APS/DFD Seattle 2004

Analytical formulas for the Kirchhoff-Routh path function

in multiply-connected domains

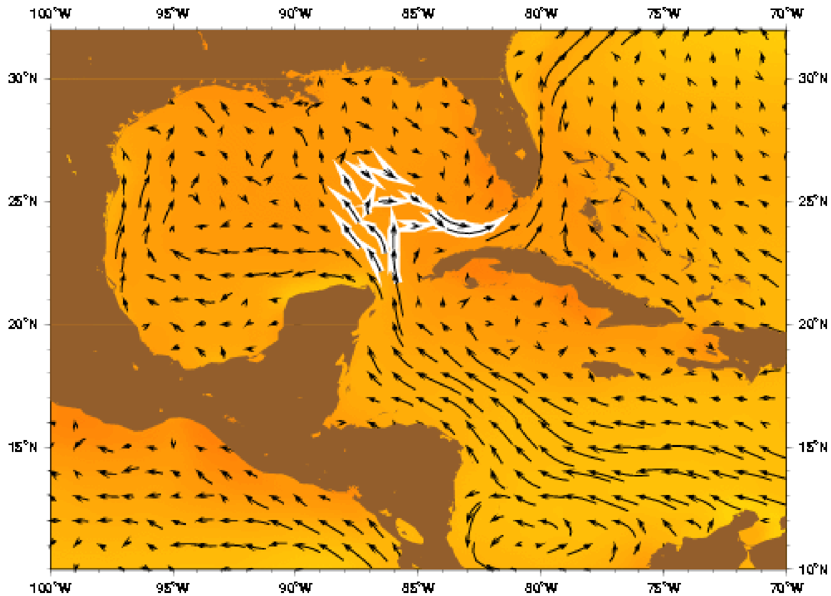
Darren Crowdy

(Imperial College London)

d.crowdy@imperial.ac.uk

(joint work with J. Marshall)

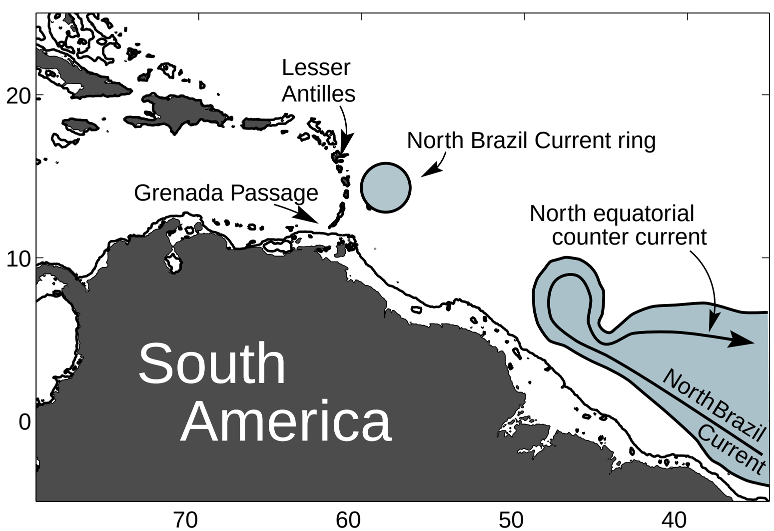
Vortex motion in multiply-connected domains



geophysical applications

e.g. ocean circulations in Caribbean

oceancurrent.rsmas.miami.edu



Simmons & Nof, "The squeezing of eddies through gaps", J. Phys. Ocean., 32, (2002).

Vortex motion in doubly-connected domains

Vortices near islands 947

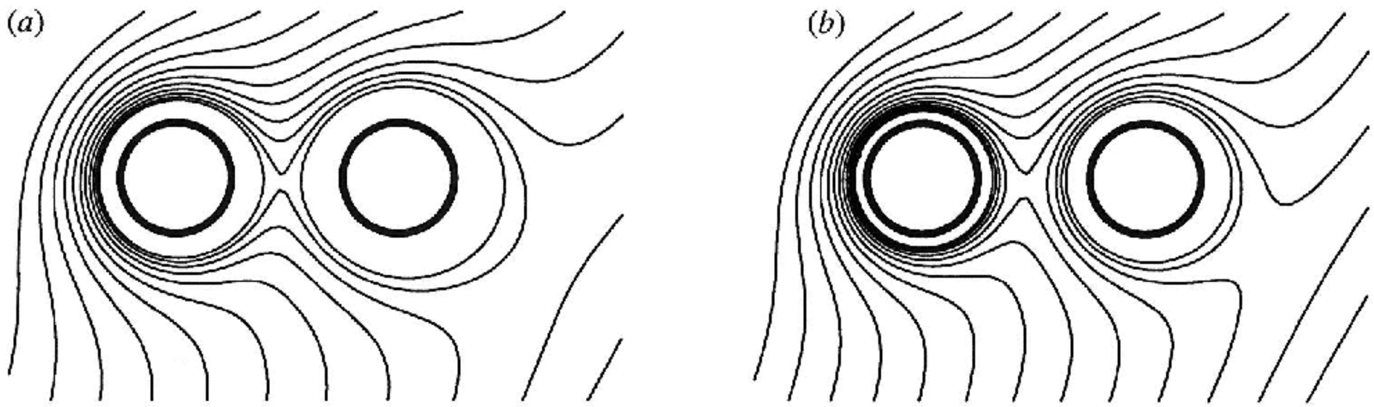


Figure 3. Trajectories for a line vortex outside two circular islands each of radius 1 with centres separated by 4. In both cases there is zero inter-island volume flux ($m = 0$) and the background flow is uniform at infinity, of speed U inclined at an angle $\pi/4$ to the x-axis. (a) Weak flow, $4\pi U/\kappa = 0.3$. The new saddle point due to the background flow is lower than the inter-island saddle point. (b) Strong flow, $4\pi U/\kappa = 0.5$. The new saddle point due to the background flow is higher than the inter-island saddle point.

