exists a continuous one-parameter family of equilibria. In the limit $\rho \to 0$, it is found that the two vortex patches become invisibly small and essentially disappear, leaving just the two point vortices. Thus these solutions can be regarded as continuations of the classical co-rotating point vortex pair. As ρ increases, so does the size of each vortex patch. As $\rho \to \rho_{crit}$ so the configuration approaches a limiting state in which the two vortex patches touch at three distinct points, the points of contact taking the form of three cusp singularities in the patch boundaries. At $\rho = \rho_{crit}$, to within numerical accuracy, it is found that the patches touch and enclose two exactly circular regions of irrotational fluid with the point vortices located at their centres. To test this, Figure 5.6 shows the critical configuration superposed with the three circles $|z \pm i| = 1$ and $|z| = 2$. The boundaries of the limiting configuration are indistinguishable from these three circles.

This feature of the limiting two-patch solutions suggests that the class of solutions can be continued even past this limiting state as follows.

Point vortices and vortex patches are related. The circulation of a patch of uniform vorticity is equal to its area multiplied by its vorticity. A point vortex may be regarded as the limit of a patch of uniform vorticity of the same circulation as the area of the patch shrinks to zero while its vorticity tends to infinity in such a way

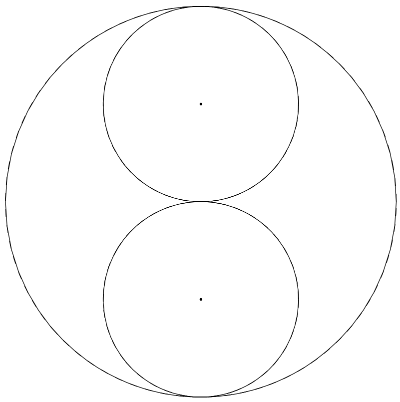

Fig. 5.6: Critical configuration for two satellite patches and two satellite point vortices for $\rho = \rho_{crit} = 0.735$ (drawn in solid lines) shown with the three circles $|z \pm i| = 1$ and $|z| = 2$ superposed (drawn with dashed lines). The solid and dashed curves are indistinguishable.

that its circulation remains fixed. Conversely, we may regard a vortex patch as a point vortex of the same circulation whose vorticity has been "smeared out" over a non-zero area. This is sometimes referred to as "desingularizing", or "regularizing" the point vortex. In fact, some equilibrium vorticity distributions involving point vortices and vortex patches can be continued to different equilibria by continuing point vortices to vortex patches or vortex patches to point vortices. However, even if an equilibrium vorticity distribution involving point vortices is describable in explicit form, any new equilibria found from it by desingularizing point vortices cannot usually be described explicitly and is normally only obtainable numerically. This is because it is not always straightforward to see into what shape one should "smear" the point vortices.

It is conceivable that any of the equilibria of the family of solutions found here could be continued to different equilibria by desingularizing the point vortices. As stated, these new equilibria would probably only be obtainable numerically and not in explicit form. However, in the particular case of the critical equilibria, the streamlines around each of the point vortices are exactly circular. Furthermore numerical computations (-see [24]) show that in this limiting configuration, the circulations of each point vortex and each vortex patch become equal, in fact to π. Thus in this case, replacing each point vortex by a Rankine vortex patch of the same circulation π the resulting vorticity distribution is an equilibrium, and in fact one which may be described *explicitly* since the boundaries and vorticities of *all* the patches are known. If the radius of one of these Rankine vortices is r then obviously since its circulation must be π, we deduce it must have vorticity $\omega(r)$ where

$$\omega_r = \frac{1}{r^2}. \tag{5.33}$$

Figure 5.7 shows several configurations for different values of ρ. The seventh diagram in Figure 5.7 shows the configuration where for both Rankine vortices $r = 0.4$. r may of course take any value from 0 up to a maximum of 1 which is when the boundaries of the Rankine vortex meets the circular boundaries of the satellite patches. When $r = 1$, the enclosed Rankine vortex has vorticity 1. But this is of course also the vorticity of the satellite vortex patches. Hence, if both Rankine vortices have radius 1, then the equilibrium is essentially a *single* circular Rankine vortex of radius 2, uniform vorticity $\omega = 1$ and total circulation 4π. This is shown in the eighth diagram in Figure 5.7.

So we have in fact shown that two classical equilibrium solutions of vortex dynamics, that is the co-rotating point vortex pair and the Rankine vortex patch, are in fact connected by a continuous branch of non-trivial vortex equilibria, and moreover, the entire branch of solutions is describable in *explicit* mathematical form (although of course to obtain the intermediate solutions consisting of two point vortices and two vortex patches, the parameters appearing in these formulae must be determined numerically).

5.1.6 Solutions with more than two patches.

We can construct similar vortical solutions with more than two patches. The explicit formulae we present in this section for a class of solutions of this form described above are also presented in [25], and the reader is referred here for further discussion of these.

Consider an N-fold rotationally symmetric configuration consisting of N point vortices and N vortex patches $\{D_j | j = 1, .., N\}$ surrounding an additional central vortex patch D_0. In particular, in the co-rotating frame, the positions of the point vortices

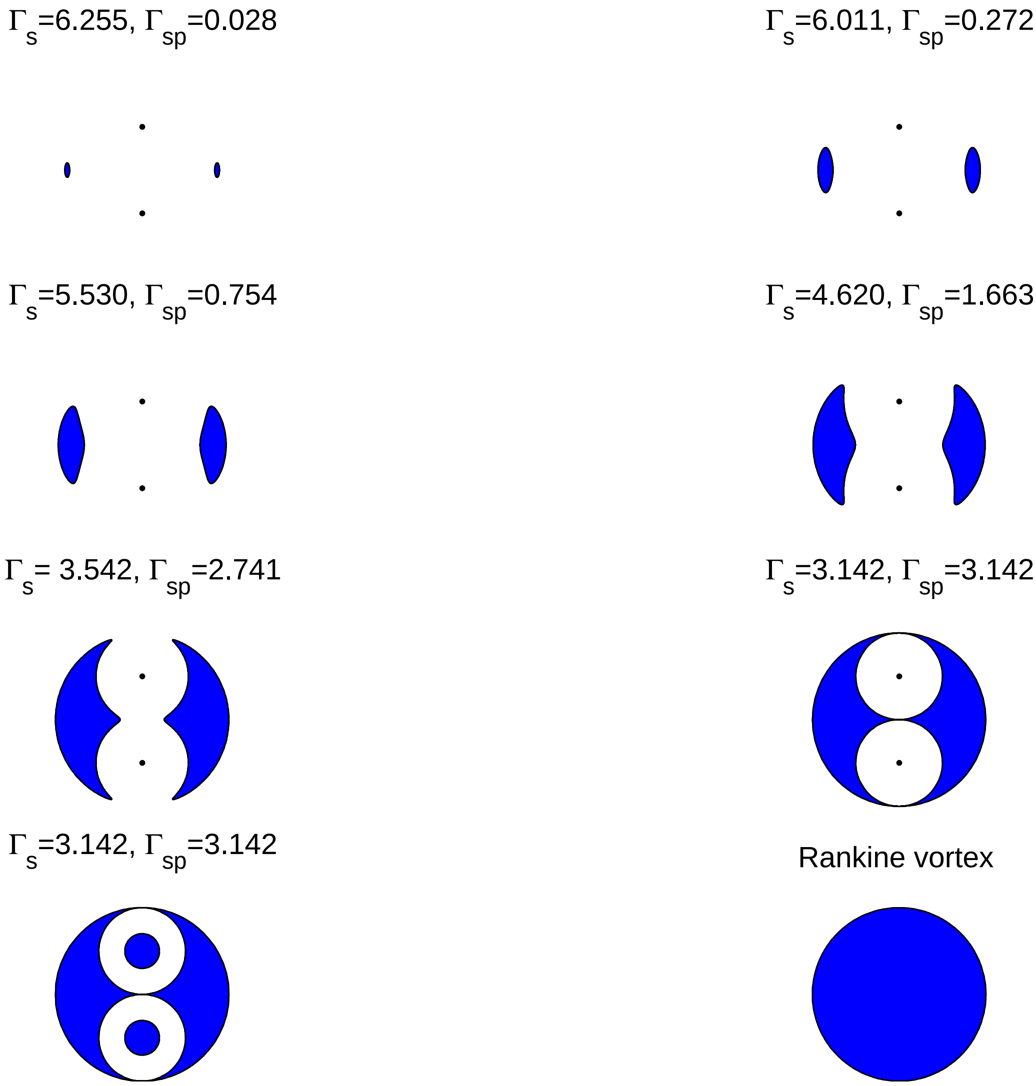

Fig. 5.7: A continuous branch of rotating vortex arrays connecting the co-rotating point-vortex pair to the Rankine vortex. The sixth figure is the limiting state of the solution where $\rho = \rho_{crit}$ and where the two patches develop cusps and touch at 3 distinct points enclosing circular irrotational regions centred on the point vortices. The last two figures illustrate the desingularization of the point vortices to form a single Rankine vortex. The circulations Γ_s and Γ_{sp} of respectively the satellite point vortices and vortex patches are also shown.

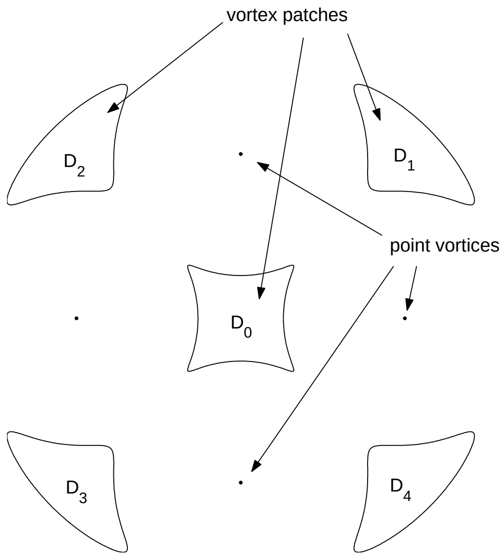

Fig. 5.8: Schematic showing the vortex configurations under consideration in the case $N = 4$. A central vortex patch is surrounded by an alternating distribution of N point vortices and N satellite vortex patches. The central vortex patch is D_0, the N satellite patches are $\{D_j | j = 1, .., N\}$.

are $\{z_j | j = 1, ..., N\}$ where

$$z_j = e^{2\pi(j-1)/N}, \quad j = 1, .., N, \tag{5.34}$$

while the centroid of D_0 is at the origin and the for $j = 1, .., N$ the centroid of D_j is on the ray

$$\arg[z] = \pi(1 + 2(j-1))/N. \tag{5.35}$$

Figure 5.8 shows a schematic in the case $N = 4$. Note that explicit formulae for similar vortical solutions *without* a central vortex patch have been derived in [16].

We parameterize the unbounded $N + 1$-connected region D exterior to the patches in terms of a conformal map $z(\zeta)$ from a circular domain D_ζ bounded by the unit circle $|\zeta| = 1$ labelled C_0 and N inner circles which we label $\{C_j | j = 1, ..., N\}$, where for $j = 0, 1, ..., N$, C_j maps onto ∂D_j. For $j = 1, ..., N$ let δ_j and q_j respectively denote the centre and radius of the circle C_j. We take

$$\delta_j = \delta e^{(1+2(j-1))i\pi/N}, \quad q_j = q, \quad \text{for } j = 1, .., N. \tag{5.36}$$

for some real parameters δ and q to be determined.

We take $z(\zeta)$ to be of the form

$$z(\zeta) = R\frac{\omega'(\zeta,\infty)\omega_N(\zeta,\beta)}{\omega(\zeta,0)\omega_N(\zeta,\alpha)} \tag{5.37}$$

where α, β are real parameters and R is a multiplicative constant, and where $\omega_N(\zeta,\gamma)$ is a product of N prime functions given by

$$\omega_N(\zeta,\gamma) = \prod_{j=1}^{N} \omega(\zeta, \gamma e^{2\pi i(j-1)/N}). \tag{5.38}$$

and where recall we define $\omega'(\zeta,\infty)$ by (1.36). This map has poles and zeros in F at the points $\{\alpha_j | j = 1, ..., N+1\}$ and $\{\beta_j | j = 1, ..., N\}$ respectively. Note that we require $\alpha > 1$, $\beta > 1$ so that these poles and zeros are contained in $\hat{D}_\zeta$ rather than D_ζ. $z(\zeta)$ also has a pole at $\zeta = 0$ and a zero at $\zeta = \infty$.

The map (5.37) depends on the parameters

$$q,\ \delta,\ \alpha,\ \beta,\ R. \tag{5.39}$$

First note that of course we must guarantee that $z(\zeta)$ is automorphic with respect the the Schottky group Θ. Recalling (2.9), note that the poles and zeros of an automorphic function of the form (5.37) must satisfy conditions of the form (2.10). In this case, due to symmetry, these conditions are in fact all the same. Also, we specify that the satellite point vortices are unit distance from the centroid of the central vortex patch. We need only impose this normalization at one of the point vortices since by the form of the map (5.37) and the symmetry of D_ζ the other point vortices are symmetrically placed. Also we must satisfy the stationarity conditions on the point vortices. Due to the symmetry of the vorticity distribution, if in the co-rotating frame one point vortex is stationary then the others will necessarily be too, and the stationarity conditions on the point vortices are all the same.

These three conditions are the only conditions we must impose. So we are free to pick two of the five parameters (5.39), and then the remaining three are determined by these three conditions.

Vortical solutions of this form have been found to exist for all integers $N \geq 3$ (note that the solutions of [16] were only found to exist for $N \geq 3$).

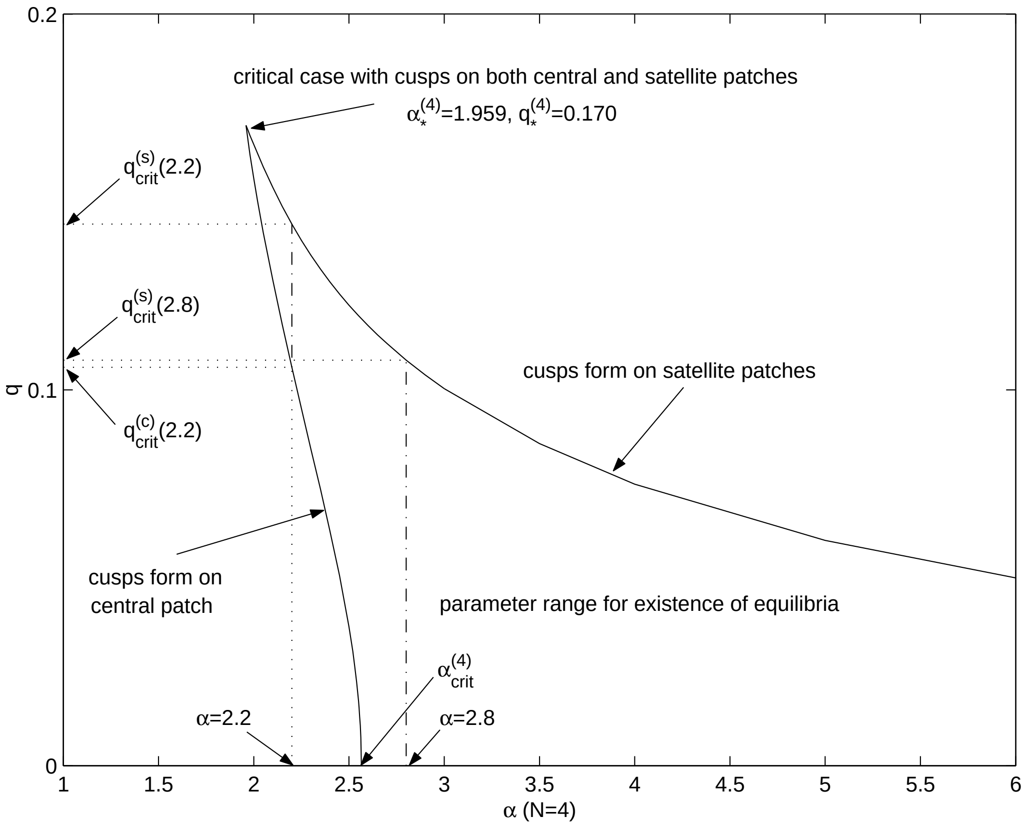

Fig. 5.9: (α, q)–parameter space indicating region where equilibria exist in the case $N = 4$. The vertical dot-dash line at $\alpha = 2.8$ shows that solutions exist for $q \in [0, q^{(s)}_{crit}(2.8)]$. Typical configurations along this dot-dash line are shown in Figure 5.10. The vertical dot-dash line at $\alpha = 2.2$ shows that solutions exist for $q \in [q^{(c)}_{crit}(2.2), q^{(s)}_{crit}(2.2)]$. Typical configurations along this dotted line are shown in Figure 5.11. Also clearly marked is the intersection point of the two branches of limiting solution where the limiting state exhibits cusps in the boundaries of both the central and satellite patches. This critical configuration is shown in Figure 5.12.

Roughly speaking the parameter q governs the size of the satellite vortex patches. In the range of parameter values for which solutions exist it is found that as q increases so the area of the satellite patches increases. When $q = 0$ so the satellite vortex patches disappear, and the solutions reduce exactly to solutions already obtained by Crowdy [16]. Thus these new solutions can be obtained as continuations of those found by Crowdy [16].

For clarity, we shall now present the solutions for $N = 4$ in detail. Figure 5.9 shows the (α, q)-parameter space for which equilibria have been found to exist.

When $q = 0$, the solutions reduce to solutions of [16] with no satellite patches. In this case, in [16] it is shown that in the limit $\alpha \to \infty$, the central patch becomes vanishingly small, while as α decreases, the size of the central patch generally gets bigger until α reaches a minimum possible value at which the central patch has developed cusp singularities in its boundary. This value is found in [16] to be $\alpha_{crit}^{(4)} = 2.565$. So for $q = 0$, equilibria exist for α in the interval

$$\alpha \in [\alpha_{crit}^{(4)}, \infty). \tag{5.40}$$

For fixed $\alpha > \alpha_{crit}^{(4)}$, as q increases from 0 the area of the satellite patches increases. In general, it is found that equilibria exist for q values below some critical value where the limiting vortex configuration exhibits cusp singularities on the satellite patches. This critical value of q, which is a function of α, is denoted $q_{crit}^{(s)}(\alpha)$. So, for fixed $\alpha > \alpha_{crit}^{(4)}$, it is found that equilibria exist for q in the interval

$$q \in [0, q_{crit}^{(s)}(\alpha)]. \tag{5.41}$$

Figure 5.10 shows vortex configurations for such a value $\alpha = 2.8$ for different values of q in the interval $q \in [0, q_{crit}^{(s)}(2.8)]$.

It is found that equilibria exist for α less than $\alpha_{crit}^{(4)}$ for $q > 0$. Indeed, they exist in a range which is denoted

$$\alpha \in [\alpha_*^{(4)}, \alpha_{crit}^{(4)}]. \tag{5.42}$$

For α in this range it is found that, while no solutions exist for q just above zero, there is nevertheless a window of q-values for which solutions do exist. The limiting solutions are found to exhibit cusp singularities in the boundaries of either the central or the satellite vortex patches. For a fixed α in the range (5.42), this window of admissible q values (which depend on α) will be denoted

$$q \in [q_{crit}^{(c)}(\alpha), q_{crit}^{(s)}(\alpha)]. \tag{5.43}$$

It is found that the lowest admissible value of q yields a limiting state where the central patch exhibits four cusp singularities in its boundary while for the highest admissible value of q, the four satellite vortices exhibit boundary cusps. Figure 5.11 shows a range of vortex configurations for such a value $\alpha = 2.2$ for different values of q in the interval $q \in [q_{crit}^{(c)}(2.2), q_{crit}^{(s)}(2.2)]$. This range of q values is clearly indicated by a dotted vertical line (the line $\alpha = 2.2$) in Figure 5.9 intersecting the region of parameter space where solutions exist.

