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Generalizing the Kutta-Joukowski lift theorem to multiple aerofoils: an analytical approach

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Basic aerodynamics

Circulation κ + Uniform flow U = LIFT!

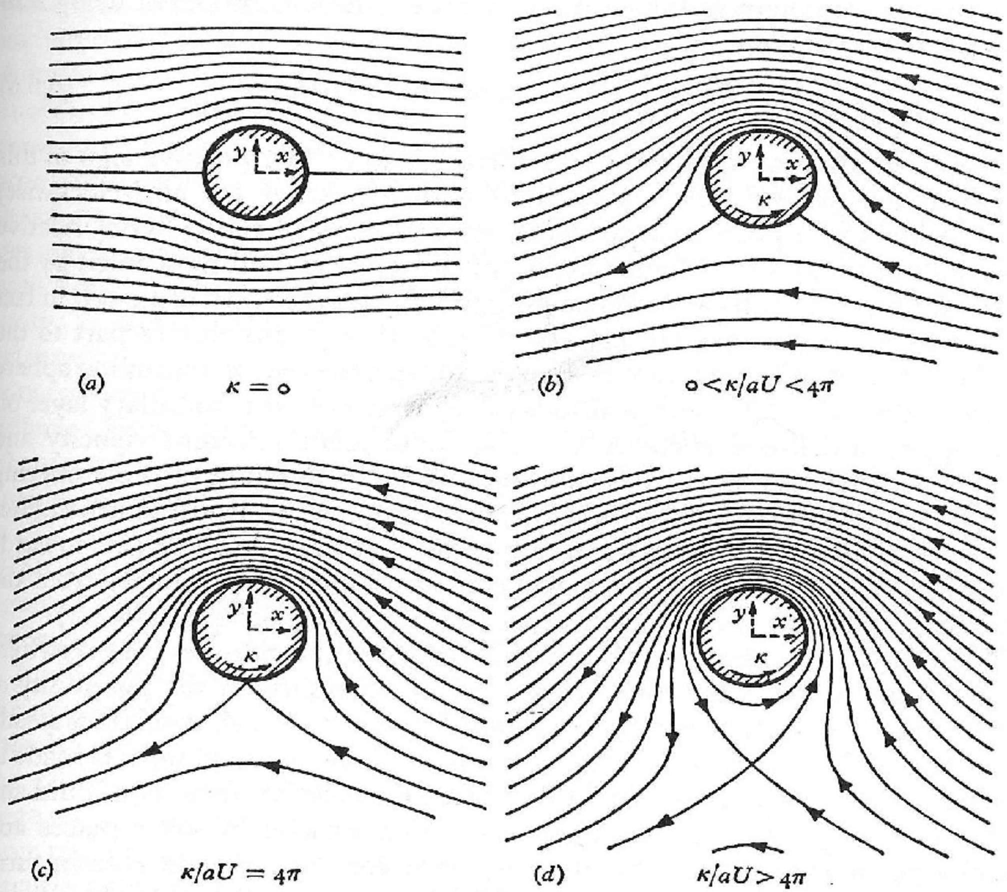


Figure 6.6.1. Streamlines for irrotational flow due to a circular cylinder held in a stream of uniform velocity $(-U, 0)$ at infinity, with circulation κ (anti-clockwise positive) round the cylinder.

from Batchelor (1967)

Kutta-Joukowski: lift force = - $\rho\kappa U$

Lift independent of aerofoil shape

Multiple aerofoils?

Single aerofoil case is rather boring!