Johnson & McDonald, "The motion of a vortex near two circular cylinders", *Proc. Roy. Soc. A*, **460, (2004).**

Q: How to extend this to any finite number of islands?

Vortex motion through gaps in walls

Phys. Fluids, Vol. 16, No. 2, February 2004

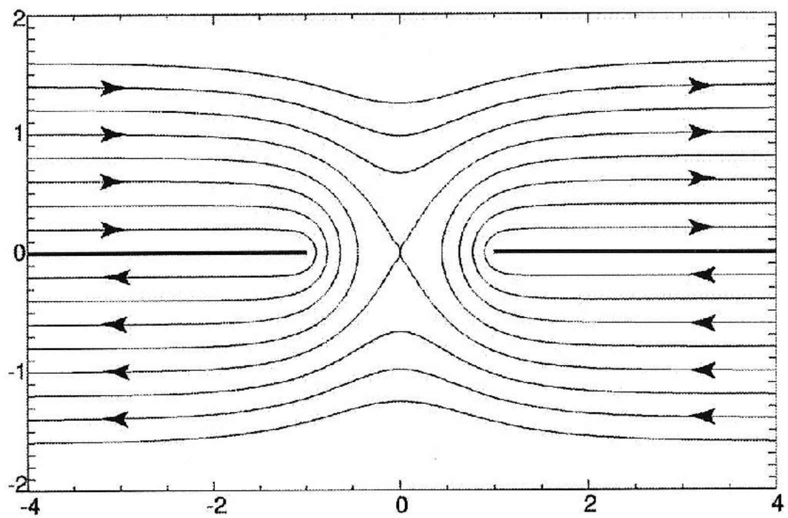


FIG. 3. The paths of a line vortex near a gap. Here, and in all succeeding figures, the figure is the z plane with the $x=\Re z$ axis horizontal, the $y=\Im z$ axis vertical; the headlands are bold lines and arrowheads give the direction of motion of a vortex with positive circulation. Negative vortices move in the opposite direction. Vortices that at large $|x|$ are further than half the gap width from the wall jump the gap. Vortices starting closer to the wall at large $|x|$ pass through the gap.

Johnson & McDonald,

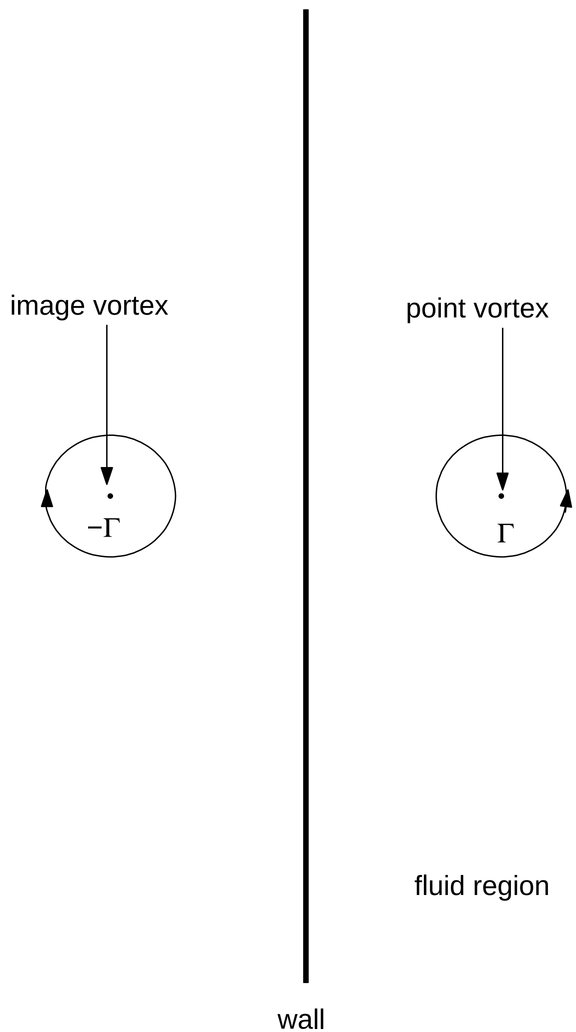
“The motion of a vortex

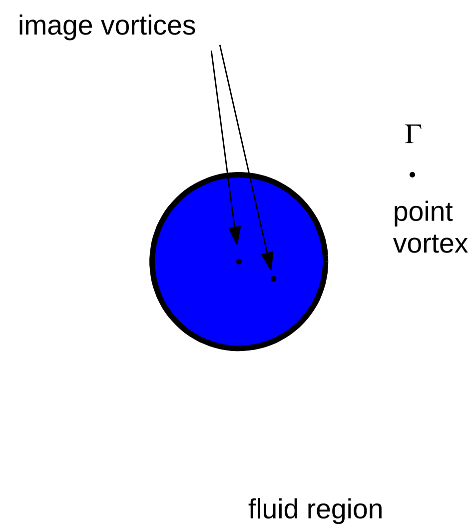
near a gap in a wall”,

Phys. Fluids, **16**, **2004**

Q: How to extend this to any finite number of gaps?