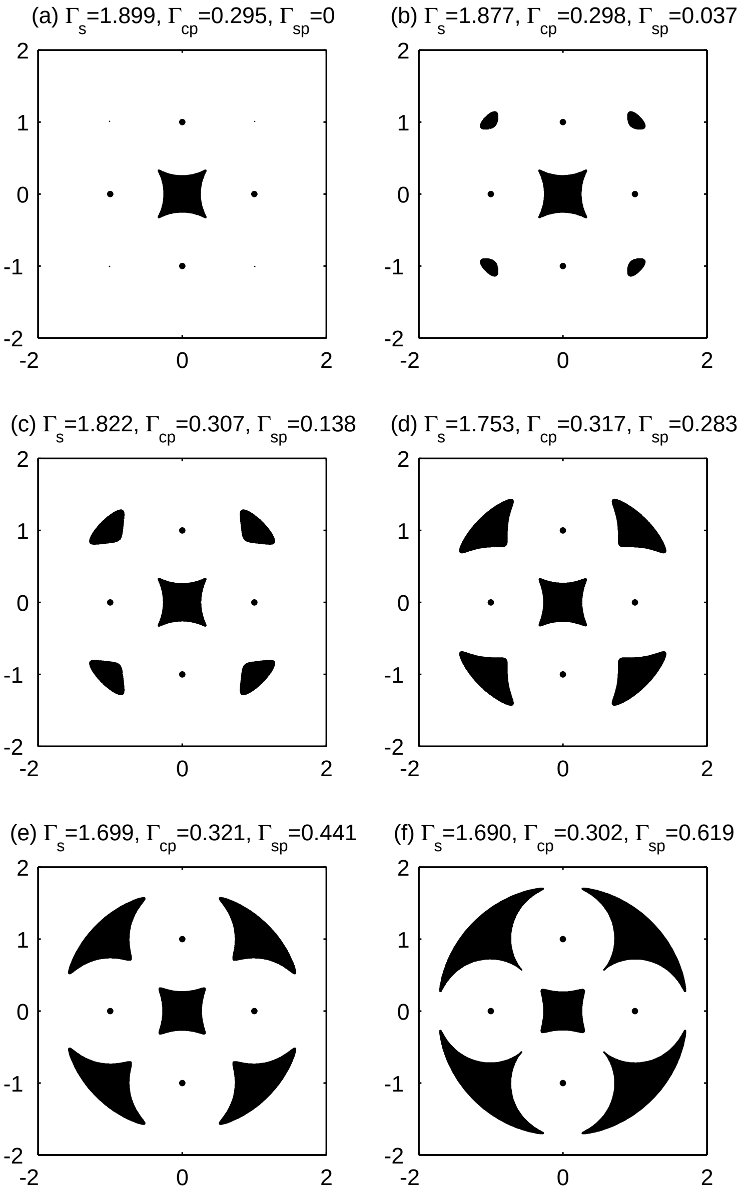

Fig. 5.10: Vortical equilibria for $\alpha = 2.8$ and $q = 0, 0.02, 0.04, 0.06, 0.08$ and $q = q_{crit}^{(s)} = 0.108$. The circulations of the point vortices Γ_s, the central patch Γ_{cp} and satellite patches Γ_{sp} are shown. Figure (a) shows a solution found in [16]. Figure (f) is the limiting state where four cusps form simultaneously on the satellite patches at the points closest to the central patch.

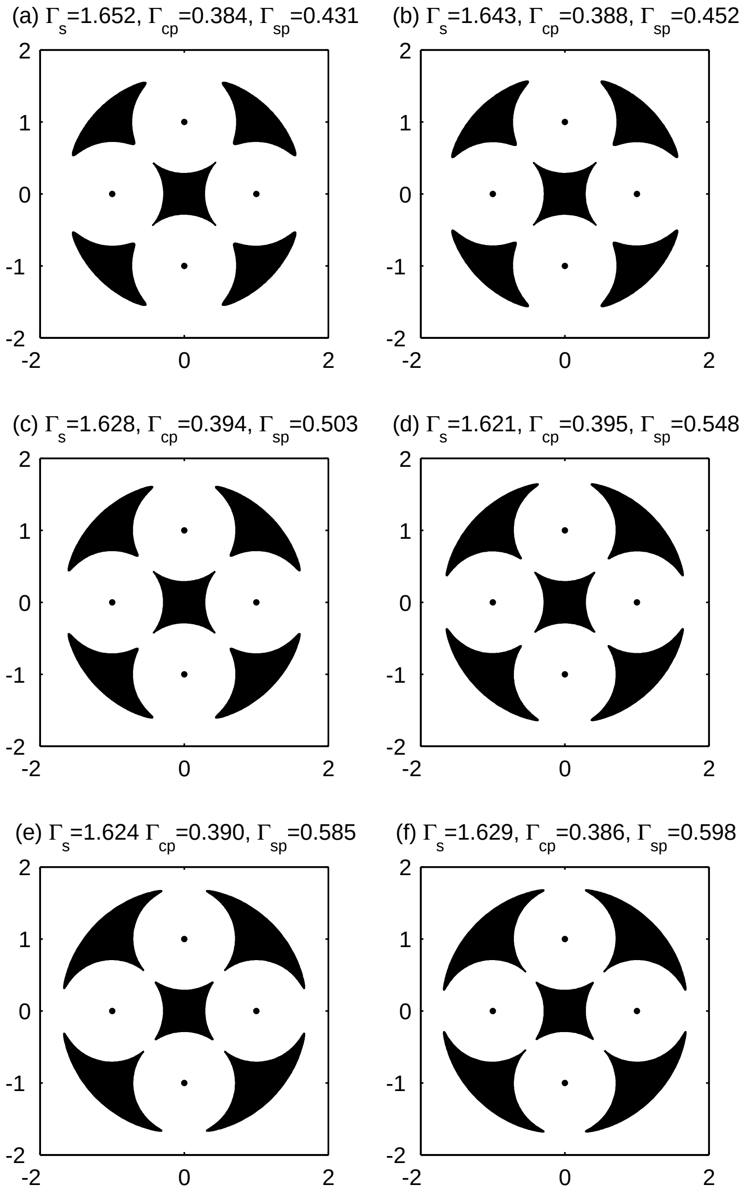

Fig. 5.11: Vortical equilibria for $\alpha = 2.2$ and $q = q_{crit}^{(c)} = 0.106,\ 0.11, 0.12, 0.13, 0.14$ and $q_{crit}^{(s)} = 0.144$. The circulations of the point vortices Γ_s, the central patch Γ_{cp} and satellite patches Γ_{sp} are shown. In Figure (a), the limiting configuration exhibits four symmetric cusps in the boundaries of the central patch while in Figure (f) the limiting configuration exhibits cusps in each of the four satellite patches at the points closest to the central patch.

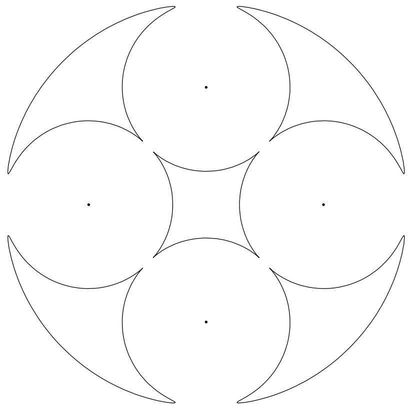

Fig. 5.12: Critical configuration, exhibiting the simultaneous formation of cusps on both the central and satellite patches, corresponding to parameters $N = 4$, $\alpha = \alpha_*^{(4)} = 1.959, q = q_*^{(4)} = 0.170$. The cusps do not touch.

There exists a critical choice of parameters $\alpha = \alpha_*^{(4)}$ and $q = q_*^{(4)}$ at which the two solution branches just described meet. In the corresponding critical state, cusps form simultaneously on both the central and satellite vortex patches. This point in the (α, q)-parameter space is clearly marked on Figure 5.9. The critical configuration is shown in Figure 5.12. The cusps which form on the central and satellite patches in this limit do not touch each other.

The solutions for $N = 4$ have been described in detail but the features of solutions for other values of $N \geq 3$ are qualitatively similar.

Finally, note that numerical checks have been carried out ([24],[25]) to test these solutions presented in §5.1.5 and §5.1.6 are indeed equilibrium solutions. A detailed analysis of the stability of these solutions remains to be performed, but preliminary investigations ([24],[25]) suggest that they are stable.

5.2 Vortex motion in bounded domains.

As an application of the explicit formulae for Green's functions of multiply connected domains presented in chapter 3, we now consider the problem of describing the motion of point vortices in general multiply connected domains. The results we shall present also appear in [26], [27] and [28], and the reader is referred to these references for further discussion.

The motion of point vortices in a completely unbounded plane is fairly straightforward to describe. It is well-known ([61]) that the equations of motion of the point vortices form a Hamiltonian system, where an explicit form for the Hamiltonian is known. Furthermore, given the instantaneous positions of the point vortices, it is straightforward to write down a streamfunction which describes the overall flow. However, a basic question to ask is what is the effect on the motion of the point vortices if a solid boundary is introduced into the plane?

The problem of finding explicit formulae describing the motion of an arbitrary number of point vortices in an arbitrary domain with boundaries is one which has previously only been resolved in very simple cases. The simplest example is of a single point vortex in a half-plane bounded by an infinite straight wall. It is well-known that the effect of the wall is to cause the point vortex to move at constant speed in a direction parallel to the wall. Another very simple example is the motion of a point vortex in the region exterior to a circular cylinder. In both these cases it is straightforward to write down explicit forms for the streamfunctions of the flow. One method of arriving at these forms is by the "method of images". In these simple cases the distribution of "image" vortices is straightforward. However, this is not so for domains with boundaries of more complex geometry and so in these cases a more sophisticated approach is required.

It turns out that the motion of point vortices is also Hamiltonian in bounded domains and may be described in terms of what is known as the Kirchoff-Routh path function (henceforth referred to as the path function) which is essentially the Hamiltonian of the motion. The motion of a single vortex in bounded simply connected domains is relatively well-studied. A number of examples involving simply connected fluid regions are given in Chapter 3 of [56] and in [61]. However, for domains of connectivity $N > 1$ very few results are known. Among these we mention recent results obtained by Johnson and McDonald [46], [47], [48] for the motion of vortices in a domain exterior to two circular cylinders and also in domains bounded by an infinite straight wall with gaps in it. They use elliptic function theory to derive these results.

We shall consider the general problem of the motion of an arbitrary number of point vortices in an arbitrary bounded domain. Making use of results derived by Lin [44], [45] we construct explicit formulae for the path function describing the motion of the point vortices.

Finally, we point out that the study of point vortex motion in bounded domains has applications to physical problems. For example, oceanographers are interested in understanding the motion of ocean eddies in the presence of coastal barriers (see for example [33], [57], [64], [65]), and point vortices in planar domains provide basic models for studying these ocean flows.

5.2.1 Mathematical formulation

The flows we shall be considering are two-dimensional flows of ideal inviscid fluids in a multiply connected domain D with fixed boundaries in a z-plane. Such flows obey the Euler equations. Furthermore we suppose that the flows are irrotational except for point vortices. We may describe such flows in terms of a streamfunction $\psi(z, \overline{z})$ or equivalently a complex potential $W(z)$. $\psi(z, \overline{z})$ is a real-valued function which is harmonic in D except for logarithmic singularities at the point vortices. $W(z)$ is the analytic extension of $\psi(z, \overline{z})$ such that

$$W(z) = \phi(z, \bar{z}) + i\psi(z, \bar{z}) \tag{5.44}$$

where $\phi(z, \bar{z})$ is the associated velocity potential. $W(z)$ is analytic (though note not necessarily single-valued) everywhere in D except for logarithmic singularities at the point vortices.

Note also that for a boundary C in the flow domain, introducing the usual coordinates n and s at each point z on C where these are directed respectively in the directions of the normal and tangent of C at z, then the circulation round C in terms of the streamfunction is given by

$$\oint_C \frac{\partial \psi}{\partial n} ds \tag{5.45}$$

or equivalently,

$$\mathrm{Re}[[W(z)]_C] \tag{5.46}$$

where $[W(z)]_C$ denotes the change in $W(z)$ on traversing C.

5.2.2 The path function

As stated above, the motion of any number of point vortices in a unbounded plane is Hamiltonian and can be described in terms of a path function. But in fact Lin [44] established that the motion of any number of point vortices in a domain with any number of boundaries is also Hamiltonian and there exists a generalized path function in terms of which one may write the equations of motion of the point vortices. Specifically, if there are M point vortices of strengths $\{\Gamma_k | k = 1, ..., M\}$ and respective positions $\{z_k, k = 1, ..., M\}$ in a domain D with fixed boundaries then there exists a path function $H(\{z_k, \overline{z_k}\})$ such that

$$\Gamma_k \frac{dx_k}{dt} = \frac{\partial H}{\partial y_k}, \quad \Gamma_k \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}, \tag{5.47}$$

where $z_k = x_k + y_k$. Note that in rescaled coordinates $(\sqrt{\Gamma_k} x_k, \sqrt{\Gamma_k} y_k)$, (5.47) is a Hamiltonian system in canonical form.

5.2.3 Transformation properties.

Suppose $z(\zeta)$ is a one-to-one conformal map of a given region D_ζ in a ζ-plane onto a region D in a z-plane. Suppose that there are point vortices of strengths $\{\Gamma_k | k = 1, ..., M\}$ in D_ζ at points $\{\zeta_k, k = 1, ..., M\}$ and in D at the points $\{z_k, k = 1, ..., M\}$, where for $k = 1, ..., M$, $z_k = z(\zeta_k)$. Then denoting the complex potentials of the flows due to these point vortices in D_ζ and D as $W^{(\zeta)}(\zeta)$ and $W^z(z)$ respectively, it is straightforward to show that

$$W^{(z)}(z) = W^{(\zeta)}(\zeta(z)) \tag{5.48}$$

where note that the one-to-one conformal map $z(\zeta)$ can in principle be inverted so that ζ may be written in terms of z.

Note additionally that recalling (5.46) it follows from (5.48) that the circulation round a boundary ∂D_j of D is the same as the circulation round the boundary C_j of D_ζ which maps onto ∂D_j under $z(\zeta)$.

What about the transformation properties of the associated path functions? If $H^{(\zeta)}$ and $H^{(z)}$ respectively denote the path functions for the point vortices in D_ζ and D_z, Lin [45] showed that these are in fact related by the formula

$$H^{(z)}(z_1, ..., z_M) = H^{(\zeta)}(\zeta_1, ..., \zeta_M) + \sum_{k=1}^{M} \frac{\Gamma_k^2}{4\pi} \log |z_\zeta(\zeta_k)| \tag{5.49}$$

Hence, if we can find an explicit formula for the path function for some canonical class of multiply connected domains then explicit expressions for the path functions in all domains follow up to conformal mapping. Note that even when $z(\zeta)$ is not known analytically (and must be computed numerically, for example), this function is independent of the instantaneous point vortex positions and can be computed once and for all at the start of any calculation (assuming the boundaries of the flow domain are not changing in time) and so we may still use (5.49) to compute the associated path function. We shall construct explicit formulae for the path functions for the canonical class of circular domains.

5.2.4 The path function for circular domains.

Lin [44] introduced a special Green's function $\mathcal{G}(\zeta; \alpha)$ with respect to the two points ζ and α in a general domain D with fixed boundaries. $\mathcal{G}(\zeta; \alpha)$ is the streamfunction of the flow induced by a point vortex of unit strength at the point $\zeta = \alpha$ in an incompressible fluid in D. Flucher and Gustafsson [39] refer to Lin's special Green's function as the *hydrodynamic* Green's function and we will adopt this terminology. Lin [44] showed that the path function for point vortex motion D can be expressed in terms of the hydrodynamic Green's function of D.

Suppose now D_ζ is a $N+1$ connected circular domain of the usual form we have considered in previous chapters, i.e. bounded by the circle $|\zeta| = 1$ which we denote C_0 and N other circles $\{C_i | i = 1, ..., N\}$ which lie in the interior of C_0. In the case where the flow is such that the circulations round the inner boundaries $\{C_i | i = 1, ..., N\}$ of D_ζ are *zero*, the hydrodynamic Green's function is in fact given by

$$\mathcal{G}(\zeta; \alpha) = \frac{1}{2\pi} G_0(\zeta; \alpha) \tag{5.50}$$

where $G_0(\zeta; \alpha)$ is the modified Green's function presented in chapter 3 and which is given explicitly by (3.39). Thus an explicit representation for $\mathcal{G}(\zeta; \alpha)$ is

$$\mathcal{G}(\zeta; \alpha) = -\frac{1}{4\pi} \log \left| \frac{\omega(\zeta, \alpha)\overline{\omega}(\zeta^{-1}, \alpha^{-1})}{\omega(\zeta, \overline{\alpha}^{-1})\overline{\omega}(\zeta^{-1}, \overline{\alpha})} \right| . \tag{5.51}$$

So, an explicit expression for the associated complex potential $W(\zeta;\alpha)$ is

$$W(\zeta;\alpha) = -\frac{i}{4\pi}\log\left(\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\overline{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\overline{\alpha})}\right). \tag{5.52}$$

Note that on use of (3.40), it is also possible to write $\mathcal{G}(\zeta;\alpha)$ and $W(\zeta;\alpha)$ in the alternative equivalent forms

$$\mathcal{G}(\zeta;\alpha) = -\frac{1}{2\pi}\log\left|\frac{1}{\alpha}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})}\right|, \quad W(\zeta;\alpha) = -\frac{i}{2\pi}\log\left(\frac{1}{|\alpha|}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})}\right). \tag{5.53}$$

Note that $\mathcal{G}(\zeta;\alpha)$ is the streamfunction of the flow in D_ζ induced by a point vortex at $\zeta=\alpha$ of unit circulation. And indeed, from the properties of $G_0(\zeta;\alpha)$ described in §3.6.1 it is clear that close enough to the point $\zeta=\alpha$, where the effects of the boundaries of D_ζ are insignificant, $\mathcal{G}(\zeta;\alpha)$ resembles the streamfunction of the flow induced by a unit strength point vortex at this point. In fact we can write,

$$g(\zeta;\alpha) = \mathcal{G}(\zeta;\alpha) + \frac{1}{2\pi}\log|\zeta-\alpha| \tag{5.54}$$

where $g(\zeta;\alpha)$ is harmonic in D_ζ. Furthermore, the property (3.6) implies that the boundaries of D_ζ are streamlines of the flow corresponding to $\mathcal{G}(\zeta;\alpha)$, and also, recalling (5.45) we see that (3.7) implies that the circulations round the inner circles $\{C_i|i=1,...,N\}$ are zero, though by fixing a value for $\mathcal{G}(\zeta;\alpha)$ on C_0 this constrains the circulation round C_0 and this will generally be non-zero. If we wanted non-zero circulations round *all* the boundaries of D_ζ we could simply add to (5.51) a suitable linear combination of the N harmonic measures $\{\sigma_k(\zeta)|k=1,...,N\}$ defined in §3.6.2. The resulting function will still be a streamfunction with the same flow singularities since recall the harmonic measures are harmonic in D_ζ and constant on all the boundaries of D_ζ. But for $k\in\{1,...,N\}$ the addition of some constant multiple of $\sigma_k(\zeta)$ will contribute a non-zero circulation around C_k.