Much more interesting, and non-trivial, is the case of multiple aerofoils

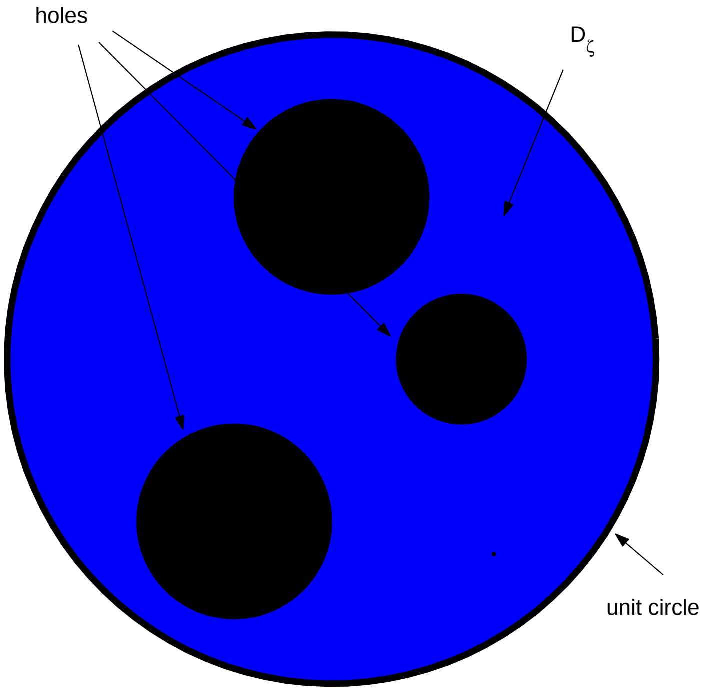
In contrast to single aerofoil case:

- “interference forces” exist between aerofoils
- forces on individual aerofoils depend, in nontrivial way, on global geometry, aerofoil shapes and relative circulations
- forces can exist between aerofoils even without circulations around them

Important to have an analytical theory of multi-aerofoil case

Introduce "circular domains"

Introduce a (canonical) *multiply connected* circular domain D_ζ



circle centers $\{\delta_j\}$

circle radii $\{q_j\}$

Fact: a domain of this type can be conformally mapped to the fluid region outside *any* collection of aerofoils.

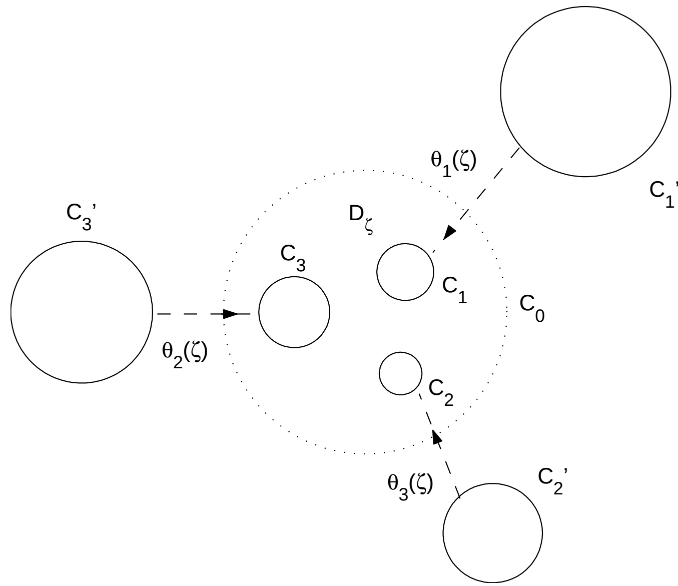
Let conformal map be $z(\zeta)$. Let $\zeta = \beta$ map to $z = \infty$.

Example: $z(\zeta) = \frac{a\zeta+b}{\zeta-\beta}$ maps to the fluid outside circular aerofoils

The "Schottky-Klein prime function" $\omega(\zeta, \gamma)$

Associated with such a domain is a special function $\omega(\zeta, \gamma)$

depending on $\{q_j, \delta_j | j = 1, .., M\}$



$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma) = (\zeta-\gamma) \prod_{\theta_j \in \Theta''} \frac{(\theta_j(\zeta)-\gamma)(\theta_j(\gamma)-\zeta)}{(\theta_j(\zeta)-\zeta)(\theta_j(\gamma)-\gamma)}$$

The maps $\theta_j(\zeta)$ are simply Möbius maps of the form

$$\theta_j(\zeta) = \frac{a_j\zeta + b_j}{c_j\zeta + d_j}$$

Note: easy to truncate and compute!