Vortex motion with boundaries: famous results





Method of images **Milne-Thomson circle theorem**

Hydrodynamic Green's function G – Lin (1941a)

Lin (1941a) proved existence of a special Green's function:

(a) $G(\zeta;\alpha)$ **has a logarithmic singularity at** $\zeta = \alpha$**;**

(corresponds to point vortex at $\zeta = \alpha$**)**

(b) Let C_j **be interior boundaries of islands.**

$$G = 0, \quad \text{on outer boundary}$$

$$G = \beta_j(\alpha), \quad \text{on } C_j,\ j = 1, ..., M$$

(corresponds to streamline conditions at boundaries)

(c)

$$\oint_{C_j} \frac{\partial G}{\partial n} ds = 0, \quad j = 1, ..., M$$

(corresponds to zero-circulation around islands)

Kirchhoff-Routh path function or Hamiltonian

Then the Hamiltonian for the motion of N vortices is

$$H^{(\zeta)}(\{\zeta_k\}) = \sum_{k>l}\sum \Gamma_k \Gamma_l G(\zeta_k;\zeta_l) - \frac{1}{2}\sum_k \Gamma_k^2 g(\zeta_k;\zeta_k)$$

where g is function such that

$$G(\zeta;\alpha) = -\frac{1}{2\pi}\log|\zeta-\alpha| - g(\zeta;\alpha)$$

Problem: until now, nobody has explicitly constructed G

Conformal mapping (Lin, 1941b)

Having found $H^{(\zeta)}$ in a multiply-connected domain D_ζ, the Hamiltonian $H^{(z)}$ in any domain D_z to which D_ζ is conformally mapped by $z(\zeta)$ is given by

$$H^{(z)}(\{z_k\}) = H^{(\zeta)}(\{\zeta_k\}) + \sum_k \frac{\Gamma_k^2}{4\pi} \log |z_\zeta(\zeta_k)|$$

where

$$z_k = z(\zeta_k)$$

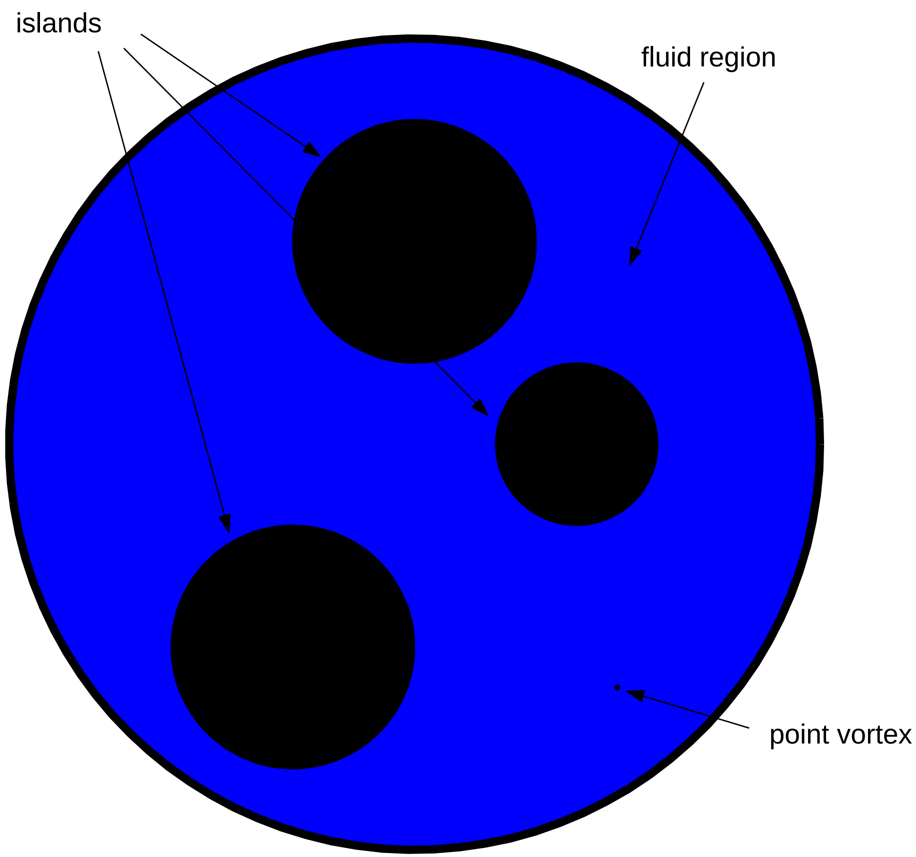
What needs to be done?