Now, from [45] the path function $H(\zeta_1,...,\zeta_M)$ for point vortices in D_ζ at $\zeta_1,...,\zeta_M$, where for $k=1,...,M$ the circulation of the point vortex at ζ_k is Γ_k, is given by

$$\begin{aligned} H(\zeta_1,...,\zeta_M) = \sum_{k=1}^{M}\Gamma_k\psi_0(\zeta_k) + \sum_{\substack{k_1,k_2=1\\k_1>k_2}}^{M}\Gamma_{k_1}\Gamma_{k_2}\mathcal{G}(\zeta_{k_1};\zeta_{k_2}) \\ + \frac{1}{2}\sum_{k=1}^{M}\Gamma_k^2 g(\zeta_k;\zeta_k). \end{aligned} \tag{5.55}$$

where $g(\zeta;\alpha)$ is defined by (5.54) and $\psi_0(\zeta)$ denotes the streamfunction for any background flow in D_ζ induced by agencies other than the point vortices. In the subsequent examples we shall impose no background flows. Furthermore, we shall impose that if the image domain D is either finite with an outer boundary ∂D_0, or infinite but with a boundary ∂D_0 which extends to infinity, then in both these cases we take the exceptional boundary ∂D_0 to be the image of C_0 so that the circulation is non-zero round this boundary but zero round all the others. But if D is an infinite domain and none of its boundaries extend to infinity, that is they all bound finite "islands" of the plane, then we shall insist that the circulation round all these boundaries is zero. This corresponds to a case where for example all point vortices start off at infinity so that with no imposed background flows the flow around the islands is initially trivial so that all circulations round the islands are initially zero and thus must remain zero for all subsequent times by Kelvin's circulation theorem. So how can we render the circulation around C_0 and hence ∂D_0 equal to zero? To do this we add in D_ζ another point vortex of equal but opposite circulation of the total circulation of the original M point vortices. We place this additional vortex at the point $\zeta = \zeta_\infty$ which maps to $z = \infty$. Thus in the case where there is $M = 1$ point vortices of circulation Γ, the complex potential becomes

$$\begin{aligned} W(\zeta) &= -\frac{i\Gamma}{4\pi}\log\left(\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\overline{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\overline{\alpha})}\right) + \frac{i\Gamma}{4\pi}\log\left(\frac{\omega(\zeta,\zeta_\infty)\overline{\omega}(\zeta^{-1},\zeta_\infty^{-1})}{\omega(\zeta,\overline{\zeta_\infty}^{-1})\overline{\omega}(\zeta^{-1},\overline{\zeta_\infty})}\right) \\ &= -\frac{i\Gamma}{4\pi}\log\left(\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})\omega(\zeta,\overline{\zeta_\infty}^{-1})\overline{\omega}(\zeta^{-1},\overline{\zeta_\infty})}{\omega(\zeta,\overline{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\overline{\alpha})\omega(\zeta,\zeta_\infty)\overline{\omega}(\zeta^{-1},\zeta_\infty^{-1})}\right) \end{aligned} \tag{5.56}$$

where the branch of $W(\zeta)$ is chosen so that a branch cut joins α to ζ_∞ inside D_ζ while another branch cut joins $\bar{\alpha}^{-1}$ to $\overline{\zeta_\infty}^{-1}$ in $\hat{D}_\zeta$ (with analogous choices of cuts being made in all other images of the fundamental region) so that indeed the circulation round C_0 and hence ∂D_0 is also zero. Note that Johnson and McDonald [46] employ this same approach to render the circulation around all boundaries of a multiply connected flow domain equal to zero.

Finally, using conformal maps from circular domains we can use the formulae above to plot point vortex trajectories in a variety of bounded domains.

We shall focus on the case of the motion of a single vortex of strength Γ at $\zeta = \alpha$. We point out that the Hamiltonian of any Hamiltonian system is a conserved quantity.

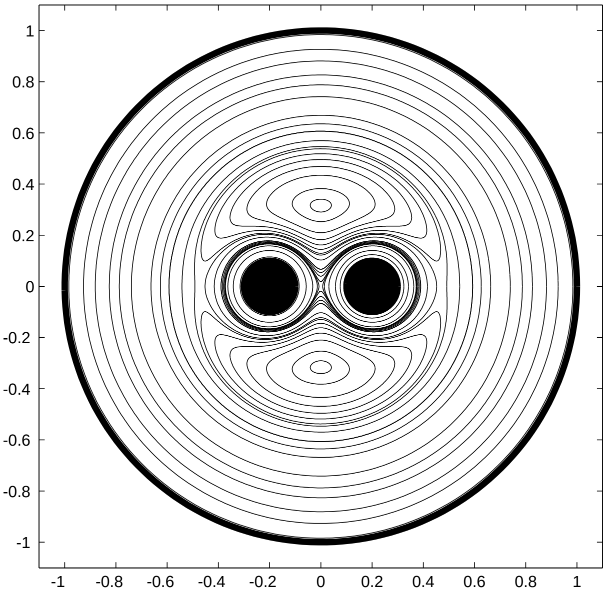

Fig. 5.13: Typical vortex paths in the unit disc with two cylinders of radius 0.1 centred at ± 0.2.

Thus in the case of a single vortex, the trajectories of the vortex are simply the level-lines of the path function.

When just a single vortex is present, the double sum in (5.55) disappears and the path function for a circulation Γ point vortex in the absence of any background flows due to other agencies is given by,

$$H^{(\zeta)}(\alpha) = -\frac{\Gamma^2}{2} g(\alpha, \alpha). \tag{5.57}$$

Some algebraic manipulations reveal

$$g(\alpha, \alpha) = \frac{1}{4\pi} \log \left| \frac{\omega'(\alpha, \alpha)\overline{\omega}'(\alpha^{-1}, \alpha^{-1})}{\alpha^2 \omega(\alpha, \bar{\alpha}^{-1})\overline{\omega}(\alpha^{-1}, \bar{\alpha})} \right|. \tag{5.58}$$

On use of (5.58), we have

$$H^{(\zeta)}(\alpha) = -\frac{\Gamma^2}{8\pi} \log \left| \frac{\omega'(\alpha, \alpha)\overline{\omega}'(\alpha^{-1}, \alpha^{-1})}{\alpha^2 \omega(\alpha, \overline{\alpha}^{-1})\overline{\omega}(\alpha^{-1}, \overline{\alpha})} \right|. \tag{5.59}$$

Using (5.59) we can plot the possible trajectories of a single point vortex in circular domains of the form of D_ζ. Figures 5.13–5.15 show such examples.

Now let $z(\zeta)$ be a conformal map from the circular domain to a conformally equivalent multiply connected domain D. Then, by (5.49), the Hamiltonian in the z-plane is given by

$$H^{(z)}(z_\alpha) = H^{(\zeta)}(\alpha) + \frac{\Gamma^2}{4\pi} \log |z_\zeta(\alpha)| \tag{5.60}$$

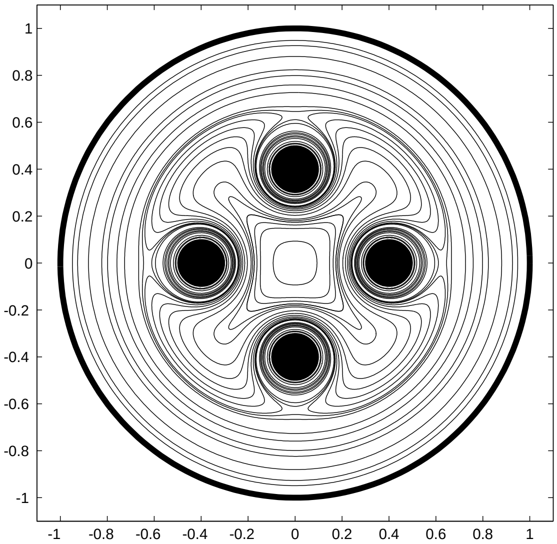

Fig. 5.14: Typical vortex paths in the unit disc with four cylinders of radius 0.1 centred at $\pm 0.4, \pm 0.4i$.

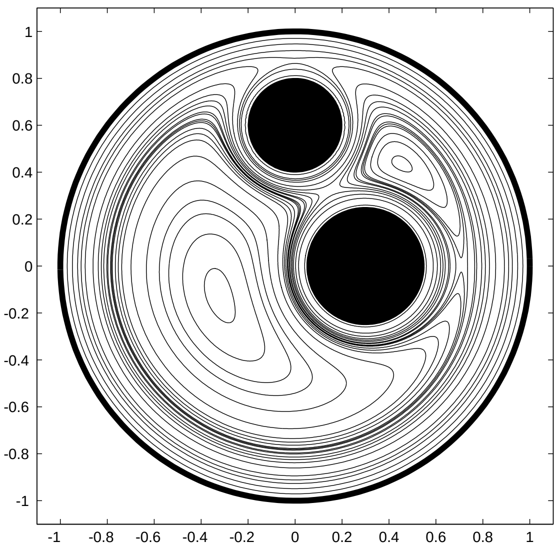

Fig. 5.15: Typical vortex paths in the unit disc with two cylinders, one of radius 0.25 centred at 0.3, another of radius 0.2 centred at $0.6i$.

where $z_\alpha = z(\alpha)$. Equivalently,

$$H^{(z)}(z_\alpha) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{1}{\alpha^2}\frac{\omega'(\alpha,\alpha)\overline{\omega}'(\alpha^{-1},\alpha^{-1})}{\omega(\alpha,\bar{\alpha}^{-1})\overline{\omega}(\alpha^{-1},\bar{\alpha})}\frac{1}{z_\zeta(\alpha)^2}\right|. \tag{5.61}$$

If the conformal map $z(\zeta)$ is known explicitly, (5.61) gives the Hamiltonian in explicit form.

In the case where we impose that the circulation round C_0 is zero by adding an additional point vortex of strength $-\Gamma$ at some point ζ_∞, the complex potential is given by (5.56) and the path function is

$$H^{(z)}(z_\alpha) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{1}{\alpha^2}\frac{\omega'(\alpha,\alpha)\overline{\omega}'(\alpha^{-1},\alpha^{-1})\omega^2(\alpha,\bar{\zeta}_\infty^{-1})\overline{\omega}^2(\alpha^{-1},\bar{\zeta}_\infty)}{\omega(\alpha,\bar{\alpha}^{-1})\overline{\omega}(\alpha^{-1},\bar{\alpha})\omega^2(\alpha,\zeta_\infty)\overline{\omega}^2(\alpha^{-1},\zeta_\infty^{-1})}\frac{1}{z_\zeta(\alpha)^2}\right|. \tag{5.62}$$

5.2.5 Vortex motion around circular boundaries.

Let us consider the motion of a single point vortex around an arbitrary finite number of circular cylinders either in a half-plane bounded by an infinite straight line or in an otherwise unbounded plane in the special case when the circulations around all the cylinders are zero. Such domains could be regarded as modelling groups of islands either near a coastline or far from shore. Note that the examples we shall present in this section also appear in [27].

Recall that Mobius transformations map circles to circles. It is straightforward to show that the particular Möbius transformation

$$z(\zeta) = \frac{1-\zeta}{1+\zeta}. \tag{5.63}$$

maps C_0 onto the imaginary axis and any other inner circular boundaries of D_ζ onto circles in the right-half plane. Using this conformal map from the circular domain D_ζ and the formula (5.61) we can compute the path function for the image domain D. Figures 5.16–5.19 show a variety of such examples.

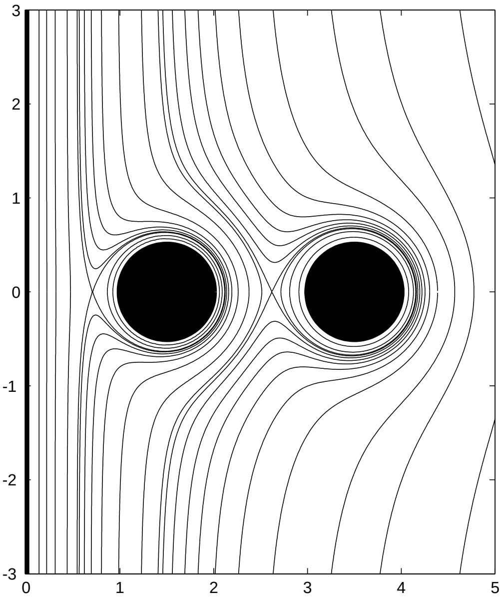

Fig. 5.16: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis and two cylinders.

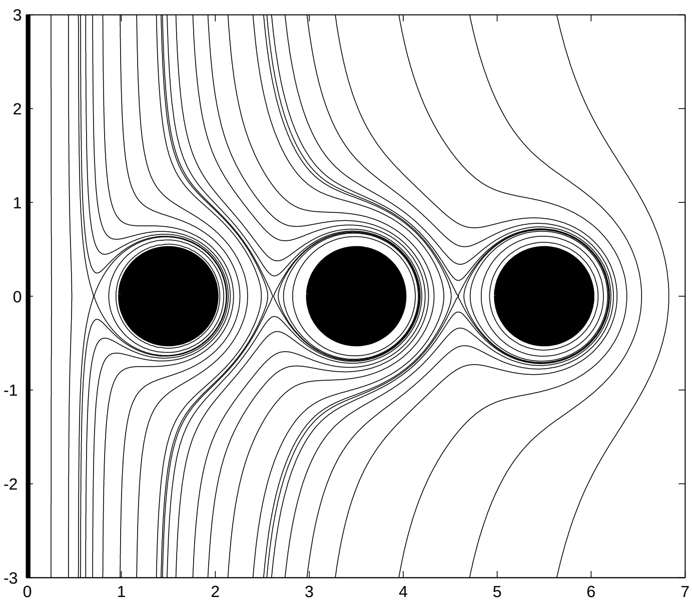

Fig. 5.17: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis and three cylinders.

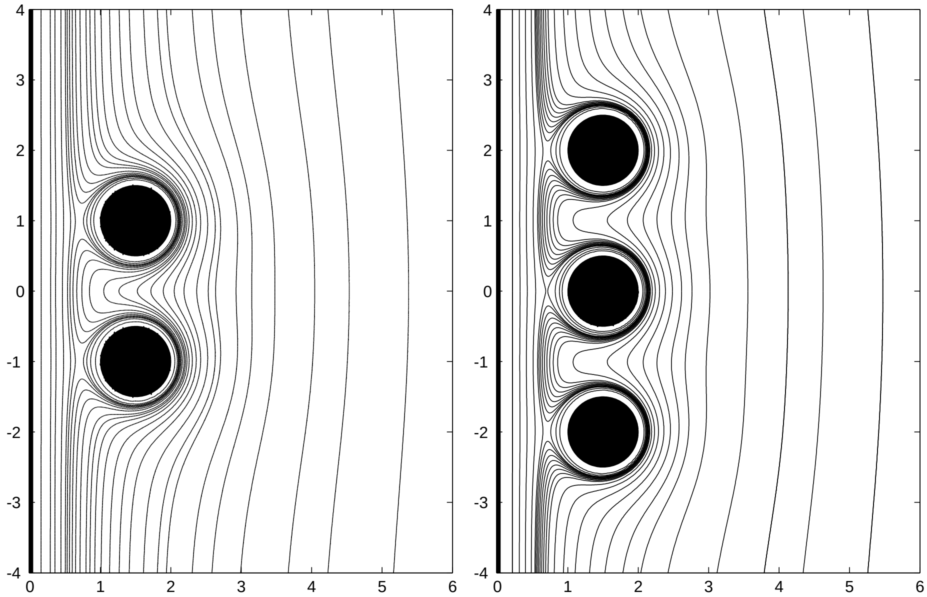

Fig. 5.18: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis and two and three cylinders.

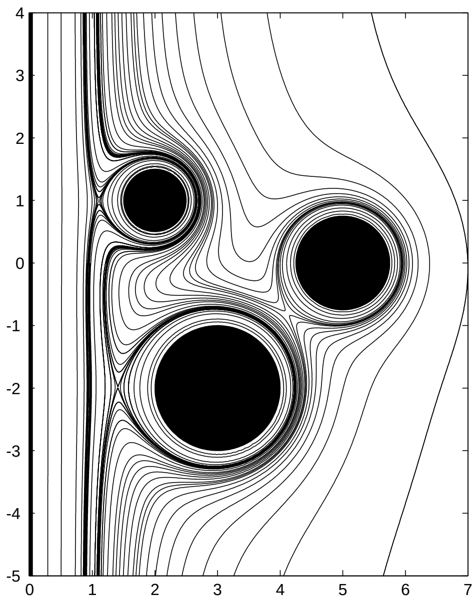

Fig. 5.19: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis and three cylinders.

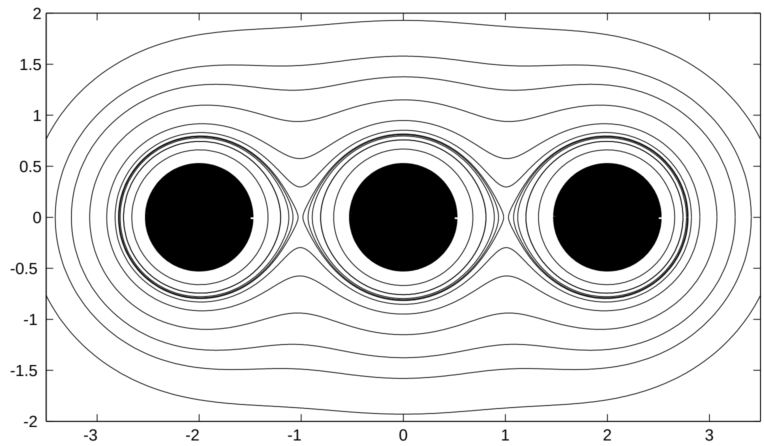

Fig. 5.20: Typical trajectories for a single vortex in the region exterior to three cylinders.

Now suppose D is the $N+1$-connected region exterior to $N+1$ circular cylinders, so none of its boundaries extend to infinity. This could be regarded as modelling $N+1$ islands far from shore. The circulation around these islands will be taken to be zero.

The conformal map from a circular domain D_ζ to such a domain is of the form

$$z(\zeta) = \frac{a}{\zeta} + b \tag{5.64}$$

where the real parameter a is chosen to fix the radius of the island corresponding to the image of C_0 and the (generally complex) parameter b is chosen to appropriately locate its centre. Clearly, $\zeta = 0$ maps to physical infinity. Thus, using (5.62), taking $\zeta_\infty = 0$, we can compute the path function for such domains. Figures 5.20 and 5.21 illustrate examples.