Key result 1: uniform flow past multiple cylinders?

The complex potential $W_U(\zeta)$ for uniform flow past any number of *circular* aerofoils can be written in terms of $\omega(\zeta,\gamma)$:

$$z(\zeta) = \frac{a}{\zeta - \beta} \qquad \text{(maps circles to circles)}$$

$$W_U(\zeta) = \phi + i\psi = Ua\left(e^{i\chi}\frac{\partial}{\partial\bar{\beta}} - e^{-i\chi}\frac{\partial}{\partial\beta}\right)\log\left(\frac{\omega(\zeta,\beta)}{|\beta|\omega(\zeta,\bar{\beta}^{-1})}\right)$$

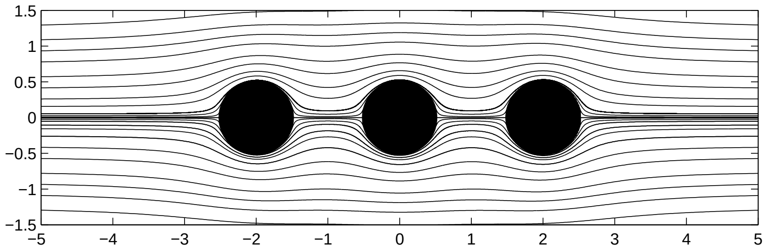
where U is speed and χ is angle of uniform flow. β maps to $z = \infty$.

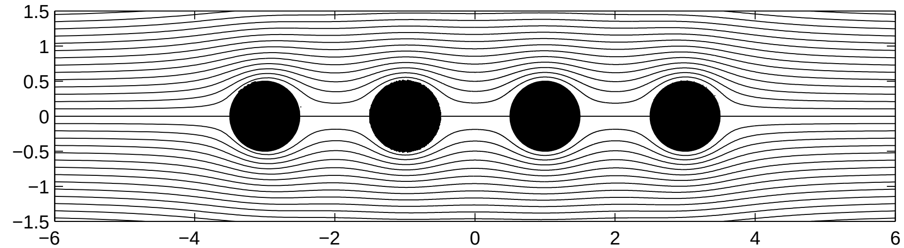
In case of a single cylinder, with $\beta = 0$, formulas reduce to

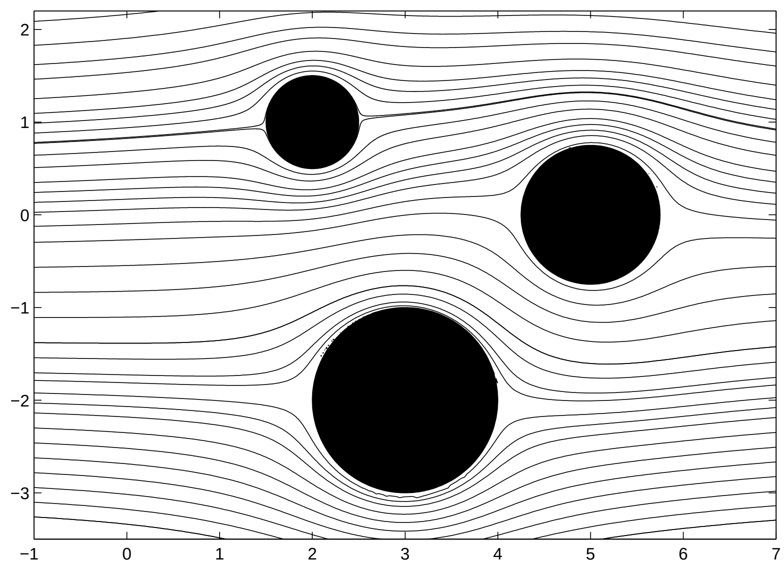
$$z(\zeta) = \frac{a}{\zeta}, \quad W_U(\zeta) = U\left(\zeta + \frac{1}{\zeta}\right)$$

and mapping is from unit ζ-disc. This is well-known classic result.