Progress hinges on two questions:

- **can we find explicit formulas for G, and hence $H^{(\zeta)}$, in some "canonical" set of multiply-connected domains D_ζ?**
- **can we find useful conformal maps $z(\zeta)$ from these domains to flow domains of (geophysical) interest?**

Point vortices in multiply-connected domains

Introduce a *multiply-connected* circular domain D_ζ with a point vortex



circle centers $\{\delta_j\}$

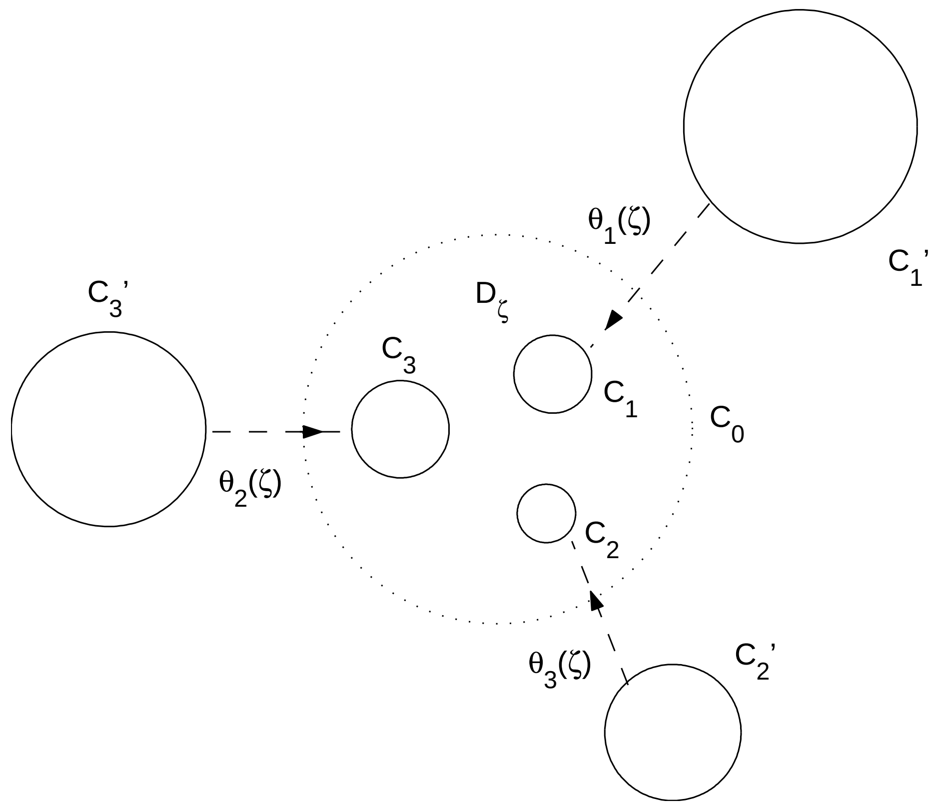
circle radii $\{q_j\}$

"canonical multiply-connected domains"

What are the point vortex trajectories?

The Schottky-Klein prime function

Given D_ζ, can construct an associated special function $\omega(\zeta;\alpha)$ depending on $\{q_j, \delta_j | j = 1, .., M\}$



$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma) = (\zeta-\gamma)\prod_{\theta_i \in \Theta''} \frac{(\theta_i(\zeta)-\gamma)(\theta_i(\gamma)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\gamma)-\gamma)}$$

Explicit expression for G

In terms of this prime function, an explicit formula for G is

$$G(\zeta;\alpha) = -\frac{1}{4\pi}\log\left|\frac{\omega(\zeta;\alpha)\overline{\omega}(\zeta^{-1};\alpha^{-1})}{\omega(\zeta;\bar{\alpha}^{-1})\overline{\omega}(\zeta^{-1};\bar{\alpha})}\right|$$

Hence, for single vortex at $\alpha(t)$, Hamiltonian is

$$H^{(\zeta)}(\alpha,\bar{\alpha}) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{\omega'(\alpha,\alpha)\overline{\omega}'(\bar{\alpha}^{-1},\bar{\alpha}^{-1})}{\alpha^2\omega(\alpha,\bar{\alpha}^{-1})\overline{\omega}(\alpha^{-1},\bar{\alpha})}\right|$$

Consequence

Up to knowledge of $z(\zeta)$, we have explicit formulas for the Hamiltonians in ANY multiply-connected domain

Crowdy & Marshall, *Proc. Roy. Soc. A.*, (submitted)

Circular islands in bounded or unbounded oceans

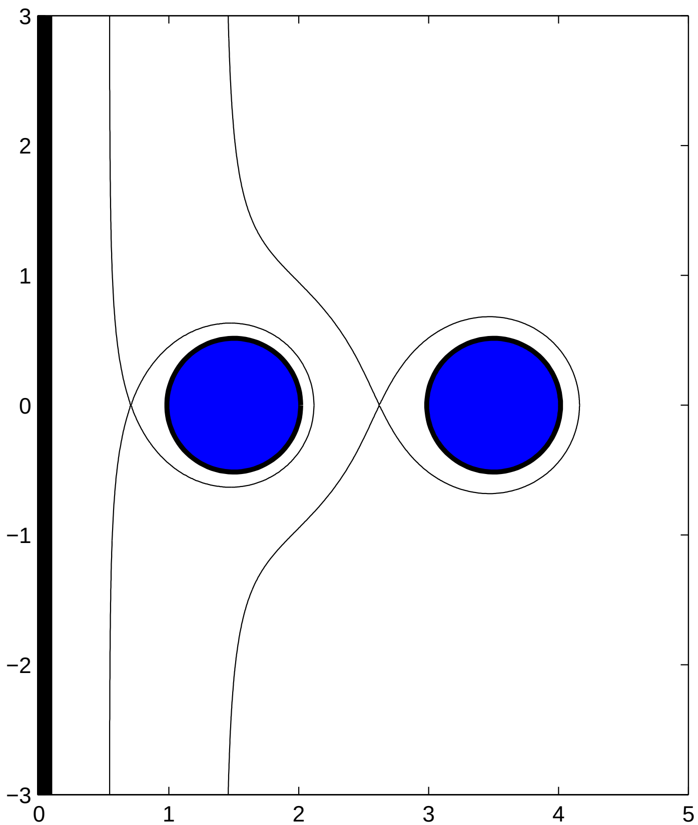
It is well-known that Mobius mappings of the form