5.2.6 Vortex motion around straight boundaries.

In this section we consider the motion of a single point vortex around an arbitrary number of finite straight-line slits, either in a half-plane bounded by an infinite straight line or in an otherwise unbounded plane, or possibly lying between two semi-infinite lines. We consider the case where the circulations around the slits are all zero. Such domains can be regarded as modelling groups of islands either in the presence of a coastline or far from shore, or a chain of islands between two headlands. We use conformal maps from circular domains to such slit domains of

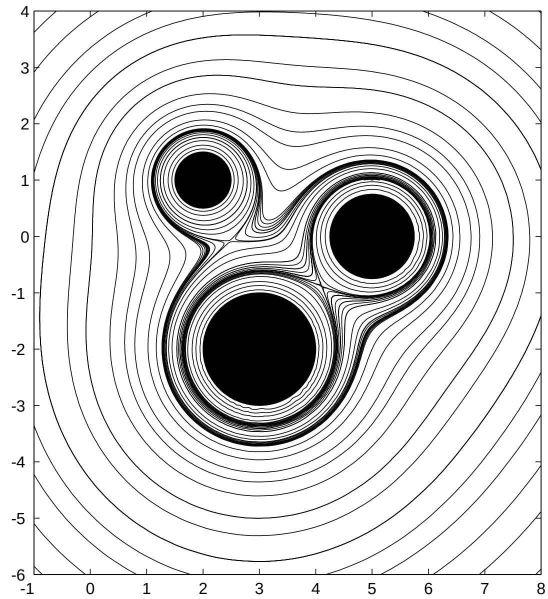

Fig. 5.21: Typical trajectories for a single vortex in the region exterior to three cylinders.

the type presented in appendix F. Note that the examples we shall present in this section also appear in [28].

Using the conformal map given by (G.1) together with (5.61) we can consider point vortex motion in a half-plane with slits. Examples are shown in figures 5.22–5.25.

Now using the conformal maps presented in §G.2 we can consider the case of point vortex motion in a region exterior to a chain of finite slits. Figure 5.26 shows typical point vortex trajectories in such a domain.

Johnson and McDonald have discussed the general problem of vortex motion around a chain of slits between two semi-infinite axes. In [47] the problem of point vortex motion through a single gap in an infinite straight axis is solved explicitly while the generalization to two gaps is solved in [48]. We now consider the general case of any number of gaps using the conformal maps presented in §G.2. Figures 5.27 and 5.28 show two such examples.

Note that referring back to the familiar "method of images", it is worth pointing out that the prime functions that appear in the formulae for the Hamiltonians automatically do everything necessary to place an appropriate distribution of infinitely many "image vortices" throughout the plane in such a way that the conditions on all the disjoint boundaries are simultaneously satisfied.

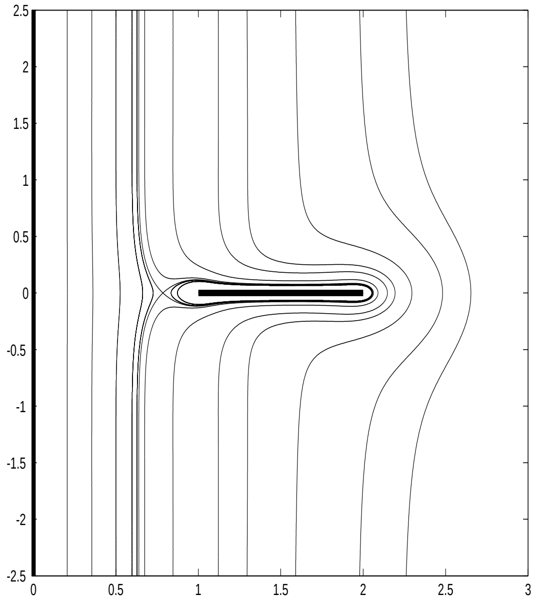

Fig. 5.22: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis with a single slit along the real axis.

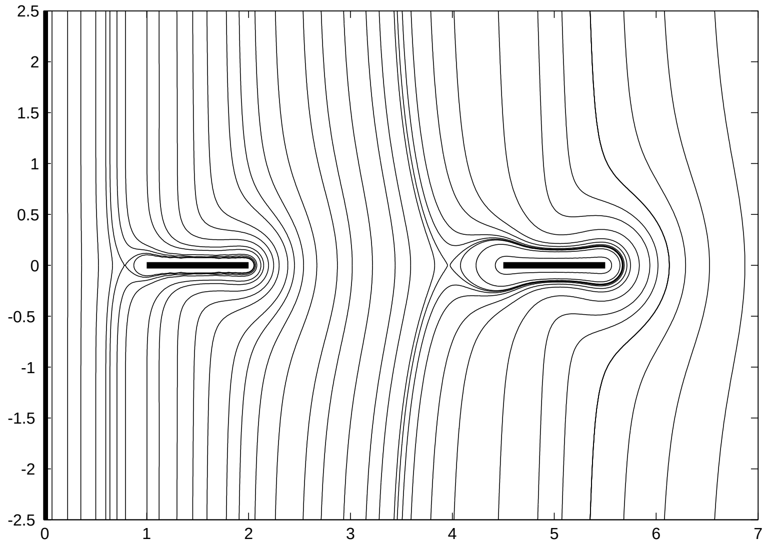

Fig. 5.23: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis with two slits along the real axis.

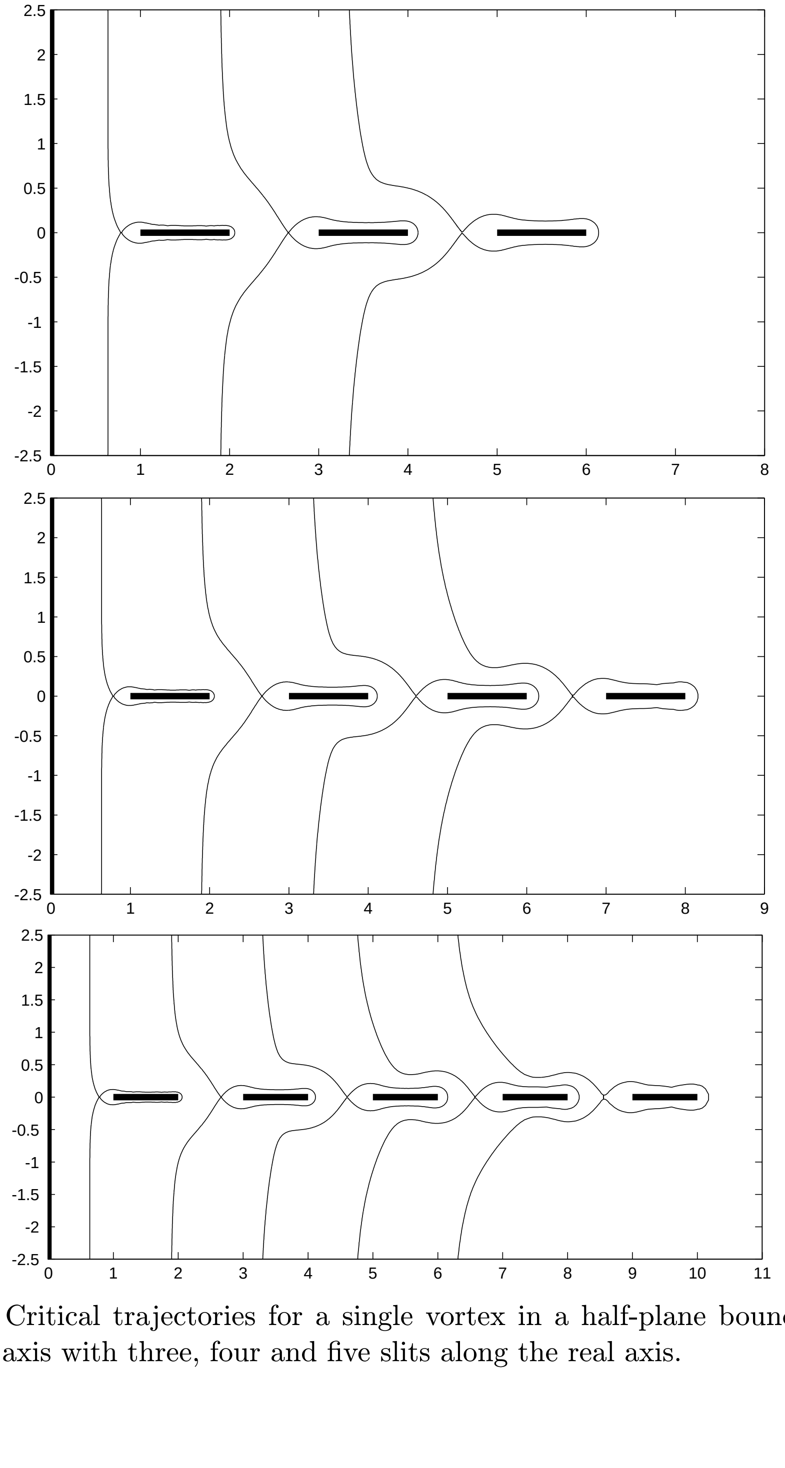

Fig. 5.24: Critical trajectories for a single vortex in a half-plane bounded by the imaginary axis with three, four and five slits along the real axis.

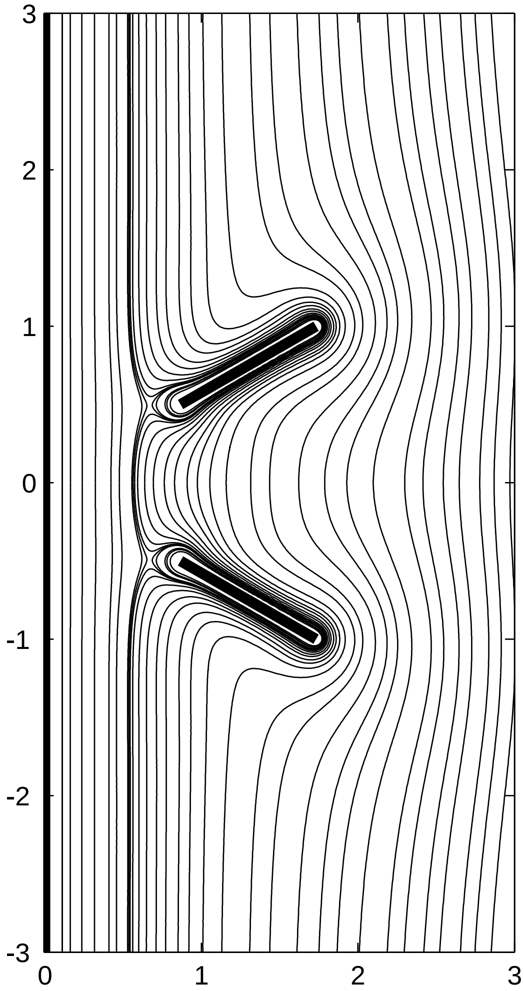

Fig. 5.25: Typical trajectories for a single vortex in a half-plane bounded by the imaginary axis and two slits of making angles of $\pm\pi/6$ with the real axis.

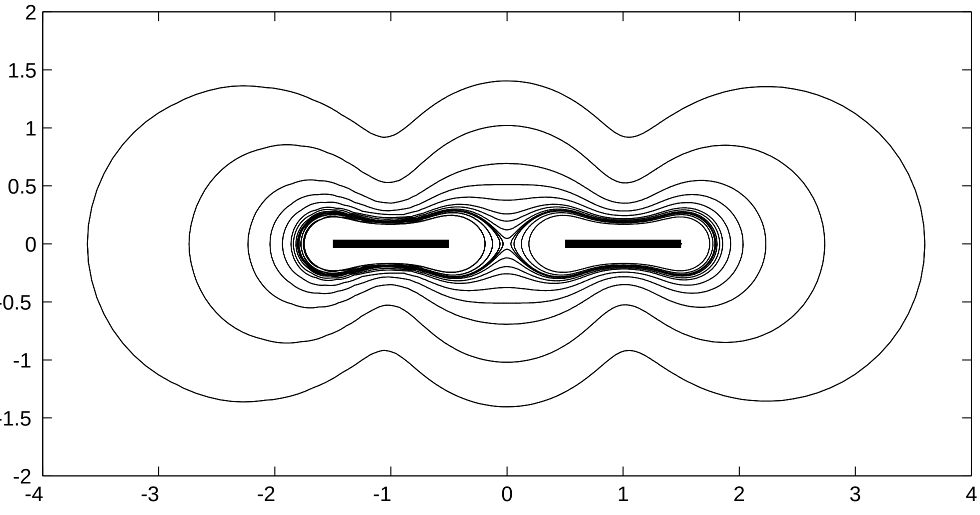

Fig. 5.26: Typical trajectories for a single vortex in the region exterior to two finite-length slits.

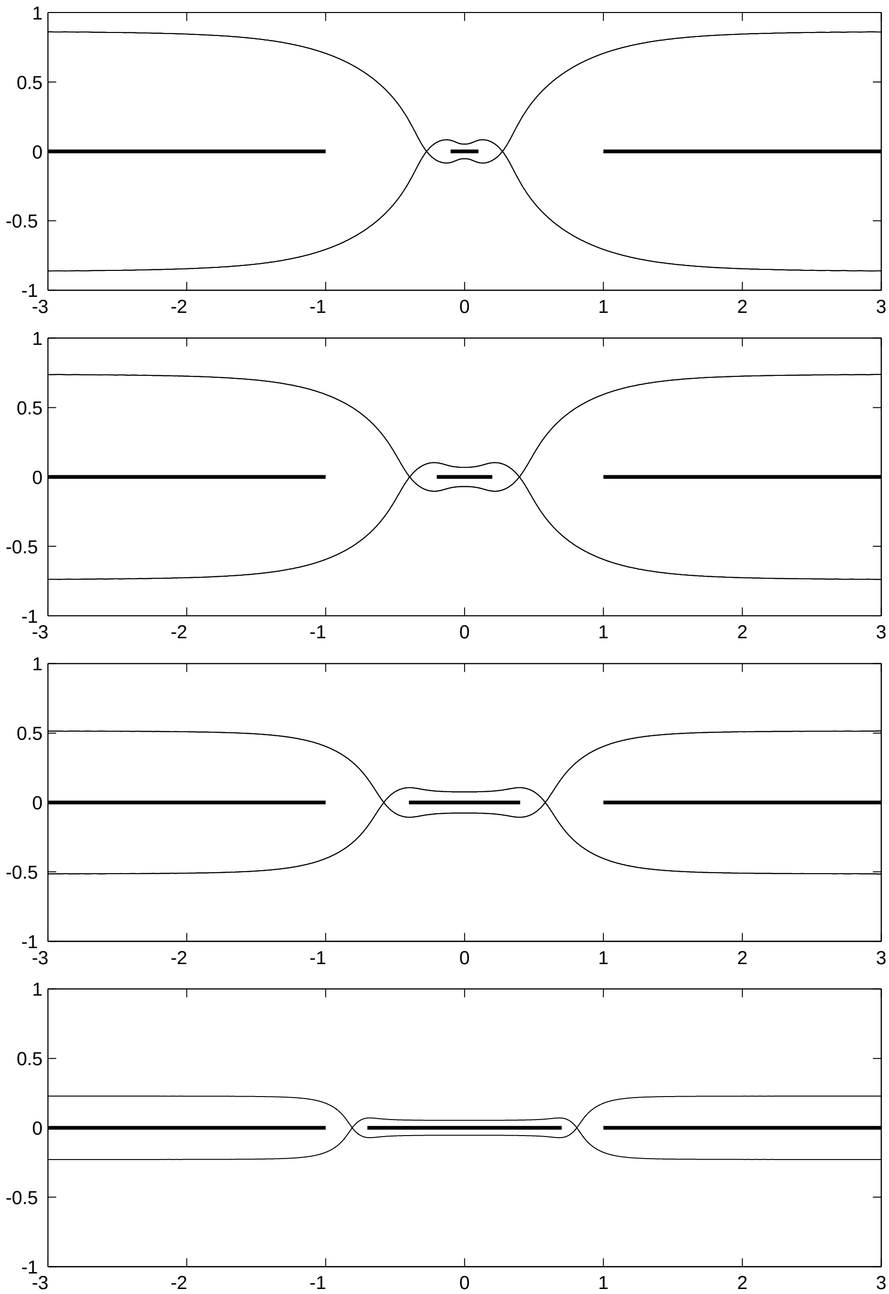

Fig. 5.27: Critical trajectories of a single vortex in the domain bounded by the real axis with two gaps and a slit of lengths $0.2, 0.4, 0.8$ and 1.4.

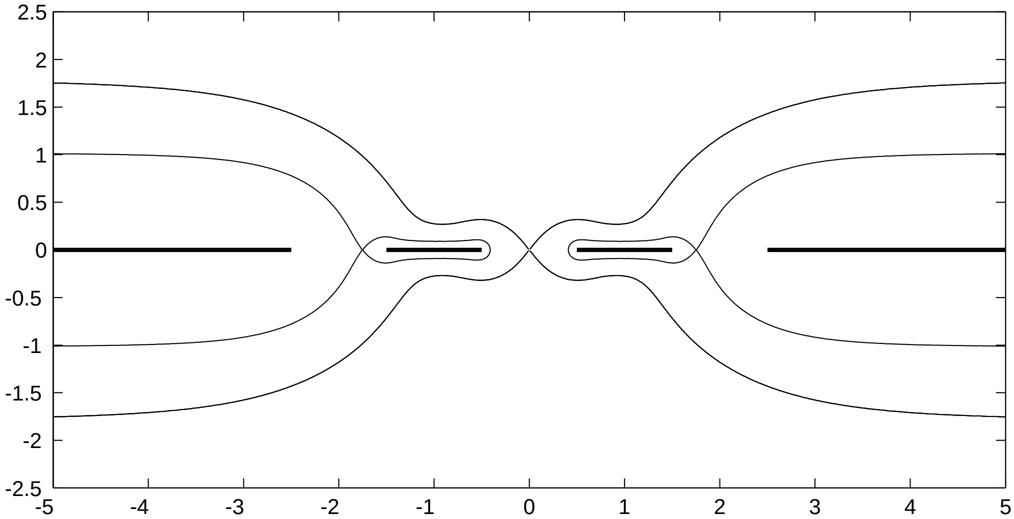

Fig. 5.28: Critical trajectories for a single vortex in the domain bounded by the real axis with three gaps. The slits are between -1.5 and -0.5 and between 0.5 and 1.5.

CHAPTER 6

Conclusion.

Among all multiply connected domains, circular domains are a canonical class in the sense that every multiply connected domain D is conformally equivalent to some circular domain D_ζ. Given a circular domain D_ζ, we have shown how one may explicitly construct an associated Schottky group Θ and hence prime function $\omega(\zeta, \gamma)$. In terms of $\omega(\zeta, \gamma)$ one may construct explicit formulae for a number of important functions. These in turn may be used to address a number of problems of fluid dynamics.

We have shown how to construct functions automorphic with respect to the associated Schottky group Θ. Using these we may parameterize multiply connected quadrature domains. Construction of quadrature domains using these conformal mappings has been checked independently using a method of construction based on the fact that the boundaries of quadrature domains are algebraic curves. Furthermore, these parameterizations of quadrature domains lead to explicit descriptions of vortical solutions of the two-dimensional Euler equations exhibiting multiple point vortices and vortex patches. However, a full stability analysis of these solutions remains to be performed.

We have also obtained explicit formulae for the modified Green's functions, harmonic measures and Green's functions of multiply connected circular domains. Such formulae for arbitrary domains follow by conformal mapping. In terms of these modified Green's formulae one may construct the path functions for the motion of point vortices in bounded domains. It would be of interest to generalize the formulae herein to this more general case where background flows are incorporated. However, Lin [44] has demonstrated that this circumstance simply implies the addition of a further contribution to the total Hamiltonian - one that does not depend on the instantaneous point vortex positions. Thus this additional contribution can be determined (e.g. numerically) at the start of any calculation and will remain fixed during the calculation if the contribution from the external agencies is time-invariant. The explicit formulas presented here can still be employed to give the “point vortex contribution” to the Hamiltonian. In this way, the formulas employed here should simplify the numerical computation of vortex trajectories even in the presence of background flows.