Example streamline distributions:







Contours of $\mathrm{Im}[W_U(\zeta)]$ plotted using new formulae

But, to get lift, also need to add circulation around aerofoils...

Key result 2: circulation around the aerofoils

To add a non-zero circulation κ_k around the k-th island, the required complex potential is

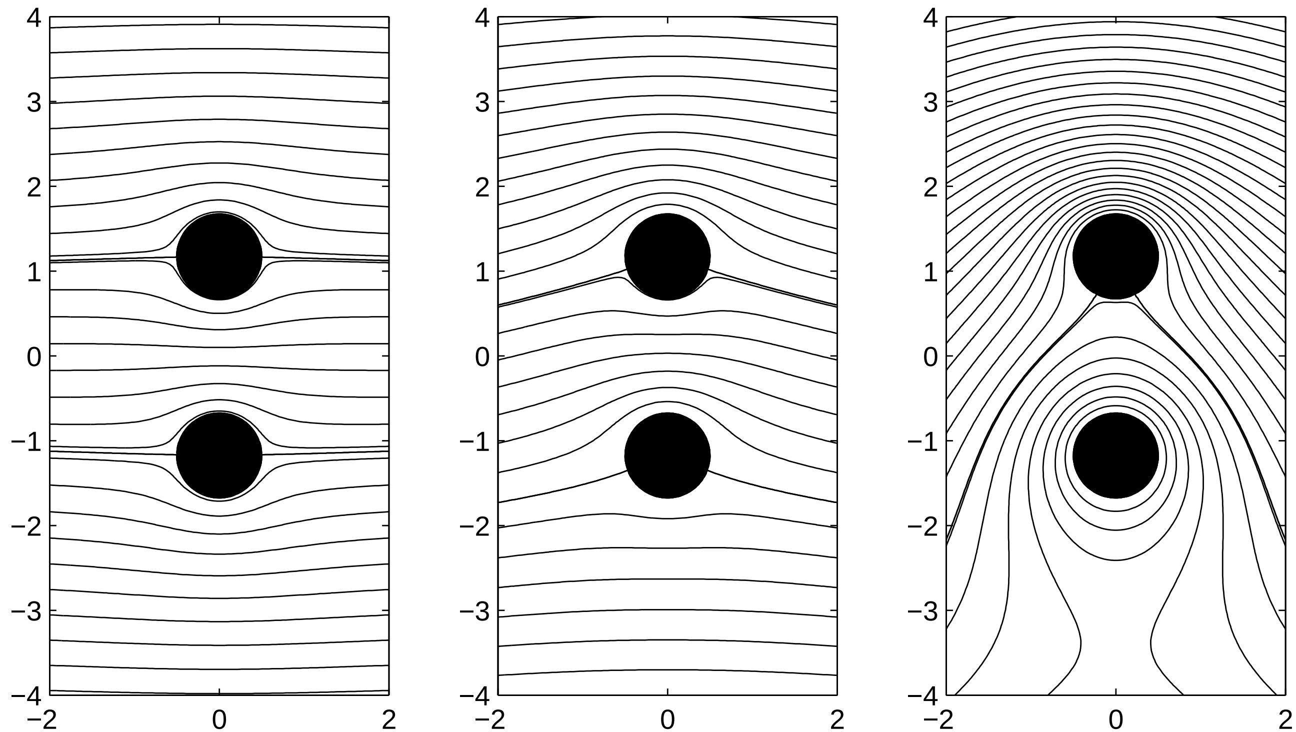
$$W_\kappa(\zeta) = \frac{i}{2\pi}\sum_{k=1}^{M} \kappa_k \log \frac{\omega(\zeta,\beta)}{\omega(\zeta,\theta_k(\bar{\beta}^{-1}))}$$

The total complex potential $W_T(\zeta)$ therefore takes the form

$$W_T(\zeta) = W_U(\zeta) + W_\kappa(\zeta)$$

MAIN RESULT: The complex potential for ANY number of aerofoils can be written in terms of the prime function $\omega(\zeta,\gamma)$