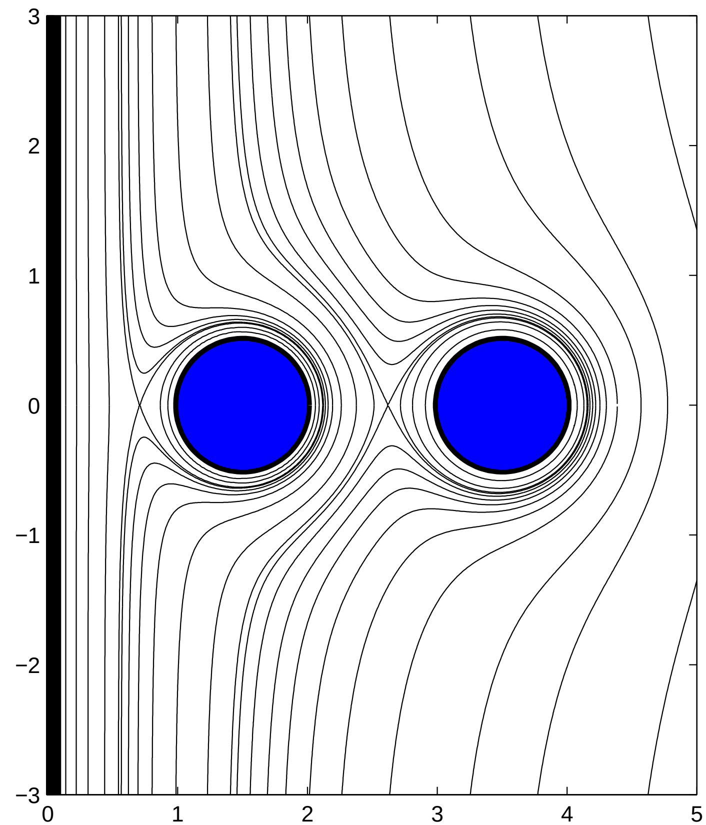
$$z(\zeta) = \frac{a\zeta + b}{c\zeta + d}$$

where a, b, c and $d \in \mathbb{C}$ map circles to circles.

Thus, mappings to unbounded oceans with circular islands or islands off a coastline map from D_ζ using Mobius mappings.