We have also derived explicit formulae for the conformal maps from the canonical class of multiply connected circular domains to various other commonly studied canonical classes of multiply connected slit domains.

Note that an issue which remains to be resolved is that of determining the conditions under which the representation of the prime function as an infinite product converges. This is an area for future work. A better understanding of the convergence properties of this representation may improve the numerical efficiency of the formulae presented in this thesis.

APPENDIX **A**

Reflections in circles.

A.1 Definition.

Consider a circle C_i of centre δ_i and radius q_i in the ζ-plane. We define the reflection of a point ζ in C_i as

$$r_{C_i(\zeta)} = \delta_i + \frac{q_i^2}{\overline{\zeta} - \overline{\delta_i}} \tag{A.1}$$

Note that for points ζ on C_i,

$$|\zeta - \delta_i| = q_i, \tag{A.2}$$

and thus

$$\bar{\zeta} = \overline{\delta_i} + \frac{q_i^2}{\zeta - \delta_i}. \tag{A.3}$$

It follows from (A.3) and (A.1) that for points ζ on C_i, $r_{C_i(\zeta)} = \zeta$, i.e. as would be expected reflection in C_i leaves each point on C_i invariant. Also,it is straightforward to check that $r_{C_i(\zeta)}$ is a self-inverse transformation.

We may write

$$\begin{aligned} r_{C_i(\zeta)} &= \overline{\phi_i(\zeta)} \\ &= \overline{\phi_i}(\bar{\zeta}) \end{aligned} \tag{A.4}$$

where we define

$$\phi_i(\zeta) = \overline{\delta_i} + \frac{q_i^2}{\zeta - \delta_i} \tag{A.5}$$

and the conjugate function $\overline{\phi_i}(\zeta)$ by

$$\overline{\phi_i}(\zeta) = \overline{\phi_i(\overline{\zeta})} \tag{A.6}$$

Note that for points ζ on C_i,

$$\overline{\zeta} = \phi_i(\zeta). \tag{A.7}$$

It is straightforward to see that we may write

$$\phi_i(\zeta) = \frac{\overline{\delta_i}\zeta + q_i^2 - |\delta_i|^2}{\zeta - \delta_i} \tag{A.8}$$

and,

$$r_{C_i(\zeta)} = \frac{\delta_i\overline{\zeta} + q_i^2 - |\delta_i|^2}{\overline{\zeta} - \overline{\delta_i}}. \tag{A.9}$$

A.2 The image of a circle under reflection in another circle.

What is the image of a circle under reflection in another circle? Take a circle C_i of centre δ_i, radius q_i and a circle C_j of centre δ_j, radius q_j. Let $\zeta' = r_{C_i}(\zeta)$ denote the reflection of ζ in the circle C_i. Since reflection in a circle is self-inverse then of course $\zeta = r_{C_i}(\zeta')$. So from the definition (A.1),

$$\zeta = \delta_i + \frac{q_i^2}{\overline{\zeta'} - \overline{\delta_i}} \tag{A.10}$$

Now, for points ζ on C_j,

$$(\zeta - \delta_j)(\overline{\zeta} - \overline{\delta_j}) = q_j^2. \tag{A.11}$$

Substituting (A.10) for ζ in (A.11) leads to the following equation for points ζ' on $r_{C_i}(C_j)$,

$$\left(\delta_i + \frac{q_i^2}{\overline{\zeta'} - \overline{\delta_i}} - \delta_j\right)\left(\overline{\delta_i} + \frac{q_i^2}{\zeta' - \delta_i} - \overline{\delta_j}\right) = q_j^2. \tag{A.12}$$

Following straightforward manipulations, (A.12) can be shown to simplify to

$$\left|\zeta' + \frac{q_i^2(\delta_i - \delta_j)}{|\delta_i - \delta_j|^2 - q_j^2} - \delta_i\right| = \frac{q_i^2 q_j}{||\delta_i - \delta_j|^2 - q_j^2|}, \tag{A.13}$$

which of course represents a circle of centre δ_j', radius q_j' given by

$$\begin{aligned} \delta_j' &= \delta_i - \frac{q_i^2(\delta_i - \delta_j)}{|\delta_i - \delta_j|^2 - q_j^2}; \\ q_j' &= \frac{q_i^2 q_j}{||\delta_i - \delta_j|^2 - q_j^2|}. \end{aligned} \tag{A.14}$$

APPENDIX **B**

Möbius transformations

B.1 Definition.

A very important class of transformations of the extended complex plane are **Möbius** maps (also called bilinear maps, or linear-fractional maps). These are discussed in most books on complex analysis, for example [54], [1]. A particularly clear description of Möbius maps is also provided by [52].

A general Möbius map $T(\zeta)$ may be written in the form

$$T(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \qquad ad - bc \neq 0 \tag{B.1}$$

where $a, b, c, d \in \mathbb{C}$. Note that the condition $ad - bc \neq 0$ is necessary, as otherwise $T(\zeta)$ reduces to a constant. This can be seen by simply rearranging (B.1) into the form

$$T(\zeta) = \frac{a}{c} + \frac{bc - ad}{c(c\zeta + d)}. \tag{B.2}$$

Note that there is an alternative way of representing Möbius transformations in terms of matrices as follows. To any point ζ in the plane we may associate a vector v_ζ say, by writing

$$\zeta = \frac{\zeta_1}{\zeta_2}, \tag{B.3}$$

for some ζ_1, ζ_2 (an obvious choice being $\zeta_1 = \zeta, \zeta_2 = 1$) and then taking

$$v_\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}. \tag{B.4}$$

And to a Möbius map $T(\zeta)$ of the form (B.1) we may associate a matrix M_T say, where

$$M_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{B.5}$$

Then the vector given by the action of M_T on v_ζ is,

$$M_T v_\zeta = \begin{pmatrix} a\zeta_1 + b\zeta_2 \\ c\zeta_1 + d\zeta_2 \end{pmatrix} \tag{B.6}$$

But by the above association this is the vector associated with the point

$$\frac{a\zeta_1 + b\zeta_2}{c\zeta_1 + d\zeta_2} \tag{B.7}$$

which by (B.3) is clearly the same as

$$\frac{a\zeta + b}{c\zeta + d} \tag{B.8}$$

i.e. $T(\zeta)$.

Note that the determinant of the matrix M_T is the quantity $ad - bc$. This quantity is thus sometimes referred to as the **determinant** of the Möbius map T. By definition (B.1) this quantity is non-zero. We can of course always normalize a Möbius map so that its determinant is whatever (finite, non-zero) value we wish.

Note furthermore that by the above association between Möbius transformations and matrices, M_T is invertible, and its inverse is in fact the matrix associated with T^{-1}; the identity matrix is associated with the identity transformation; and for two Möbius transformations the matrix associated with their composition is precisely the composition of their associated matrices.

This representation of Möbius transformations in terms of matrices is particularly helpful for computational purposes since programs such as MATLAB can perform matrix manipulations very quickly. Indeed most of the computations carried out for the thesis were performed in precisely this way.

B.2 Decomposition of a Möbius transformation.

Consider a Möbius map of the form (B.1). If $c = 0$ then this is a linear transformation. Otherwise, we may write

$$\frac{a\zeta + b}{c\zeta + d} = \frac{a}{c} + \frac{\frac{1}{c^2}(bc - ad)}{\zeta + \frac{d}{c}} \tag{B.9}$$

Thus, letting ζ' denote the transformed point, this shows that the Möbius transformation may be decomposed into the following three successive transformations:

$$\zeta_1 = \zeta + \frac{d}{c} \tag{B.10}$$

$$\zeta_2 = \frac{1}{\zeta_1} \tag{B.11}$$

$$\zeta' = \frac{a}{c} + \left(\frac{1}{c^2}(bc - ad)\right)\zeta_2 \tag{B.12}$$

The map (B.11) is an inversion while the maps (B.10), (B.12) are both linear transformations.

B.3 Image of a circle under a Möbius map.

The image a circle C under a Möbius map is another circle C'. In the case where C and C' are both non-degenerates circles (i.e. neither is an infinite straight line) we now give explicit formulae for the centre and radius of C'.

Suppose C has centre δ, radius q. C is described by the equation

$$(\zeta - \delta)(\bar{\zeta} - \bar{\delta}) = q^2 \tag{B.13}$$

Under a Möbius map (B.1) C maps to another circle C'. Let us first consider separately the cases where the map is either a linear transformation or an inversion.

B.3.1 Image of a circle under a linear transformation.

Consider a general linear transformation.

$$\zeta' = a\zeta + b. \tag{B.14}$$

So,

$$\zeta = a^{-1}\zeta' - ba^{-1}. \tag{B.15}$$

Thus substituting for ζ in (B.13) using (B.15), it is straightforward to show that C' is given by

$$\left|\zeta' - (a\delta + b)\right|^2 = (|a|q)^2 \tag{B.16}$$

which is a circle of centre δ' and radius q' where

$$\delta' = a\delta + b, \qquad q' = |a|q. \tag{B.17}$$

B.3.2 Image of a circle under an inversion.

Consider an inversion

$$\zeta^{'} = \zeta^{-1} \tag{B.18}$$

So,

$$\zeta = \zeta^{\prime -1} \tag{B.19}$$

Thus, substituting for ζ in (B.13) using (B.19), it is straightforward to show that C' is given by

$$\left| \zeta' - \frac{\overline{\delta}}{|\delta|^2 - q^2} \right|^2 = \frac{q^2}{(|\delta|^2 - q^2)^2}. \tag{B.20}$$

which is a circle of centre δ' and radius q' where

$$\delta' = \frac{\overline{\delta}}{|\delta|^2 - q^2}, \qquad q' = \frac{q}{||\delta|^2 - q^2|} \tag{B.21}$$

B.3.3 Image of a circle under a general Möbius transformation.

Now, to determine the image C' of C under a general Möbius map (B.1), consider the map as a composition of linear transformations and an inversion as in §B.2 and use the formulae (B.17), (B.21).

The first map (B.10) in the composition is a linear transformation and maps C to a circle C_1 with centre δ_1, radius q_1 which by (B.17) are

$$\delta_1 = \delta + \frac{d}{c}, \qquad q_1 = q. \tag{B.22}$$

The second map (B.11) is an inversion and maps C_1 to a circle C_2 with centre δ_2, radius q_2 which by (B.21) are

$$\delta_2 = \frac{\overline{\delta_1}}{|\delta_1|^2 - q_1^2}, \qquad q_2 = \frac{q_1}{||\delta_1|^2 - q_1^2|}. \tag{B.23}$$

The final map (B.11) is another linear transformation and maps C_2 to the circle C' with centre $\delta^{'}$ and radius $q^{'}$ which by (B.17) are

$$\delta' = \left(\frac{1}{c^2}(bc - ad)\delta_2 + \frac{a}{c} \right), \qquad q' = \left| \frac{1}{c^2}(bc - ad) \right| q_2. \tag{B.24}$$

Thus, combining the formulae (B.22),(B.22), (B.24) we get

$$\begin{aligned} \delta' &= \left(\overline{\delta} + \frac{\overline{d}}{\overline{c}}\right)\left(\frac{bc-ad}{c^2\left(\left|\delta+\frac{d}{c}\right|^2 - q^2\right)}\right), \\ q' &= q\left|\frac{bc-ad}{c^2\left(\left|\delta+\frac{d}{c}\right|^2 - q^2\right)}\right|. \end{aligned} \tag{B.25}$$

B.4 Fixed points of a Möbius transformation.

The fixed points of a Möbius transformation $T(\zeta)$ of the form (B.1) are of course the solutions of the equation

$$\zeta = \frac{a\zeta + b}{c\zeta + d}, \tag{B.26}$$

or

$$c\zeta^2 + (d-a)\zeta - b = 0. \tag{B.27}$$

Clearly, if $c = 0$ then $T(\zeta)$ has just one fixed point (namely $b(d-a)^{-1}$, or ∞ if $d = a$). But if $c \neq 0$ then $T(\zeta)$ has two fixed points given by

$$\frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c}. \tag{B.28}$$

If $(d-a)^2 - 4bc = 0$, these two points coincide, but otherwise they are distinct.

Suppose now that $T(\zeta)$ is a Möbius map with two distinct fixed points, labelled A and B say. It can be shown ([54],[1],[52]) that an alternative representation to (B.1) for $T(\zeta)$ is

$$\frac{T(\zeta) - B}{T(\zeta) - A} = \lambda\frac{\zeta - B}{\zeta - A} \tag{B.29}$$

where λ is some complex constant, $\lambda \neq 0, \infty$. Consider any point ζ in the extended complex plane other than A or B. It can be shown ([54],[1],[52]) that as $k \to \infty$, if $T^k(\zeta)$ converges then its limit is one of the fixed points A or B, and furthermore its limit as $k \to -\infty$ is the other fixed point. We refer to these limits as the **sink** and **source** of T, respectively. In this case T is called a **loxodromic** Möbius map. This case corresponds to $\lambda \neq 1$ in the representation (B.29). In particular, $|\lambda| < 1$ corresponds to the case where B is the sink and A the source, while $|\lambda| > 1$ corresponds to the case where A is the sink and B the source. In the special case where $\lambda \neq 1$ and λ is in fact also real, then T is called a **hyperbolic** Möbius map.

APPENDIX **C**

Properties of our Schottky groups constructed from circular domains.

In this appendix we state and prove some properties of our Schottky groups constructed from circular domains. Note that these results do not hold for all Schottky groups in general.

Proposition C.0.1 *The fixed points B_j and A_j of the map θ_j given by (1.6) are reflections of one another in the unit circle.*

Proof: A_j and B_j are respectively the source and sink of θ_j. Of course $A_j = \theta_j(A_j)$, and so obviously $\overline{A_j} = \overline{\theta_j}(\overline{A_j})$. So repeating the same argument with B_j we see that the transformation $\overline{\theta_j}$ has fixed points $\overline{A_j}, \overline{B_j}$. The fixed points of θ_{-j} are of course the same as the fixed points of θ_j. So,

$$A_j = \theta_{-j}(A_j) \tag{C.1}$$

But, by (1.15),

$$\theta_{-j}(A_j) = \frac{1}{\overline{\theta}(A_j^{-1})}. \tag{C.2}$$

Hence it follows that

$$\overline{\theta_j}(A_j^{-1}) = A_j^{-1} \tag{C.3}$$

ie, A_j^{-1} is a fixed point of $\overline{\theta_j}$. But from earlier arguments we know that the fixed points of $\overline{\theta_j}$ are $\overline{A_j}, \overline{B_j}$.
And $A_j^{-1} \neq \overline{A_j}$ since otherwise it follows that $|A_j| = 1$, which is not possible.
Hence it follows that

$$A_j^{-1} = \overline{B_j}, \tag{C.4}$$

or,

$$A_j = \overline{B_j}^{-1}. \tag{C.5}$$

Thus, the fixed points A_j, B_j are reflections of one another in the unit circle. So they lie on the same ray, and the product of their moduli is 1. This completes the proof.

Proposition C.0.2 *A_j and B_j lie on the same ray as δ_j.*

Proof: A_j, B_j are the two distinct solutions of

$$\theta_j(\zeta) - \zeta = 0 \tag{C.6}$$

which using from (1.6) can be written as

$$\zeta^2 + \frac{(d_j - a_j)}{c_j}\zeta - \frac{b_j}{c_j} = 0. \tag{C.7}$$

where a_j, b_j, c_j, d_j are given by (1.6). Thus we must have

$$A_j B_j = -\frac{b_j}{c_j} = \frac{\delta_j}{\overline{\delta_j}} \tag{C.8}$$

But from Proposition C.0.1, $B_j = \overline{A_j}^{-1}$. Thus it follows from (C.8) that A_j and B_j lie on the same ray as δ_j. This completes the proof.

Proposition C.0.3 *The map θ_j given by (1.6) is a hyperbolic Möbius map.*

Proof: For $j = 1, ..., N$, since the map θ_j given by (1.6) has two distinct fixed points A_j and B_j where A_j is its source and B_j its sink, then [54] it can be written in the form

$$\frac{\theta_j(\zeta) - B_j}{\theta_j(\zeta) - A_j} = \lambda_j \frac{\zeta - B_j}{\zeta - A_j} \tag{C.9}$$

where λ_j is a non-zero complex constant and $|\lambda_j| < 1$. So

$$\lambda_j = \frac{(\theta_j(\zeta) - B_j)(\zeta - A_j)}{(\theta_j(\zeta) - A_j)(\zeta - B_j)} \tag{C.10}$$

Taking $\zeta = 0$ in this identity we get

$$\lambda_j = \frac{A_j(\theta_j(0) - B_j)}{B_j(\theta_j(0) - A_j)} \tag{C.11}$$

where from (1.6) we see that

$$\theta_j(0) = \frac{b_j}{d_j} = \delta_j \tag{C.12}$$

So,

$$\lambda_j = \frac{A_j(\delta_j - B_j)}{B_j(\delta_j - A_j)} \tag{C.13}$$

But from Propositions C.0.1 and C.0.2, A_j, B_j are reflections of one another in the unit circle and furthermore they lie on the same ray as δ_j. Thus, we deduce

$$\lambda_j = |A_j|^2 \frac{|\delta_j| - |B_j|}{|\delta_j| - |A_j|} \tag{C.14}$$

which is clearly real. This completes the proof.

APPENDIX **D**

Poincaré theta series.

Consider a Schottky group Θ. We may express each map $\theta \in \Theta$ in the form (B.1) with *unit* determinant, i.e.

$$\theta(\zeta) = \frac{a\zeta + b}{c\zeta + d} \quad \text{where } ad - bc = 1 \tag{D.1}$$

Then, for $Q(\zeta)$ a rational function and m an integer consider the infinite series

$$\sum_{\theta \in \Theta} \frac{Q(\theta(\zeta))}{(c\zeta + d)^{2m}} \tag{D.2}$$

Such a series (D.2) is called a **Poincaré theta series**. A Poincaré theta series (D.2) defined with integer m is sometimes referred to as a $(-2m)$-dimensional Poincaré theta series.

In all cases for $m > 1$, and in certain cases for $m = 1$ it can be shown [6],[10],[9] that this series converges for all ζ except at (i) the singular points of Θ; (ii) the points $\zeta = -d/c$ (which it is straightforward to see are all the images of the point $\zeta = \infty$ under maps in Θ, including the point $\zeta = \infty$ itself) - at these points it has polar singularities; (iii) the singularities of $Q(\zeta)$ and the images of these points under maps in Θ - at these points it has polar singularities. Thus when (D.2) converges it represents a single-valued continuous function of ζ.

For a Schottky group Θ, the representation () for the prime function $\omega(\zeta, \gamma)$ can be derived directly from consideration of a (-2)-dimensional Poincaré theta series of the form (D.2) with $m = 1$ and $Q(\zeta) = (\zeta - a)^{-1}$, where a is some ordinary point of Θ (see [10],[6] for further details). Under conditions where such a series converges the infinite product (1.22) converges too. Some criteria sufficient for convergence are known (-see [10],[6],[9]); roughly speaking these state that convergence is guaranteed if the circles $\{C_i | i = 1, ..., N, -1, ..., -N\}$ defining the Schottky group are far enough apart.