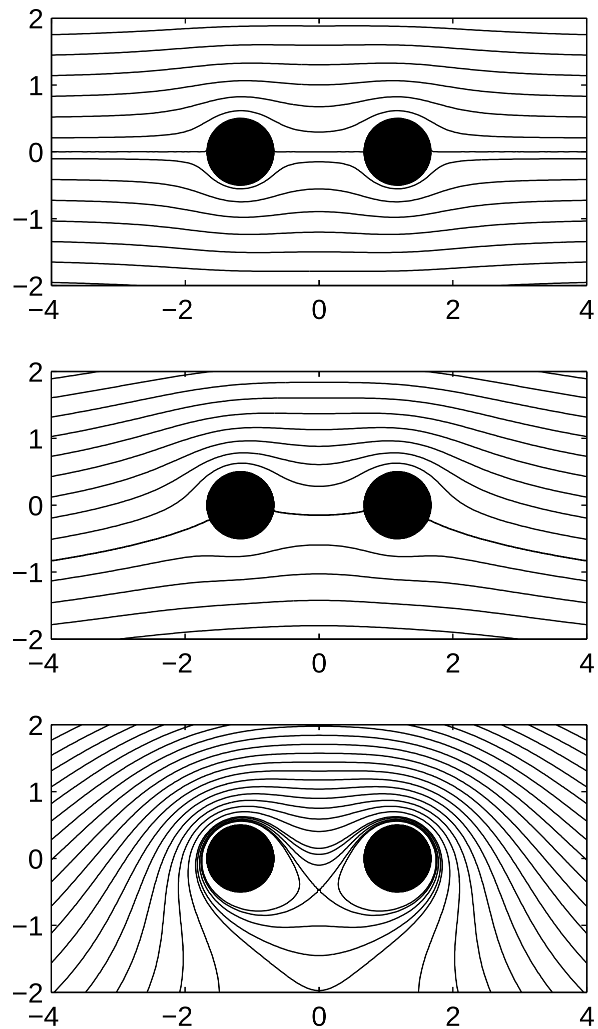
Two (biplane) aerofoils (unstaggered stack)



Two aerofoils with gradually increasing circulation

Note: there is an attractive force between aerofoils even if $\kappa_j = 0$

Two (biplane) aerofoils (in tandem)



Two aerofoils with gradually increasing circulation

There is a repelling force between aerofoils even if $\kappa_j = 0$

Some of the “doubly connected” literature:

W.M. Hicks, On the motion of two cylinders in a fluid, *Q. J. Pure Appl. Math.***, (1879)**

A. G. Greenhill, Functional images in Cartesians, *Q. J. Pure Appl. Math.***, (1882)**

M. Lagally, The frictionless flow in the region around 2 circles, *ZAMM***, (1929).**

C. Ferrari, Sulla trasformazione conforme di due cerchi in due profili alari, *Mem. Real. Accad. Sci. Torino***, (1930)**