Example 1: two circular islands off a coastline

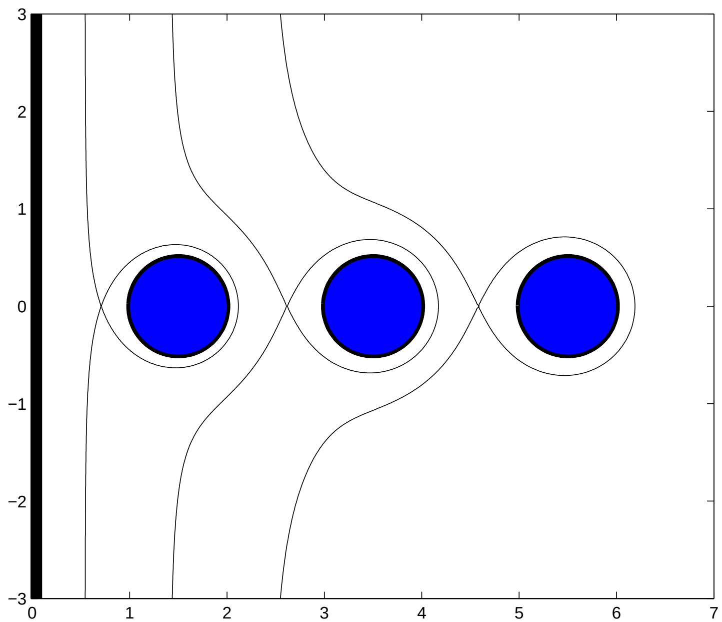


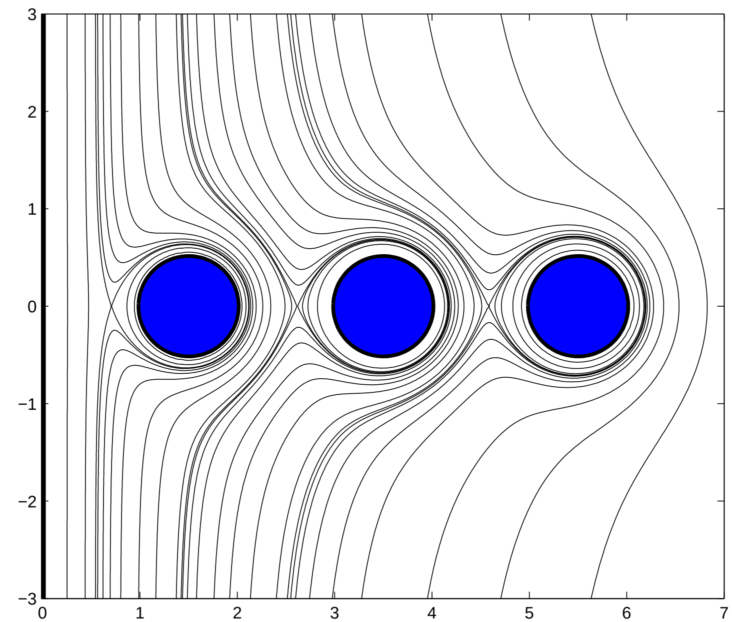


critical trajectories **other trajectories**

This example uses $z(\zeta) = \frac{1-\zeta}{1+\zeta}$

Example 2: three circular islands off a coastline

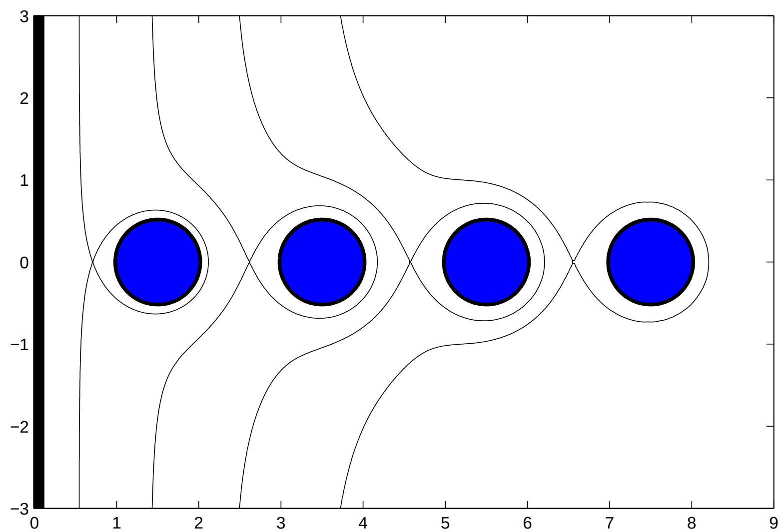




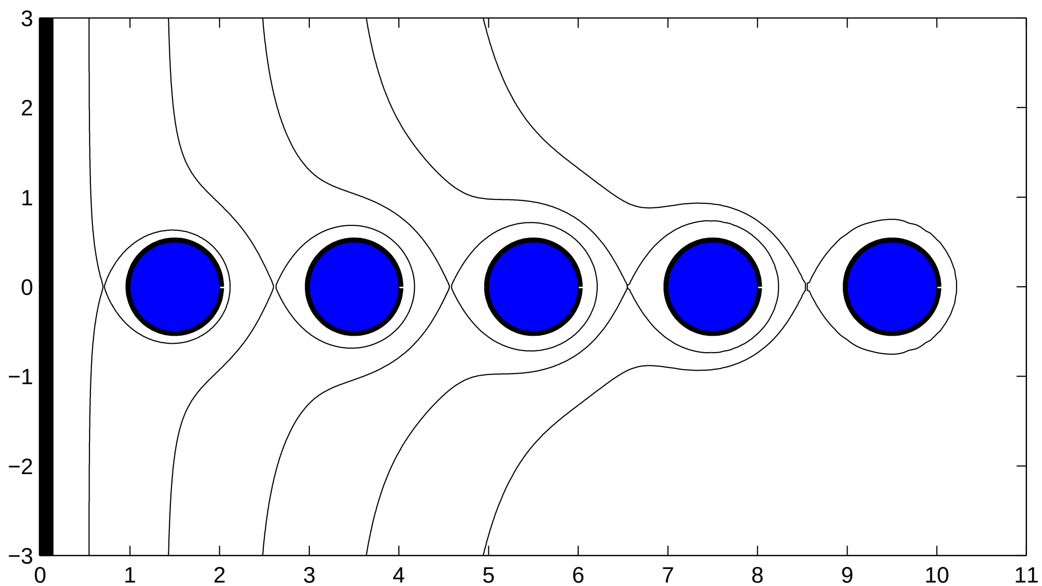
critical trajectories

other trajectories

Example 3: more circular islands off a coastline

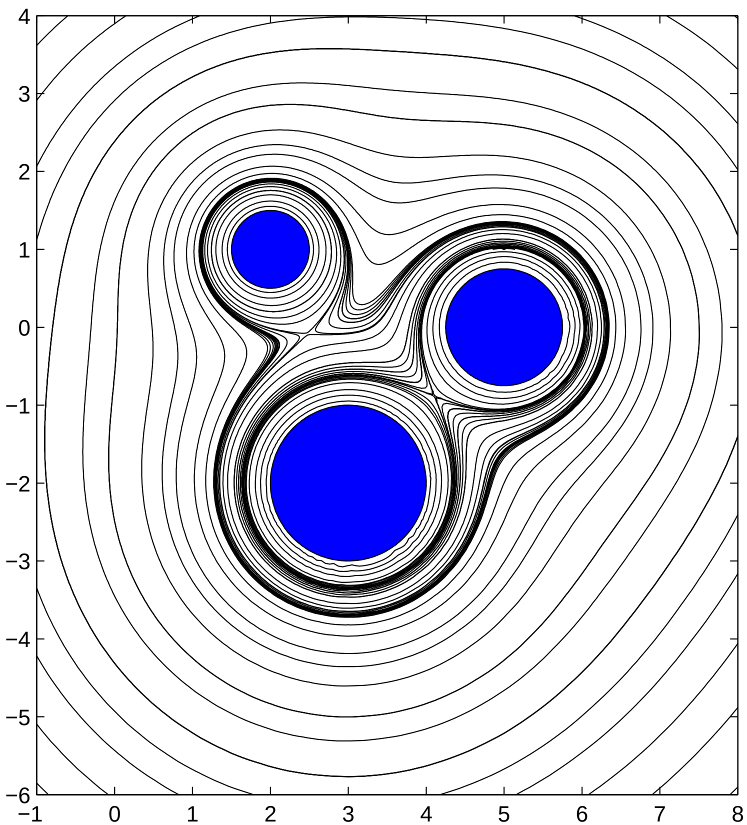


critical trajectories

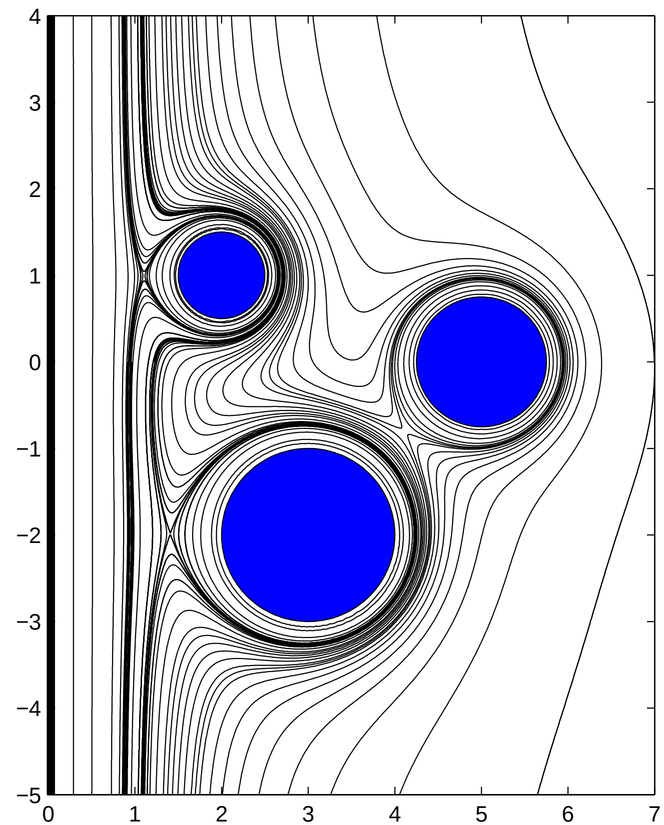


critical trajectories

Example 4: random circular islands



unbounded ocean



with coastline

Crowdy and Marshall, *Phys. Fluids* (submitted)

What about vortex motion through gaps in walls?

Phys. Fluids, Vol. 16, No. 2, February 2004