We briefly point out that in order to carry out computations with Poincaré theta series, one must of course truncate the infinite sum (D.2) to a finite sum. As in truncating the infinite product representation (1.22) for the prime function, this may be done in a very natural way by including all maps in Θ of up to some chosen level and truncating the contribution to the product from all higher-level maps.

APPENDIX E

Representing automorphic functions in terms of Poincaré theta series.

In chapter 2 we describe the representation a function automorphic with respect to a Schottky group in terms of the associated prime function. We shall now briefly describe an alternative representation of such a function in terms of Poincaré theta series. Indeed it was the consideration of automorphic functions that led Poincaré to study these series, and the construction we describe is precisely the one suggested by him. Further details are given in [10],[6],[8].

From the definition (D.2) of a Poincaré theta series $\mathcal{P}(\zeta)$ of a Schottky group Θ it is straightforward to show that for any map $\theta \in \Theta$ represented in the form

$$\theta(\zeta) = \frac{a\zeta + b}{c\zeta + d} \tag{E.1}$$

we have

$$\mathcal{P}(\theta(\zeta)) = (c\zeta + d)^{2m}\mathcal{P}(\zeta) \tag{E.2}$$

Suppose $\mathcal{P}_n(\zeta)$ and $\mathcal{P}_d(\zeta)$ are two Poincaré series of the form (D.2) with the same value of m but different choices for the rational function $Q(\zeta)$, say $Q_n(\zeta)$,$Q_d(\zeta)$ respectively. Then from (E.2) it is clear that their ratio,

$$\frac{\mathcal{P}_n(\zeta)}{\mathcal{P}_d(\zeta)} \tag{E.3}$$

represents a function which is automorphic with respect to Θ.

For the purposes of their numerical computation we find the representation of automorphic functions in terms of the prime function to be preferable to that in terms of Poincaré theta series for the following reasons.

(i) Presented with an automorphic function, its zeros are not explicit if it is represented in terms of Poincaré theta series, but are so if it is represented in terms of the prime function. Furthermore, note that for a Poincaré theta series (D.2), labelling the number of its zeros and poles in the fundamental region as Z and P respectively, it is known [60] that

$$Z - P = 2mN \tag{E.4}$$

where N is the order of the associated Schottky group. Having chosen $Q_d(\zeta)$ we have no control over the positions of the zeros of $\mathcal{P}_d(\zeta)$ in F and they will give rise to possibly unwanted poles of the ratio (E.3) unless we choose $Q_n(\zeta)$ in such a way that these are also zeros of $\mathcal{P}_n(\zeta)$. However, these zeros of $\mathcal{P}_d(\zeta)$ are not known explicitly and must therefore be found (numerically) as part of the solution. Furthermore, note that clearly a zero of $Q_n(\zeta)$ is not necessarily a zero of $\mathcal{P}_n(\zeta)$. Thus it is not straightforward to pick $Q_n(\zeta)$ so that $\mathcal{P}_n(\zeta)$ possesses the same zeros as $\mathcal{P}_d(\zeta)$ in F.

(ii) A particular advantage of using the prime function representation of automorphic functions concerns changes of topology, specifically, changes in the connectivity N of the domain. Any function $f(\zeta)$ with a given distribution of poles $\{\alpha_k | k = 1, ..., M\}$ and automorphic with respect to some Schottky group Θ may be represented in the form (2.2) regardless of Θ. If Θ changes, for example if its order N changes, then of course the prime function changes, but the form of the representation of $f(\zeta)$ in terms of this prime function does not. However, such changes in Θ *do* alter the form of the representation of $f(\zeta)$ in terms of Poincaré theta series. Note that by (E.4), any changes in N mean that the number of zeros in the fundamental region of $\mathcal{P}_d(\zeta)$ alters, and thus we must modify $Q_n(\zeta)$ in order that the zeros of $\mathcal{P}_d(\zeta)$ do not give rise to unwanted poles of the ratio (E.3). Note that such changes in topology are relevant for example, in the context of constructing conformal maps to quadrature domains which represent regions of fluid, where the number of holes in the fluid region may change as it evolves.

(iii) Finally, it must be mentioned that in the computations performed for §5.1 it appeared that the Poincaré theta series representations converged better than the prime function representations, by which we mean they reached convergence for fewer maps in the truncation of the infinite group Θ. However, as is illustrated in the figures produced, even at very low levels of truncation the two different representations appear to agree.

APPENDIX **F**

Properties of $R_j(\zeta;\alpha)$

In this appendix, properties of the prime function are used to establish properties of the function $R_j(\zeta;\alpha)$ given by (3.31) in terms of which we write our representation (3.27) for the modified Green's function $G_j(\zeta;\alpha)$ of a general circular domain.

F.1 An alternative expression.

For $j = 0, 1, ..., N$, let us show that $R_j(\zeta;\alpha)$ can be expressed as the function $\hat{R}_j(\zeta;\alpha)$ given by (3.29) multiplied by a factor which is independent of ζ, though not necessarily α. Comparing (3.31) and (3.29) we see that this is so if

$$\frac{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\overline{\omega}(\phi_j(\zeta),\overline{\alpha})} \tag{F.1}$$

is $\hat{R}_j(\zeta;\alpha)$ multiplied by such a factor.

We have

$$\begin{aligned}
\frac{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\overline{\omega}(\phi_j(\zeta),\overline{\alpha})} &= \frac{\phi_j(\alpha)}{\overline{\alpha}}\frac{\omega((\phi_j(\zeta))^{-1},(\phi_j(\alpha))^{-1})}{\omega((\phi_j(\zeta))^{-1},\overline{\alpha}^{-1})} \quad \text{by (1.62)} \\
&= \frac{\phi_j(\alpha)}{\overline{\alpha}}\frac{\omega((\phi_j(\zeta))^{-1},(\phi_j(\alpha))^{-1})}{\omega((\phi_j(\zeta))^{-1},(\phi_j(\overline{\phi_j(\alpha)}))^{-1})} \quad \text{by (A.4)} \\
&= \frac{\phi_j(\alpha)}{\overline{\alpha}}\frac{\omega(\theta_{-j}(\zeta),\theta_{-j}(\alpha))}{\omega(\theta_{-j}(\zeta),\theta_{-j}(\overline{\phi_j(\alpha)}))} \quad \text{by (1.9)} \\
&= \frac{\phi_j(\alpha)}{\overline{\alpha}}\left(\frac{c\overline{\phi_j(\alpha)}+d}{c\alpha+d}\right)\hat{R}_j(\zeta) \quad \text{by (1.56) and (3.29)}
\end{aligned} \tag{F.2}$$

where $c,\ d$ are coefficients of $\theta_{-j}(\zeta)$ written in the form

$$\theta_{-j}(\zeta) = \frac{a\zeta+b}{c\zeta+d} \tag{F.3}$$

Thus, (F.1) is clearly $\hat{R}_j(\zeta;\alpha)$ multiplied by a factor which is independent of ζ, though not α. Hence $R_j(\zeta;\alpha)$ is also $\hat{R}_j(\zeta;\alpha)$ multiplied by such a factor.

F.2 Boundary values of $R_j(\zeta;\alpha)$.

Let us consider the values of $R_j(\zeta;\alpha)$ for ζ on the boundaries of the circular domain D_ζ. Begin by considering the complex conjugate of $R_j(\zeta;\alpha)$. From (3.31) we have

$$\overline{R_j(\zeta;\alpha)} = \left(\frac{\overline{\omega}(\bar{\zeta},\bar{\alpha})\omega(\overline{\phi}_j(\bar{\zeta}),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\bar{\zeta},\phi_j(\alpha))\omega(\overline{\phi}_j(\bar{\zeta}),\alpha)}\right)^{1/2}. \tag{F.4}$$

Now suppose $\zeta \in C_j$. From (A.7) we have

$$\bar{\zeta} = \phi_j(\zeta) \tag{F.5}$$

and so from (F.4),

$$\overline{R_j(\zeta;\alpha)} = \left(\frac{\overline{\omega}(\phi_j(\zeta),\bar{\alpha})\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))\omega(\zeta,\alpha)}\right)^{1/2} = \frac{1}{R_j(\zeta;\alpha)}. \tag{F.6}$$

This confirms that

$$|R_j(\zeta;\alpha)| = 1 \quad \text{for } \zeta \in C_j. \tag{F.7}$$

Consider next $\zeta \in C_0$. From (F.4),

$$\begin{aligned}\overline{R_j(\zeta;\alpha)} &= \left(\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})\omega(\overline{\phi}_j(\zeta^{-1}),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))\omega(\overline{\phi}_j(\zeta^{-1}),\alpha)}\right)^{1/2} \\ &= \left(\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})\omega(\theta_j(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))\omega(\theta_j(\zeta),\alpha)}\right)^{1/2}.\end{aligned} \tag{F.8}$$

where we have used the fact that $\bar{\zeta} = \zeta^{-1}$ for ζ on C_0 and the identity (1.4). But from (1.49),

$$\frac{\omega(\theta_j(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_j(\zeta),\alpha)} = \mathcal{V}_j(\overline{\phi}_j(\bar{\alpha}),\alpha)\frac{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\omega(\zeta,\alpha)} \tag{F.9}$$

where recall $\mathcal{V}_j(\gamma_1,\gamma_2)$ is given by (1.41). Similarly,

$$\frac{\overline{\omega}(\overline{\theta}_j(\zeta^{-1}),\bar{\alpha})}{\overline{\omega}(\overline{\theta}_j(\zeta^{-1}),\phi_j(\alpha))} = \overline{\mathcal{V}}_j(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))} \tag{F.10}$$

or equivalently,

$$\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))} = \overline{\mathcal{V}}_j(\phi_j(\alpha),\bar{\alpha})\frac{\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}. \tag{F.11}$$

where we have used the property (1.44) of $\mathcal{V}_j(\gamma_1,\gamma_2)$.

On use of (F.9) and (F.11) in (F.8), we get

$$|R_j(\zeta;\alpha)| = |\mathcal{V}_j(\overline{\phi}_j(\bar{\alpha}),\alpha)|^{1/2} \quad \text{for } \zeta \in C_0. \tag{F.12}$$

Finally, consider points ζ on C_k where $k \neq j$. From (F.4),

$$
\begin{aligned}
\overline{R_j(\zeta;\alpha)} &= \left(\frac{\overline{\omega}(\phi_k(\zeta),\bar{\alpha})\omega(\overline{\phi}_j(\phi_k(\zeta)),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\phi_k(\zeta),\phi_j(\alpha))\omega(\overline{\phi}_j(\phi_k(\zeta)),\alpha)} \right)^{1/2} \\
&= \left(\frac{\overline{\omega}(\overline{\theta}_k(\zeta^{-1}),\bar{\alpha})}{\overline{\omega}(\overline{\theta}_k(\zeta^{-1}),\phi_j(\alpha))} \frac{\omega(\theta_j((\phi_k(\zeta))^{-1}),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_j((\phi_k(\zeta))^{-1}),\alpha)} \right)^{1/2} \\
&= \left(\frac{\mathcal{V}_j(\overline{\phi}_j(\bar{\alpha}),\alpha))}{\overline{\mathcal{V}}_k(\phi_j(\alpha),\bar{\alpha})} \frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))} \frac{\omega((\phi_k(\zeta))^{-1},\overline{\phi}_j(\bar{\alpha}))}{\omega((\phi_k(\zeta))^{-1},\alpha)} \right)^{1/2} \\
&= \left(\frac{\mathcal{V}_j(\overline{\phi}_j(\bar{\alpha}),\alpha))}{\overline{\mathcal{V}}_k(\phi_j(\alpha),\bar{\alpha})} \frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))} \frac{\omega(\theta_k^{-1}(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_k^{-1}(\zeta),\alpha)} \right)^{1/2}
\end{aligned}
\tag{F.13}
$$

But by (1.49),

$$
\frac{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\omega(\zeta,\alpha)} = \mathcal{V}_k(\overline{\phi}_j(\bar{\alpha}),\alpha) \frac{\omega(\theta_k^{-1}(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_k^{-1}(\zeta),\alpha)} \tag{F.14}
$$

Then on use of (F.11) and (F.14) in (F.13) we get

$$
|R_j(\zeta;\alpha)| = \left(\left| \frac{\mathcal{V}_j(\overline{\phi}_j(\bar{\alpha}),\alpha)}{\mathcal{V}_k(\overline{\phi}_j(\bar{\alpha}),\alpha)} \right| \right)^{1/2} \quad \text{for } \zeta \in C_k,\ k \neq j. \tag{F.15}
$$

To summarize all the above results, we conclude that for $j,k \in \{0,1,...,N\}$,

$$
|R_j(\zeta;\alpha)| = \left| \frac{\mathcal{V}_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\mathcal{V}_k(\overline{\phi_j}(\bar{\alpha}),\alpha)} \right|^{1/2} \quad \text{for } \zeta \in C_k \tag{F.16}
$$

provided we adopt the convention that $\mathcal{V}_0(\zeta,\alpha) \equiv 1$.

APPENDIX G

A particular class of slit mappings.

We shall now present some conformal maps from circular domains to domains with straight boundaries. In particular we shall construct maps to three different types of such domains: (i) "slit half-planes" bounded by a single infinite straight line and slits; (ii) domains exterior to chains of slits; (iii) domains bounded by an infinite straight line with gaps. As will be seen, these mappings can be represented in terms of the prime function associated with the circular domain. Note that these maps are special cases of Schwarz-Christoffel mappings to multiply-connected polygonal domains.

G.1 Domains bounded by slits in a half-plane.

Let D_ζ be a circular domain of the usual form. Now consider the conformal mapping given by the simple form

$$z(\zeta;\gamma_1,\gamma_2) = R\,\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}, \tag{G.1}$$

where $R \in \mathbb{C}$ is a complex constant and γ_1 and γ_2 are taken to be two distinct points on the unit ζ-circle. Let us now describe the main properties of this map (G.1). First, notice (G.1) clearly has a simple zero at $\zeta = \gamma_1$ and a simple pole at $\zeta = \gamma_2$. Now consider the argument of $z(\zeta)$ for points on the circular boundaries of D_ζ. First, for ζ on C_0,

$$\begin{aligned}
\overline{z(\zeta;\gamma_1,\gamma_2)} &= \overline{R}\frac{\overline{\omega(\zeta,\gamma_1)}}{\overline{\omega(\zeta,\gamma_2)}} \\
&= \overline{R}\frac{\overline{\gamma_1}\omega(\overline{\zeta}^{-1},\overline{\gamma_1}^{-1})}{\overline{\gamma_2}\omega(\overline{\zeta}^{-1},\overline{\gamma_2}^{-1})} \\
&= \overline{R}\frac{\gamma_2\omega(\zeta,\gamma_1)}{\gamma_1\omega(\zeta,\gamma_2)} \\
&= \frac{\overline{R}}{R}\frac{\gamma_2}{\gamma_1}z(\zeta;\gamma_1,\gamma_2)
\end{aligned} \tag{G.2}$$

where the second equality follows from (1.62) and the third equality follows from the fact that $\gamma_1,\ \gamma_2$ and also ζ lie on the unit circle. From (G.2) it follows that the argument of $z(\zeta;\gamma_1,\gamma_2)$ is constant for ζ on C_0, except that at $\zeta=\gamma_1$ and $\zeta=\gamma_2$, since $z(\zeta;\gamma_1,\gamma_2)$ has respectively a simple zero and a simple pole, its argument jumps by π at each of these points. Thus, the image of C_0 under $z(\zeta;\gamma_1,\gamma_2)$ is an infinite straight line through the origin $z=0$, where the argument on the section of this line on one side of the origin is

$$\arg[z(\zeta;\gamma_1,\gamma_2)] = \arg[R] + \frac{1}{2}\arg\left[\frac{\gamma_1}{\gamma_2}\right], \qquad \zeta \in C_0 \tag{G.3}$$

while that on the other differs from (G.3) by π. Now, for ζ on one of the inner circles $C_j,\ j\in\{1,...,N\}$ we have

$$\begin{aligned}
\overline{z(\zeta;\gamma_1,\gamma_2)} &= \overline{R}\frac{\overline{\omega(\zeta,\gamma_1)}}{\overline{\omega(\zeta,\gamma_2)}}\\
&= \overline{R}\frac{\overline{\omega(\theta_j(\overline{\zeta}^{-1}),\gamma_1)}}{\overline{\omega(\theta_j(\overline{\zeta}^{-1}),\gamma_2)}}\\
&= \overline{R}\,\overline{\mathcal{V}_j(\gamma_1,\gamma_2)}\frac{\overline{\omega(\overline{\zeta}^{-1},\gamma_1)}}{\overline{\omega(\overline{\zeta}^{-1},\gamma_2)}}\\
&= \overline{R}\,\overline{\mathcal{V}_j(\gamma_1,\gamma_2)}\frac{\overline{\omega(\overline{\zeta}^{-1},\overline{\gamma_1}^{-1})}}{\overline{\omega(\overline{\zeta}^{-1},\overline{\gamma_2}^{-1})}}\\
&= \overline{R}\,\overline{\mathcal{V}_j(\gamma_1,\gamma_2)}\frac{\gamma_2}{\gamma_1}\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}\\
&= \frac{\overline{R}}{R}\overline{\mathcal{V}_j(\gamma_1,\gamma_2)}\frac{\gamma_2}{\gamma_1}z(\zeta;\gamma_1,\gamma_2)
\end{aligned} \tag{G.4}$$

where the second equality follows from the fact that for $\zeta\in C_j,\ \ \zeta=\theta_j(\overline{\zeta}^{-1})$; the third equality follows from (1.49); the fourth equality follows from the fact that γ_1 and γ_2 lie on the unit circle; and the fifth equality follows from (1.62). It follows from (G.4) that the argument of $z(\zeta;\gamma_1,\gamma_2)$ is constant for ζ on C_j. Notice also that, taking the modulus of both sides of (G.4) shows

$$|\mathcal{V}_j(\gamma_1,\gamma_2)| = 1, \qquad \text{for } \gamma_1,\ \gamma_2 \in C_0 \tag{G.5}$$

Thus, in fact

$$\arg[z(\zeta;\gamma_1,\gamma_2)] = \arg[R] + \frac{1}{2}\arg\left[\frac{\gamma_1}{\gamma_2}\right] + \frac{1}{2}\arg\Big[\mathcal{V}_j(\gamma_1,\gamma_2)\Big], \qquad \zeta\in C_j \tag{G.6}$$

Thus $z(\zeta;\gamma_1,\gamma_2)$ maps each of the interior circles $\{C_j|j=1,...,M\}$ to a finite-length radial slit on some ray emanating from the origin $z=0$. For $j=1,...,N$, notice from (G.3) and (G.6) that the angle made by the slit image of C_j with the infinite straight line image of C_0 is

$$\frac{1}{2}\arg\Big[\mathcal{V}_j(\gamma_1,\gamma_2)\Big] \tag{G.7}$$

Suppose we fix $\gamma_1,\ \gamma_2$ and R. This fixes the orientation of the infinite straight line image of C_0. Now suppose we wish the inner circles $C_1, ..., C_j$ to map to a specified distribution of slits. Specifically, suppose for $j = 1, ..., N$ we wish C_j to map to a slit oriented at an angle ϕ_j to the image of C_0 with its two endpoints at distances $a_j,\ b_j$ from the origin. In general, it is not known *a priori* which two points on C_j map to the end-points of the slit in the z-plane. The end-points of the slit in the z-plane are

$$a_j e^{i\phi_j}, \quad b_j e^{i\phi_j} \tag{G.8}$$

where the real parameters a_j, b_j and ϕ_j are specified. Let the two pre-image points on C_j corresponding to the end-points of the slit be

$$\delta_j + q_j e^{ir_j}, \quad \delta_j + q_j e^{is_j}, \tag{G.9}$$

where r_j and s_j are real parameters. The latter parameters must be found as part of the solution. The required additional equations are provided by the fact that, at these pre-image points, the derivative of the conformal mapping must vanish. Thus, the equations to be solved in this case for the parameters $\{q_j, \delta_j | j = 1, ..., M\}$, as well as the subsidiary parameters $\{r_j, s_j | j = 1, ..., M\}$, are

$$\begin{aligned} |z(\delta_j + q_j e^{ir_j})| = a_j, \quad |z'(\delta_j + q_j e^{ir_j})| = 0, \\ |z(\delta_j + q_j e^{is_j})| = b_j, \quad |z'(\delta_j + q_j e^{is_j})| = 0, \end{aligned} \tag{G.10}$$

(where $z'(\zeta)$ denotes the derivative of $z(\zeta)$ with respect to ζ), along with the real equation

$$\phi_j = \frac{1}{2}\arg\Big[\mathcal{V}_j(\gamma_1,\gamma_2)\Big] \tag{G.11}$$

which follows from (G.6). Note that (G.10) and (G.11) represent five real equations for the five real unknowns, i.e. $\text{Re}[\delta_j], \text{Im}[\delta_j], q_j, r_j$ and s_j. These nonlinear equations are readily solved using Newton's method.