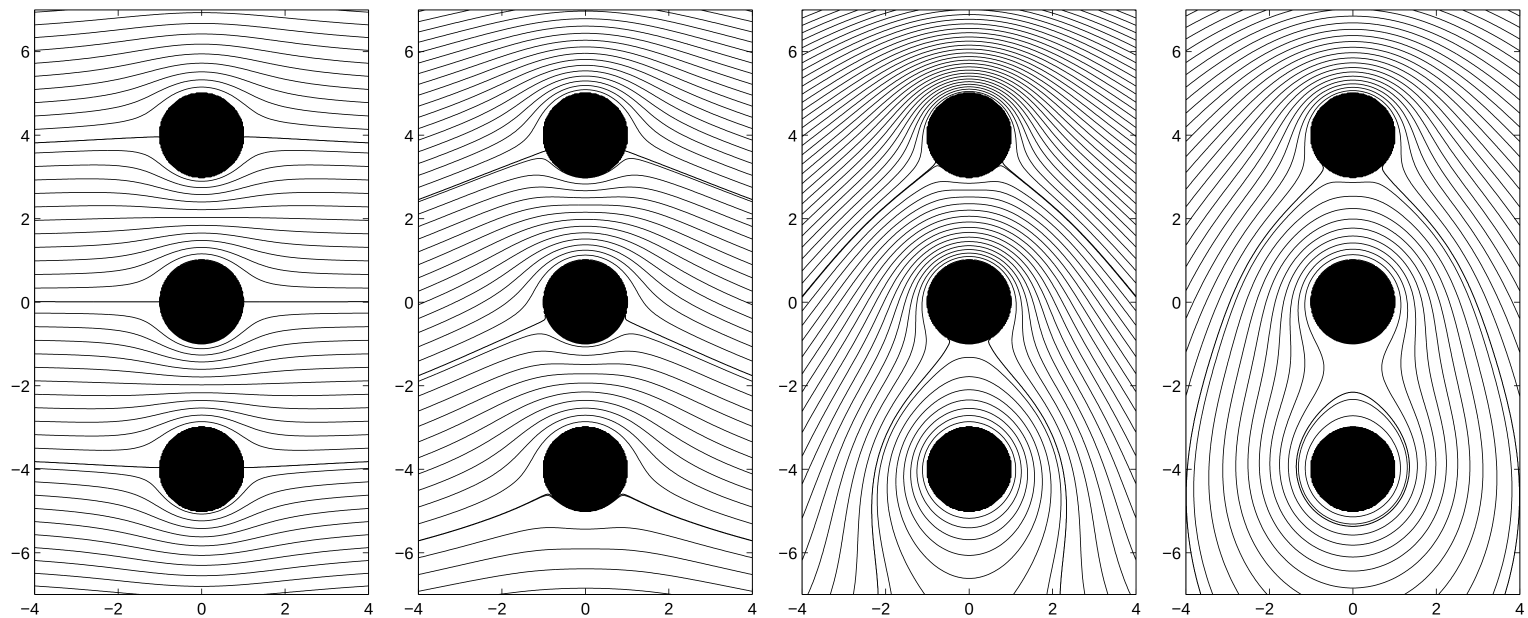
T. Yamamoto, Hydrodynamic forces on multiple circular cylinders, *J. Hydr. Div, ASCE***, (1976).**

E.R. Johnson & N. Robb McDonald, The motion of a vortex near two circular cylinders, *Proc. Roy. Soc. A***, (2004)**

Burton, D.A., Gratus, J. & Tucker, R.W., Hydrodynamic forces on two moving discs, *Theor. Appl. Mech.***, (2004)**

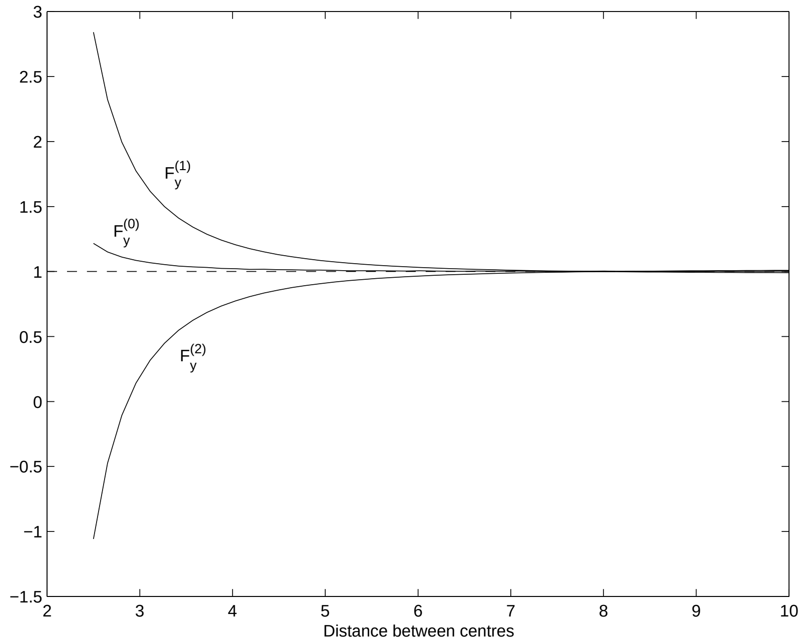
No prior analytical results for more than two aerofoils