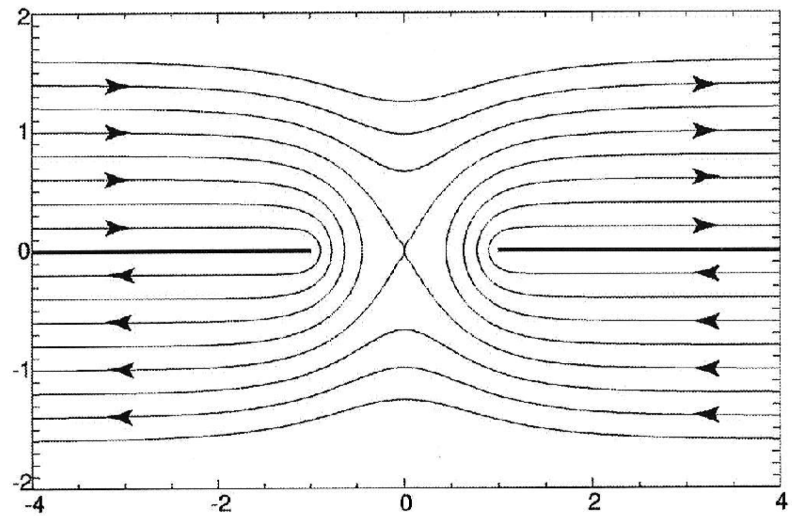


FIG. 3. The paths of a line vortex near a gap. Here, and in all succeeding figures, the figure is the z plane with the $x = \Re z$ axis horizontal, the $y = \Im z$ axis vertical; the headlands are bold lines and arrowheads give the direction of motion of a vortex with positive circulation. Negative vortices move in the opposite direction. Vortices that at large $|x|$ are further than half the gap width from the wall jump the gap. Vortices starting closer to the wall at large $|x|$ pass through the gap.

Johnson & McDonald, "The motion of a vortex near a gap in a wall",

Phys. Fluids, **16, 2004**

Conformal mappings to slit domains

It turns out that the hydrodynamic Green's function G is also relevant to the construction of conformal mappings to multiply-connected slit domains.