Now suppose we in fact pick R to be real and make the particular choices

$$\gamma_1 = 1, \quad \gamma_2 = -1 \tag{G.12}$$

Then under (G.3) C_0 maps to the imaginary z-axis. It turns out that if we pick R to be negative then the interior of C_0 maps to the right half-plane. Otherwise, if we pick R to be positive then the interior of C_0 maps to the left half-plane. Now suppose also that the inner circles $C_1, ..., C_N$ are all centred on the real ζ-axis so that the circular domain D_ζ is reflectionally symmetric in the real axis. Then,

$$\begin{aligned}
\overline{z(\zeta;\gamma_1,\gamma_2)} &= R\overline{\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}} \\
&= R\frac{\overline{\omega}(\overline{\zeta},\gamma_1)}{\overline{\omega}(\overline{\zeta},\gamma_2)} \\
&= R\frac{\omega(\overline{\zeta},\gamma_1)}{\omega(\overline{\zeta},\gamma_2)} \\
&= z(\overline{\zeta};\gamma_1,\gamma_2)
\end{aligned} \tag{G.13}$$

where the third equality follows from proposition 1.4.4. Thus it follows that points in D_ζ on the real ζ-axis map to points on the positive real z-axis, with $\zeta = \gamma_1 = 1$ mapping to the origin and $\zeta = \gamma_2 = -1$ mapping to the point at infinity. Furthermore, for $j = 1, ..., N$ the points $\delta_j \pm q_j$ where C_j crosses the real ζ-axis must map onto the real z-axis, and so since the image of C_j is a slit, this slit must lie along the real z-axis. In fact, the section of C_j in the upper half ζ-plane maps to the upper half of this slit, while the section of C_j in the lower half ζ-plane maps to the lower half. And if we label the endpoints of the slit as a_j and b_j, where $|a_j| < |b_j|$, we have

$$z(\delta_j + q_j) = a_j \quad \text{and} \quad z(\delta_j - q_j) = b_j, \quad j = 1, ..., M. \tag{G.14}$$

So in this particular case, to solve for the two real parameters δ_j and q_j we need only solve the two real equations (G.14), rather than the five real equations (G.10) and (G.11).

G.2 Domains bounded by slits in a line

We shall now describe a conformal mapping from a circular domain D_ζ to a domain D bounded by slits in a line. We construct this map as a sequence of maps, including the map (G.1). Specifically, consider the map given by the sequence of mappings

$$\begin{aligned} \zeta_1(\zeta) &= R\frac{\omega(\zeta,1)}{\omega(\zeta,-1)}; \\ \zeta_2(\zeta_1) &= \frac{1-\zeta_1}{1+\zeta_1}; \\ \zeta_3(\zeta_2) &= \frac{L}{2}\left(\frac{1}{\zeta_2}+\zeta_2\right). \end{aligned} \tag{G.15}$$

where R and L are real, and in fact R is negative. Also it is assumed that for $j = 1, ..., N$ the inner circle C_j is centred at δ_j on the real ζ-axis, of radius q_j, so that the circular domain is reflectionally symmetric in the real ζ-axis.

As just argued in the previous section, the first mapping in the sequence (G.15) maps C_0 onto the imaginary ζ_1 axis and the interior of C_0 onto the right half ζ_1 plane, while for $j = 1, ..., N$ the inner circle C_j maps onto a slit lying on the positive real ζ_1-axis.

The second map in the sequence (G.15) is a Möbius map. It is straightforward to show that this maps the imaginary ζ_1-axis onto the unit circle given by $|\zeta_2| = 1$, with $\zeta_1 = 0$ mapping to $\zeta_2 = 1$ and $\zeta_1 = \infty$ mapping to $\zeta_2 = -1$. The right half ζ_1 plane maps onto the interior of this circle, with the positive real ζ_1-axis mapping onto the section of the real ζ_2-axis between $\zeta_2 = \pm 1$. Thus the slit images of $C_1, ..., C_N$ in the ζ_1 plane map onto slits on this section of the real ζ_2 axis.

Finally, it is straightforward to check that the third map in the sequence (G.15) maps the unit ζ_2 circle onto a slit on the real ζ_3 axis with endpoints L and $-L$ which are respectively the images of $\zeta_2 = 1$ and $\zeta_2 = -1$. This slit is of course the image under the whole sequence of mappings of the unit circle C_0 of the original circular domain D_ζ. Also, the interior of the unit ζ_2 circle maps onto the whole of the plane exterior to this slit, with $\zeta_2 = 0$ mapping to $\zeta_3 = \infty$. The section of the real ζ_2 axis between $\zeta_2 = \pm 1$ maps onto the sections of the real ζ_3-axis either side of the slit between $\pm L$. Thus the slit images of $C_1, ..., C_N$ in the ζ_2 plane map onto slits on the real ζ_3-axis.

A schematic illustrating an example of this composition of mappings is shown in figure G.1.

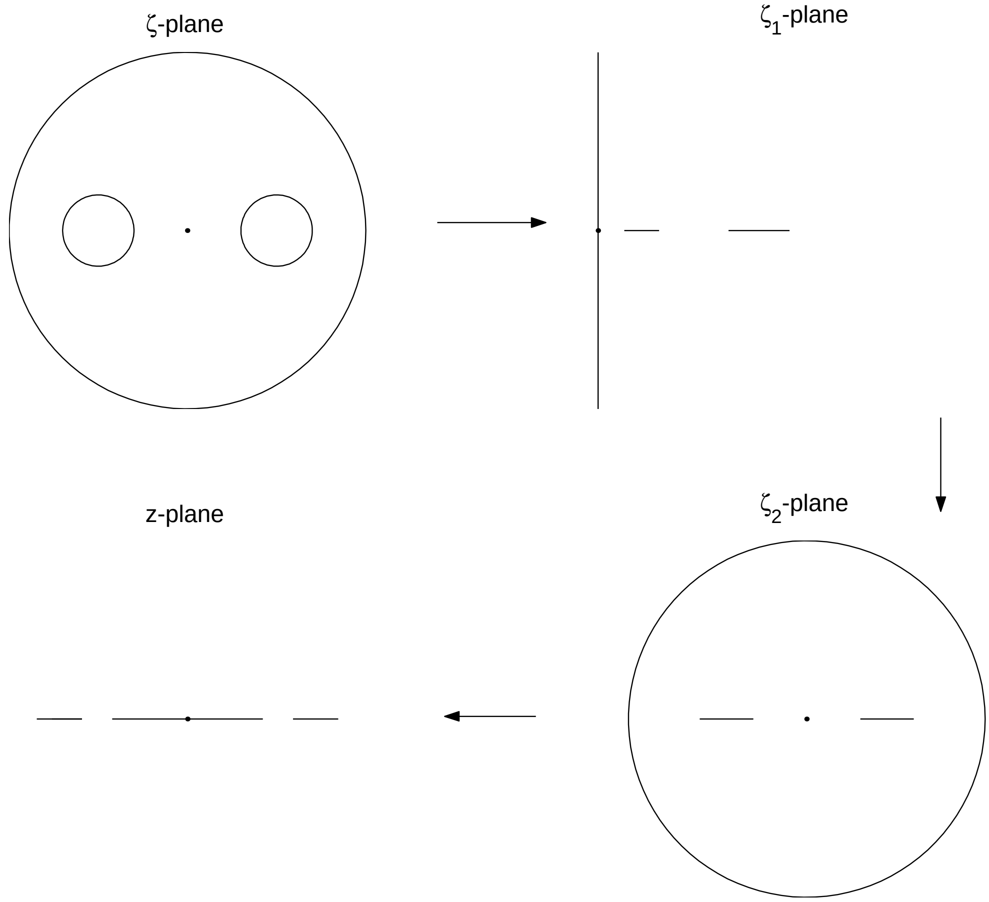

Fig. G.1: Schematic illustrating an example of the sequence of conformal mappings (G.15). The origin in each plane is shown as a dot.

Taking $z = \zeta_3$, it is straightforward to show that the composition of the sequence of maps G.15 is given by

$$z(\zeta) = 2L\frac{\omega(\zeta,-1)^2 + R^2\omega(\zeta,1)^2}{\omega(\zeta,-1)^2 - R^2\omega(\zeta,1)^2} \tag{G.16}$$

To summarize the above, for *any* choice of R or L, the map (G.16) given by the sequence of mappings (G.15) *always* maps the unit ζ circle C_0 bounding the circular domain D_ζ to a finite slit with endpoints $\pm L$ on the real z-axis, with the interior of C_0 mapping onto the whole plane exterior to this slit. The inner circles $C_1, ..., C_N$ bounding D are mapped to slits on the real z-axis.

Note that the sequence of maps (G.15) may of course be supplemented by additional mappings. For example, if we applied the Möbius map $\zeta_4(\zeta_3) = \zeta_3^{-1}$ which is simply an inversion, then the image of the unit circle C_0 in the ζ_4 plane consists of two semi-infinite slits which join up at the point at infinity and which have endpoints at $\zeta_4 = \pm L^{-1}$. The images of the inner circles $C_1, ..., C_N$ are all finite slits along the real ζ_4-axis between $\zeta_4 = \pm L^{-1}$.

Note that in particular, taking $R = -1$, (G.16) becomes

$$z(\zeta) = 2L\frac{\omega(\zeta,-1)^2 + \omega(\zeta,1)^2}{\omega(\zeta,-1)^2 - \omega(\zeta,1)^2} \tag{G.17}$$

Now suppose also that the distribution of circles $C_1, ..., C_N$ is also reflectionally symmetric in the imaginary ζ-axis. Then,

$$\begin{aligned} z(-\zeta) &= 2L\frac{\omega(-\zeta,-1)^2 + \omega(-\zeta,1)^2}{\omega(-\zeta,-1)^2 - \omega(-\zeta,1)^2} \\ &= 2L\frac{\omega(\zeta,1)^2 + \omega(\zeta,-1)^2}{\omega(\zeta,1)^2 - \omega(\zeta,-1)^2} \\ &= -z(\zeta) \end{aligned} \tag{G.18}$$

where the second equality follows from proposition 1.4.6. Thus in this case, in the z-plane, the image of C_0 is a slit with endpoints L and $-L$, while it follows from (G.18) that the distribution of the slit images of the circles $C_1, ..., C_N$ on the real z axis is reflectionally symmetric in the imaginary z-axis. Note that in this case, if N is odd, then one of these circles in the interior of C_0, say C_1, must be centred on the origin. But by proposition 1.4.6

$$\omega(0,-1)^2 - \omega(0,1)^2 = 0 \tag{G.19}$$

and thus (G.17) maps $\zeta = 0$ to $z = \infty$. Thus, in this case C_1 maps to two semi-infinite sections of the real z-axis which join up at infinity.

We point out that these maps from circular domains to slits domains provide an alternative to those presented in §4.2.4 and §4.2.5.

APPENDIX H

Further details of the vortical solutions of §5.1.3.

In this appendix we present further details of the vortical solutions of the Euler equations described in §5.1.3.

H.1 Stationarity condition

Recall from (5.29) that in the co-rotating frame the velocity field in D is given by

$$u - iv = \frac{i\omega}{2}(\overline{z} - S(z)) \quad z \in D \tag{H.1}$$

We shall now derive the conditions for the point vortices to be stationary in the co-rotating frame?

Suppose in D there is a point vortex of circulation Γ_k at $z = z_k = z(\zeta_k)$, where $\zeta_k \in D_\zeta$. Then $S(z)$ must have a simple pole at $z = z_k$. For this point vortex to be stationary the non-self-induced velocity must be zero at $z = z_k$

For z near z_k we have the Taylor series expansion,

$$S(z) = \frac{s_{-1}}{z - z_k} + s_0 + s_1(z - z_k) + \mathcal{O}(z - z_k)^2 \tag{H.2}$$

Hence for z near z_k the velocity field (H.1) is given by

$$u - iv = \frac{i\omega}{2}\left(\overline{z} - \frac{s_{-1}}{z - z_k} - s_0 - s_1(z - z_k) + \mathcal{O}(z - z_k)^2\right) \tag{H.3}$$

The contribution due to the point vortex at $z = z_k$ is of course

$$-\frac{i\omega}{2}\frac{s_{-1}}{z - z_k} \tag{H.4}$$

Thus, the non-self-induced velocity on the point vortex at $z = z_k$ is *exactly*

$$\frac{i\omega}{2}(\overline{z_k} - s_0) \tag{H.5}$$

So the condition for the point vortex at $z = z_k$ to be stationary in the co-rotating frame is

$$\overline{z_k} - s_0 = 0 \tag{H.6}$$

So we must determine the expansion (H.2). Of course we have an explicit expression for $S(z(\zeta))$ in terms of ζ. Can we use this?

The map $z(\zeta)$ is a conformal map and so may be inverted to give us ζ as a function of z. If we could invert our expression in terms of ζ for $S(z(\zeta))$ into an expression in terms of z, possibly even only valid for z near z_k, then we could determine the coefficient s_0.

Since $S(z)$ has a simple pole at $z = z_k$, $S(z(\zeta))$ must have a simple pole at a point $\zeta = \zeta_k$. So, we can write

$$S(z(\zeta)) = \frac{A(\zeta)}{\zeta - \zeta_k} \tag{H.7}$$

where $A(\zeta)$ is analytic at $\zeta = \zeta_k$. Note that this expression is valid for all ζ.

For ζ near ζ_k we have the Taylor series expansion,

$$z(\zeta) = z_k + (\zeta - \zeta_k) z_\zeta(\zeta_k) + \frac{1}{2}(\zeta - \zeta_k)^2 z_{\zeta\zeta}(\zeta_k) + \mathcal{O}(\zeta - \zeta_k)^3 \tag{H.8}$$

Alternatively, considering $\zeta(z)$, for z near z_k we have the Taylor series expansion,

$$\zeta(z) = \zeta_k + (z - z_k)\zeta_z(z_k) + \mathcal{O}(z - z_k)^2 \tag{H.9}$$

From (H.8) we have for ζ near ζ_k,

$$\zeta - \zeta_k = \frac{z - z_k}{z_\zeta(\zeta_k)} + \mathcal{O}(\zeta - \zeta_k)^2 \tag{H.10}$$

But from (H.9), for z near z_k terms of order $(\zeta - \zeta_k)^2$ written in terms of z are of order $(z - z_k)^2$. So for z near z_k we have the following expansion in terms of z

$$\zeta - \zeta_k = \frac{z - z_k}{z_\zeta(\zeta_k)} + \mathcal{O}(z - z_k)^2 \tag{H.11}$$

For ζ near ζ_k, since $A(\zeta)$ is analytic at ζ_k we have the Taylor series expansion

$$A(\zeta) = A(\zeta_k) + (\zeta - \zeta_k) A_\zeta(\zeta_k) + \mathcal{O}(\zeta - \zeta_k)^2 \tag{H.12}$$

Hence, from (H.11), we have the following expansion for $A(\zeta)$ in terms of z for z near z_k,

$$A(\zeta) = A(\zeta_k) + (z - z_k)\frac{A_\zeta(\zeta_k)}{z_\zeta(\zeta_k)} + \mathcal{O}(z - z_k)^2 \tag{H.13}$$

Now, note that from (H.8), we have the following expansion for ζ near ζ_k,

$$\frac{1}{\zeta - \zeta_k} = \frac{z_\zeta(\zeta_k)}{z - z_k} + \frac{1}{2}\left(\frac{\zeta - \zeta_k}{z - z_k}\right) z_{\zeta\zeta}(\zeta_k) + \mathcal{O}(\zeta - \zeta_k) \tag{H.14}$$

Hence, using (H.11) we have the following expansion in terms of z for z near z_k

$$\frac{1}{\zeta - \zeta_k} = \frac{z_\zeta(\zeta_k)}{z - z_k} + \frac{1}{2}\frac{z_{\zeta\zeta}(\zeta_k)}{z_\zeta(\zeta_k)} + \mathcal{O}(z - z_k) \tag{H.15}$$

It follows that, we have the following expansion in terms of z for z near z_k

$$\begin{aligned} S(z) &= \left(A(\zeta_k) + (z - z_k)\frac{A_\zeta(\zeta_k)}{z_\zeta(\zeta_k)} + \mathcal{O}(z - z_k)^2\right)\left(\frac{z_\zeta(\zeta_k)}{z - z_k} + \frac{1}{2}\frac{z_{\zeta\zeta}(\zeta_k)}{z_\zeta(\zeta_k)} + \mathcal{O}(z - z_k)\right) \\ &= \frac{A(\zeta_k) z_\zeta(\zeta_k)}{z - z_k} + \left(A_\zeta(\zeta_k) + \frac{1}{2}\frac{z_{\zeta\zeta}(\zeta_k)}{z_\zeta(\zeta_k)} A(\zeta_k)\right) + \mathcal{O}(z - z_k) \end{aligned} \tag{H.16}$$

Thus

$$s_0 = \left(A_\zeta(\zeta_k) + \frac{1}{2}\frac{z_{\zeta\zeta}(\zeta_k)}{z_\zeta(\zeta_k)} A(\zeta_k)\right) \tag{H.17}$$

So recalling (H.6), the stationarity condition is

$$\overline{z_k} - \left(A_\zeta(\zeta_k) + \frac{1}{2}\frac{z_{\zeta\zeta}(\zeta_k)}{z_\zeta(\zeta_k)} A(\zeta_k)\right) = 0 \tag{H.18}$$

H.2 Circulations

We now derive formulae for the circulations of the vortices.

H.2.1 Circulation of satellite point vortices

Recalling (H.4), the circulation Γ_k of the point vortex at $z = z_k$ is given by

$$\Gamma_k = \pi\omega s_{-1} \tag{H.19}$$

From (H.16) we have

$$s_{-1} = A(\zeta_k) z_\zeta(\zeta_k) \tag{H.20}$$

Thus,

$$\Gamma_k = \pi\omega A(\zeta_k) z_\zeta(\zeta_k) \tag{H.21}$$

H.2.2 Circulation of vortex patches

The circulation of any vortex patch is just its vorticity multiplied by its area. The area of a patch D_j of one of our solutions may be computed as follows.