Three (triplane) aerofoils (unstaggered stack)

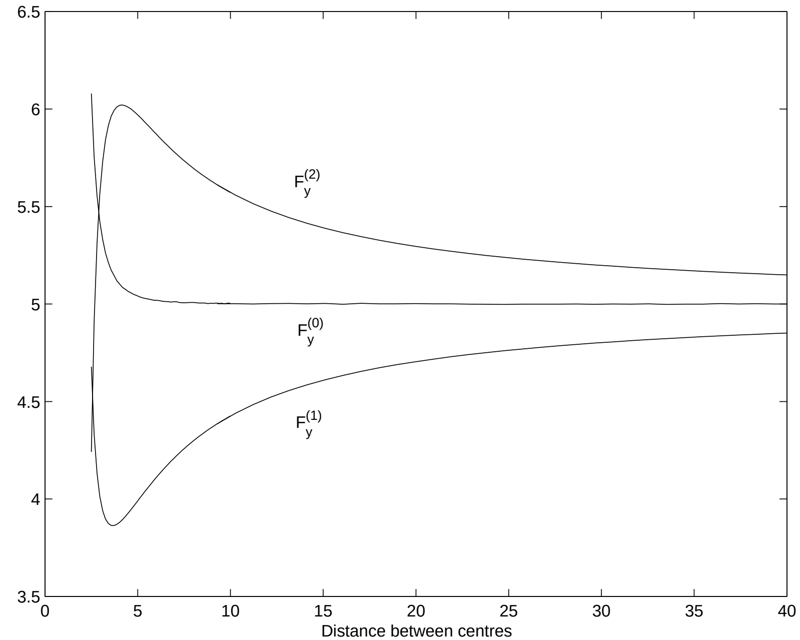


Three aerofoils with gradually increasing circulation in uniform flow

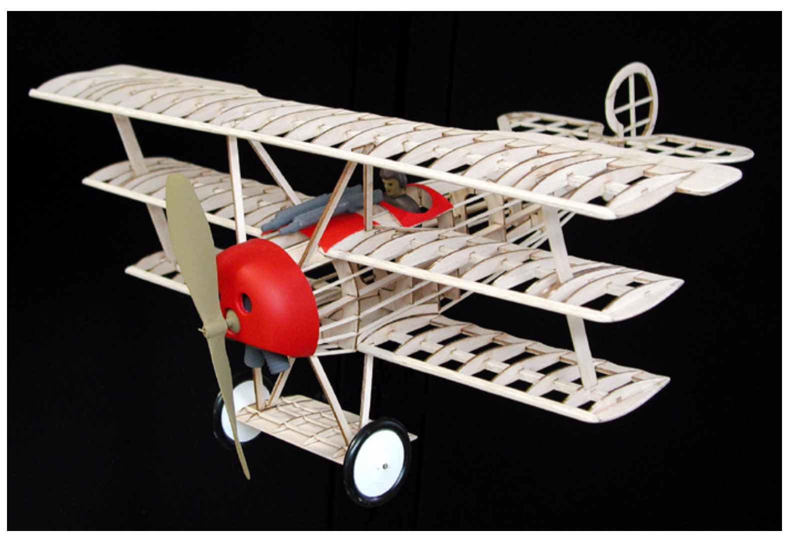
Blasius theorem to compute force distribution:



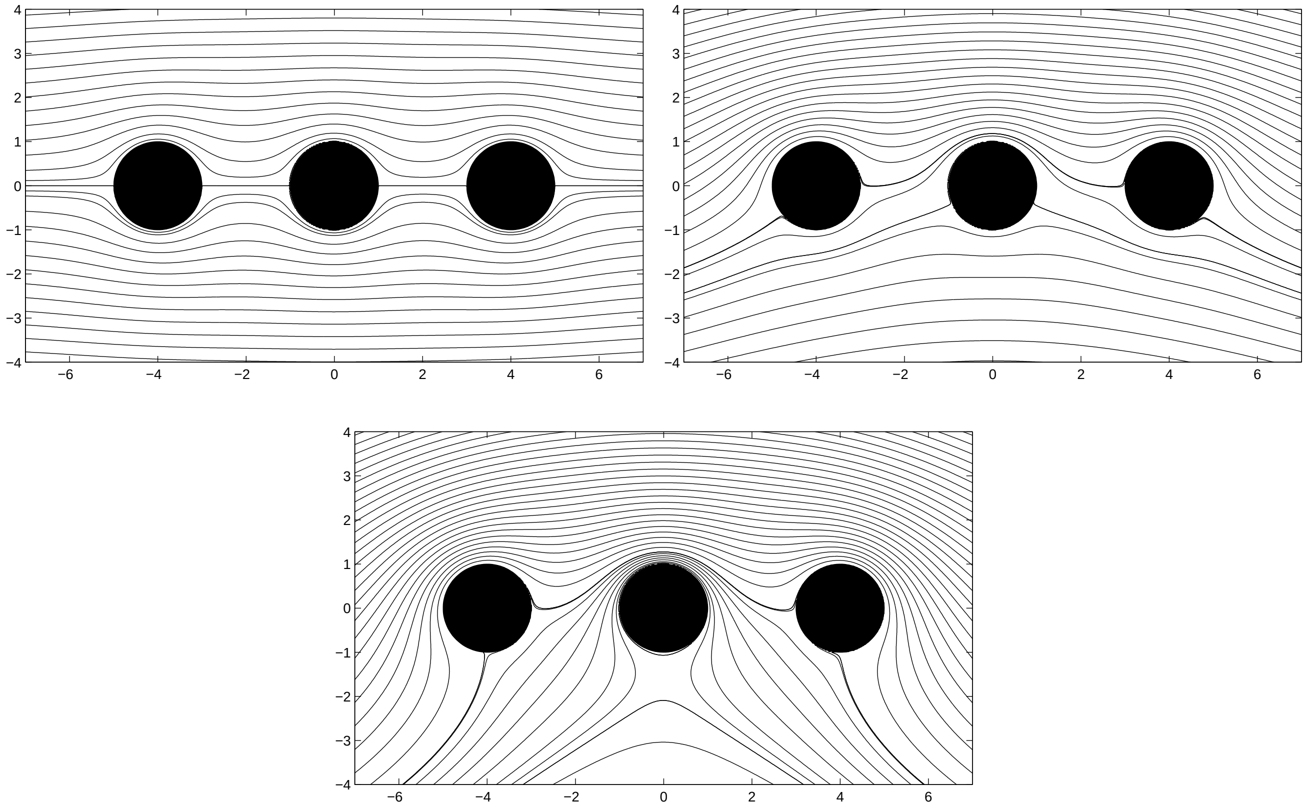
$\mathbf{\Gamma = 1}$



$\mathbf{\Gamma = 5}$



Three (triplane) aerofoils (in tandem)



Three aerofoils with gradually increasing circulation in uniform flow

Crowdy, “Calculating the lift on a finite stack of circular aerofoils”, (preprint)

Summary

- **There are now analytical formulae for the complex potential for uniform flow past any number of obstacles;**
- **These potentials can be written in a natural way in terms of a special transcendental function $\omega(\zeta, \gamma)$;**
- **The complex potentials for adding circulation around the aerofoils can also be written in a natural way in terms of this function;**
- **The streamline distribution, lift forces, interference forces, torques etc. can now be computed in a straightforward fashion without the need for boundary integral formulations.**
- **using more complex conformal maps, formulae can be applied to aerofoils of *any* shape.**