The area of D_j is of course

$$\int\int_{D_j} dxdy \tag{H.22}$$

It follows from Green's theorem that this is equal to

$$\frac{1}{2i}\oint_{\partial D_j} \overline{z} dz \tag{H.23}$$

where we integrate round ∂D_j in a positive direction (i.e., so that D_j is on the left).

Now parameterize z in terms of ζ. ∂D_j is the image of the circle C_j centre δ_j, radius q_j. Then we have

$$\text{area of } D_j = \left|\frac{1}{2}\oint_{C_j} \overline{z}(\overline{\zeta}) z_\zeta(\zeta) d\zeta\right| \tag{H.24}$$

And, for ζ on C_j,

$$\zeta = \delta_j + q_j e^{i\theta} \tag{H.25}$$

for θ between 0 and 2π. Thus, for ζ on C_j,

$$d\zeta = i q_j e^{i\theta} d\theta \tag{H.26}$$

and so

$$\text{area of } D_j = \left|\frac{q_j}{2}\int_0^{2\pi} \overline{z}(\overline{\delta_j} + q_j e^{-i\theta}) z_\zeta(\delta_j + q_j e^{i\theta}) d\theta\right| \tag{H.27}$$

H.3 Solutions with two patches.

In this section we shall provide further details of the solutions discussed in §5.1.5.

H.3.1 Derivation of conformal map.

We shall now provide a justification for the form chosen for the conformal map (5.31).

There are two point vortices in D at z_1 and z_2. Label the pre-images of these points in D_ζ as ζ_1 and ζ_2 respectively. Then it follows that $z(\zeta)$ must have simple poles at the points $\alpha_j = \overline{\zeta_j}^{-1}$, $j = 1, 2$ in $\hat{D}_\zeta$. Note also that D is infinite. Thus $z(\zeta)$ must also have a pole in D_ζ, say at α_3. These are all the poles of $z(\zeta)$ in F. It follows that $z(\zeta)$ must have three zeros in F. Note D contains $z = 0$. Thus one of these zeros must be contained in D_ζ, say at β_3. The two remaining zeros must be contained in $\hat{D}_\zeta$, say at β_1 and β_2.

We choose

$$\begin{aligned}
\alpha_1 &= \sqrt{\rho}^{-1} e^{i\phi} \\
\alpha_2 &= \sqrt{\rho}^{-1} e^{-i\phi} \\
\alpha_3 &= \sqrt{\rho} \\
\beta_1 &= \sqrt{\rho}^{-1} \\
\beta_2 &= -\sqrt{\rho}^{-1} \\
\beta_3 &= -\sqrt{\rho}
\end{aligned} \tag{H.28}$$

where ϕ is some real parameter to be determined. Notice that as required, $\alpha_3, \beta_3 \in D_\zeta$, while the remaining poles and zeros are in $\hat{D}_\zeta$. Let us justify our choices (H.28) for the poles and zeros $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ for the map $z(\zeta)$.

We consider D as the image of the annulus D_ζ bounded by the circles $|\zeta| = 1$ and $|\zeta| = \rho$, which map respectively onto ∂D_1 and ∂D_2. But it is perhaps more natural to consider D as the image of a circular domain $\tilde{D}$ say, in a parametric $\tilde{\zeta}$ plane, where $\tilde{D}$ is the infinite region exterior to two circles, $\tilde{C}_1$,$\tilde{C}_2$ of the same radius $\tilde{q}$ say centred on the real axis at say $\tilde{\delta}$ and $-\tilde{\delta}$ respectively, where $\tilde{C}_1$ maps onto ∂D_1 and $\tilde{C}_2$ maps onto ∂D_2. As we shall now demonstrate, there is such a domain $\tilde{D}$ in the $\tilde{\zeta}$-plane which is conformally equivalent to our chosen annulus domain D_ζ in the ζ-plane. In fact it can be shown that the conformal map between the two is given by

$$\tilde{\zeta} = M(\zeta) = \frac{\zeta + \sqrt{\rho}}{\zeta - \sqrt{\rho}} \tag{H.29}$$

equivalently,

$$\zeta = M^{-1}(\tilde{\zeta}) = \sqrt{\rho}\left(\frac{\tilde{\zeta}+1}{\tilde{\zeta}-1}\right) \tag{H.30}$$

Notice M is a Möbius map and so it obviously a one-to-one conformal transformation of the plane onto itself. So we may consider D as the image of $\tilde{D}$. By symmetry we expect the positions of the point vortices $z_1 = i, z_2 = -i$ to be the images of points $\tilde{\zeta}_1 = ik, \tilde{\zeta}_2 = -ik$ respectively, for some real k. What can we say about the corresponding points ζ_1 and ζ_2 in D_ζ in the ζ plane? It is in fact straightforward to show that M maps the circle described by $|\zeta| = \sqrt{\rho}$ onto the imaginary $\tilde{\zeta}$ axis, with $\zeta = \sqrt{\rho}$ mapping to $\tilde{\zeta} = \infty$, $\zeta = -\sqrt{\rho}$ mapping to $\tilde{\zeta} = 0$. And furthermore for $|\zeta| = \sqrt{\rho}$, $M(\overline{\zeta}) = \overline{M(\zeta)}$. It follows that the points $\tilde{\zeta}_1 = ik$, $\tilde{\zeta}_2 = -ik$ correspond to points $\zeta_1 = \sqrt{\rho}e^{-i\phi}$, $\zeta_2 = \sqrt{\rho}e^{i\phi}$ for some real ϕ. Hence, since $\alpha_k = \overline{\zeta_k}^{-1}$, $k = 1, 2$, we choose $\alpha_1 = \sqrt{\rho}^{-1}e^{i\phi}$ and $\alpha_2 = \sqrt{\rho}^{-1}e^{-i\phi}$. Furthermore, by symmetry we expect $z = 0$ to be the image of $\tilde{\zeta} = 0$, and $z = \infty$ to be the image of $\tilde{\zeta} = \infty$. And as stated above, under the map M, the point $\tilde{\zeta} = 0$ corresponds to $\zeta = -\sqrt{\rho}$, and $\tilde{\zeta} = \infty$ corresponds to $\zeta = \sqrt{\rho}$. Thus we choose $\alpha_3 = \sqrt{\rho}$ and $\beta_3 = -\sqrt{\rho}$.

We propose that our conformal map $z(\zeta)$ from D_ζ onto D is of the form

$$z(\zeta) = R\frac{\prod_{k=1}^{3} P(\zeta/\beta_3, \rho)}{\prod_{k=1}^{3} P(\zeta/\alpha_3, \rho)} \tag{H.31}$$

where R is some multiplicative constant. Note that this is of the form (2.18).

It remains to choose β_1, β_2. First note that the automorphicity condition associated with (H.31) is of the form (2.19). In particular it is given by,

$$\frac{\prod_{j=1}^{3} \alpha_j}{\prod_{j=1}^{3} \beta_j} = 1 \tag{H.32}$$

With the above choices for $\alpha_1, \alpha_2, \alpha_3$ and β_3 if

$$\beta_1\beta_2 = -\rho^{-1} \tag{H.33}$$

then (H.32) is satisfied for any choice of R,ρ and ϕ.

Now note that using the relations (1.97), (1.101) of $P(\zeta, \rho)$ it is possible to show that if

$$\beta_1^2 = \beta_2^2 = \rho^{-1} \tag{H.34}$$

then for all values of R, ρ and ϕ, the form (H.31) possesses the property

$$z(\rho\zeta^{-1}) = -z(\zeta) \tag{H.35}$$

And if we pick R, β_1, β_2 to be real then it is straightforward to see that for all ρ and ϕ we have

$$\overline{z}(\zeta) = z(\zeta) \tag{H.36}$$

where the conjugate function $\overline{z}(\zeta)$ is defined as usual as $\overline{z}(\zeta) = \overline{z(\overline{\zeta})}$.
And if $z(\zeta)$ has the properties (H.35) and (H.36) it can be shown that the circle $|\zeta| = \sqrt{\rho}$ maps to the imaginary z-axis. In particular, the points $\overline{\alpha_1}^{-1}$ and $\overline{\alpha_2}^{-1}$ map to points on the imaginary axis symmetrically placed either side of the origin as required. Furthermore, considering D_ζ divided into the two halves $\sqrt{\rho} < |\zeta| \leq 1$ and $\rho \leq |\zeta| < \sqrt{\rho}$, these are separated by the circle $|\zeta| = \sqrt{\rho}$ and so clearly must map to the left- and right-half z-planes. In fact, notice that the transformation $\zeta \to \rho\zeta^{-1}$ maps $\sqrt{\rho} < |\zeta| \leq 1$ onto $\rho \leq |\zeta| < \sqrt{\rho}$. By (H.35) it follows that for the images under $z(\zeta)$ of these two halves of D_ζ are rotations of one another through π. In particular, the images of the circles $|\zeta| = 1$ and $|\zeta| = \rho$ are rotations of one another through π. Thus the mapping produces two rotationally-symmetric vortex patches as required.

Thus together, (H.33) and (H.34) imply that there must be zeros at $\sqrt{\rho}^{-1}$ and $-\sqrt{\rho}^{-1}$, say $\beta_1 = \sqrt{\rho}^{-1}, \beta_2 = -\sqrt{\rho}^{-1}$.

References

[1] M.J. Ablowitz and A.S. Fokas, Complex variables, Cambridge University Press, (1997).

[2] M. Abramovitz and I. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, (1965).

[3] D.J Acheson, Elementary fluid dynamics, Oxford University Press, (1990).

[4] D. Aharonov & H. Shapiro, Domains in which analytic functions satisfy quadrature identities, *J. Anal. Math.*, **30**, 39–73, (1976).

[5] H. Aref, P.K. Newton, M.A. Stremler, T. Tokieda and D.L. Vainchtein, Vortex crystals, *Adv. Applied Mech.*, **39**, (2002).

[6] H.F. Baker, Abelian functions: Abel's theorem and the allied theory of theta functions, *Cambridge University Press*, (1897).

[7] L.L. Bauer & G.K Morikawa, Stability of rectilinear geostrophic vortices in stationary equilibrium, *Phys. Fluids*, **19**, 929–942, (1976).

[8] A.F. Beardon, A primer on Riemann surfaces, London Mathematical Society Lecture Note Series **78**, Cambridge University Press, (1984).

[9] E.D. Belokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. Its, V.B. Matveev, Algebro-geometric approach to nonlinear integrable equations, Springer Verlag, (1994).

[10] W. Burnside, On a class of automorphic functions, *Proc. London Math. Soc.*, **23**, 49–88, (1891)

[11] W. Burnside, Further note on automorphic functions, *Proc. London Math. Soc.*, (1892)

[12] D.G. Crowdy, General solutions of the 2D Liouville equation, *Int. J. Engng. Sci.*, **35**, 141, (1997).

[13] D.G. Crowdy, A note on viscous sintering and quadrature identities, *Eur. J. Appl. Math.*, **10**, 623–634, (1999).

[14] D.G. Crowdy, Circulation-induced deformations of drops and bubbles: exact two-dimensional models, *Phys. Fluids*, **11**, 2836–2845, (1999).

[15] D.G. Crowdy, A class of exact multipolar vortices, *Phys. Fluids*, **11**(9), 2556–2564, (1999).

[16] D.G. Crowdy, Multipolar vortices and algebraic curves, *Proc. Roy. Soc. A*, **457**, 2337–2359, (2001).

[17] D.G. Crowdy, Theory of exact solutions for the evolution of a fluid annulus in a rotating Hele-Shaw cell, *Q. Appl. Math.*, **LX**(1), 11–36, (2002).

[18] D.G. Crowdy, On the construction of exact multipolar equilibria of the two-dimension Euler equations, *Phys. Fluids*, **14**(1), 257–267, (2002).

[19] D.G. Crowdy, Exact solutions for rotating vortex arrays with finite-area cores, *J. Fluid Mech.*, **469**, 209–235, (2002).

[20] D.G. Crowdy, Viscous sintering of unimodal and bimodal cylindrical packings with shrinking pores, *Eur. J. Appl. Math.*, **14**, 421–445, (2003).

[21] D.G. Crowdy, Quadrature domains and fluid dynamics, *Operator theory: advances and applications*, **156**, (2005).

[22] D.G. Crowdy and H. Kang, Squeeze flow of multiply-connected fluid domains in a Hele-Shaw cell, *J. Nonlin. Sci.*, **11**, 279–304, (2001).

[23] D.G. Crowdy & J.S. Marshall, Constructing multiply connected quadrature domains, *SIAM J. Appl. Math.*, **64**, 1334–1359, (2004).

[24] D.G. Crowdy & J.S. Marshall, Growing vortex patches, *Phys. Fluids*, **16**, 3122–3129, (2004).

[25] D.G. Crowdy and J.S. Marshall, Analytical solutions for rotating vortex arrays involving multiple vortex patches, *J. Fluid Mech.*, **523**, 307–337, (2005).

[26] D.G. Crowdy & J.S. Marshall, Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains, *Proc. Roy. Soc. A*, **461**, 2477–2501, (2005).

[27] D.G. Crowdy & J.S. Marshall, On the motion of a point vortex around multiple circular islands, *Phys. Fluids*, **17**, 056602, (2005).

[28] D.G. Crowdy & J.S. Marshall, The motion of a point vortex through gaps in walls, *J. Fluid Mech.*, **551**, (2006).

[29] D.G. Crowdy & J.S. Marshall, Conformal mappings between canonical multiply connected domains, *Computational Methods and Function Theory*, (to appear)

[30] D.G. Crowdy & J.S. Marshall, Green's functions in multiply connected domains, (in preparation).

[31] D.G. Crowdy & S. Tanveer, A theory of exact solutions for annular viscous blobs, *J. Nonlinear Sci.*, **8**, 375–400, (1998). Erratum, *J. Nonlinear Sci.*, **11**, 237, (2001).

[32] P.J. Davis, The Schwarz function and its applications, Carus Mathematical Monographs, The Mathematical Association of America, Rhode Island, (1974).

[33] W.K. Dewar, Baroclinic eddy interaction with isolated topography, *J. Phys Oceanogr.*, **32**, 2789–2805, (2002).

[34] D.G. Dritschel, The stability and energetics of corotating uniform vortices, *J. Fluid Mech.*, **157**, 95–134, (1985).

[35] M. Embree & L.N. Trefethen, Green's functions for multiply connected domains via conformal mapping, *SIAM Rev.*, **41**, 745–761, (1999).

[36] M. Flucher & B. Gustafsson, Vortex motion in two dimensional hydrodynamics, TRITA-MAT-1997-MA 02, Royal Institute of Technology, (1997).

[37] G.M. Goluzin, Geometric theory of functions of a complex variable, American Mathematical Society, Providence, RI, (1969).

[38] B. Gustafsson, On the motion of a vortex in two-dimensional flow of an ideal fluid in simply and multiply connected domains, TRITA-MAT-1979-7, Royal Institute of Technology, (1979).

[39] B. Gustafsson, Quadrature identities and the Schottky double, *Acta. Appl. Math.*, **1**, 209–240, (1983).

[40] B. Gustafsson, Singular and special points on quadrature domains from an algebro-geometric point of view, *J. Anal. Math*, **51**, 91–117, (1988).

[41] B. Gustafsson & H. Shapiro, What is a quadrature domain?, p.1-25 in Quadrature domains and applications, *Operator theory: advances and applications*, **156**, Birkhäuser, (2005)

[42] Y. Hellegouarch, Invitation to the mathematics of Fermat-Wiles, Academic Press, (2002).

[43] P. Henrici, Applied and computational complex analysis, Wiley Interscience, New York, (1986).

[44] C.C. Lin, On the motion of vortices in two dimensions – I. Existence of the Kirchhoff-Routh function, *Proc. Nat. Acad. Sci.*, **27**(12), 570–575, (1941).

[45] C.C. Lin, On the motion of vortices in two dimensions – II. Some further investigations on the Kirchhoff-Routh function, *Proc. Nat. Acad. Sci.*, **27**(12), 575–577, (1941).

[46] E.R. Johnson & N.R. McDonald, The motion of a vortex near two circular cylinders, *Proc. Roy. Soc. A*, **460**, 939–954, (2004).

[47] E.R. Johnson & N.R. McDonald, The motion of a vortex near a gap in a wall, *Phys. Fluids*, **16**(2), 462–469, (2004).

[48] E.R. Johnson & N.R. McDonald, Vortices near barriers with multiple gaps, *J. Fluid Mech.*, **531**, 335–358, (2005).

[49] P. Koebe, Abhandlungen zur theorie der konformen abbildung, *Acta Mathematica*, **41**, 305–344, (1914).

[50] H. Lamb, Hydrodynamics, 6th Edition, Cambridge University, Press, (1993).

[51] G.K. Morikawa & E.V. Swenson, Interacting motion of rectilinear geostrophic vortices, *Phys. Fluids*, **14**, 1058–1073, (1971).

[52] D. Mumford, C. Series and D. Wright, Indra's Pearls: The vision of Felix Klein, Cambridge University Press, (2002).

[53] T. Needham, Visual Complex Analysis, Clarendon Press, (1997).

[54] Z. Nehari, Conformal mapping, McGraw-Hill, New York, (1952).

[55] R. Nevanlinna, Analytic functions, Springer-Verlag, New York, (1970).

[56] P.K. Newton, The N-vortex problem, Springer-Verlag, New York, (2001).

[57] D. Nof, Choked flows from the Pacific to the Indian Ocean, *J. Phys. Oceanogr.*, **25**, 1369, (1995).

[58] S. Richardson, Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, *J. Fluid Mech.*, **56**, 609–618, (1972).

[59] S. Richardson, Hele-Shaw flows with time-dependent free boundaries in which the fluid occupies a multiply-connected region, *Eur. J. Appl. Math.*, **5**, 97–122, (1994).

[60] S. Richardson, Hele-Shaw flows with time-dependent free boundaries involving a multiply-connected fluid region, *Eur. J. Appl. Math.*, **12**, 571–599, (2002).

[61] P.G. Saffman, Vortex dynamics, Cambridge University Press, (1992).

[62] M. Schiffer, Recent advances in the theory of conformal mapping. Appendix to R. Courant, Dirichlet's principle, conformal mapping and minimal surfaces.

[63] H.S. Shapiro, The Schwarz function and its generalization to higher dimension, Wiley, (1992).

[64] H.L Simmons & D. Nof, Islands as eddy splitters, *J. Mar. Res.*, **58**, 919–956, (2000).

[65] H.L Simmons & D. Nof, The squeezing of eddies through gaps, *J. Phys. Oceanogr.*, **32**, 314, (2002).

[66] J. Shen, G. Strang, & A.J. Wathen, The potential theory of several intervals and their applications, *Apl. Math. Opt.*, **44**, 67–85, (2001).

[67] J.J. Thomson, On the motion of vortex rings, Macmillan (Adams Prize Essay), (1883).

[68] G. Valiron, Cours d'analyse mathematique: Theorie des fonctions, Masson et Cie, Paris, (1947).

[69] E.T. Whittaker & G.N Watson, A course of modern analysis, Cambridge University Press, (1927).

[70] H. Widom, Extremal polynomials associated with a system of curves in the complex plane, *Adv. Math.*, **3**, 127–232, (1969).