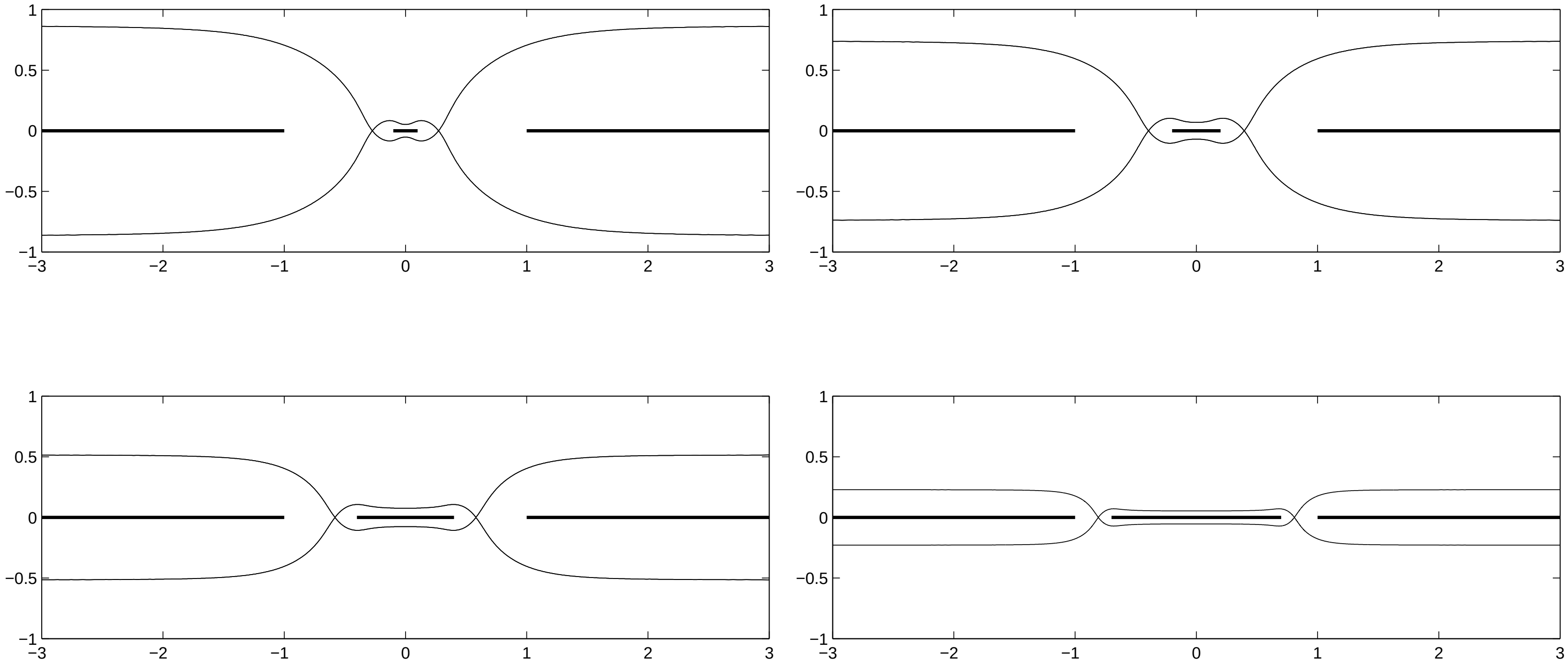
[Reference: R. Courant, "Dirichlet's Principle"]

Consequence:

we can use explicit form for G to both

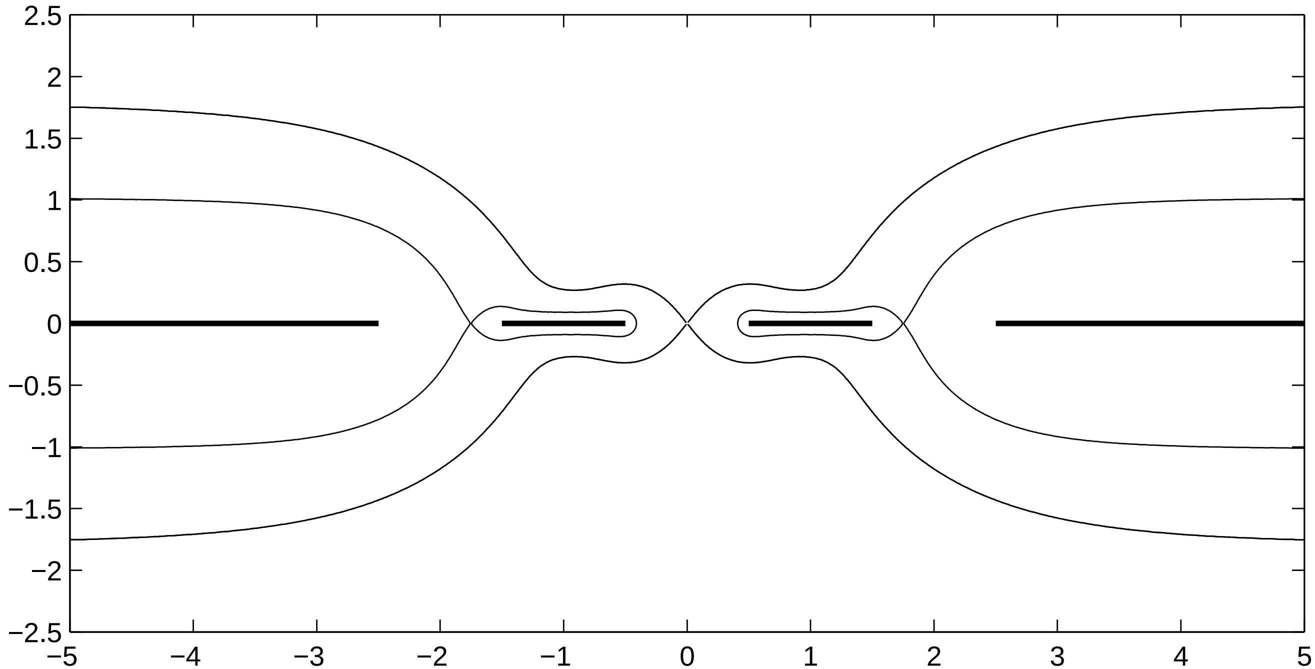
- **produce Hamiltonian;**
- **generate conformal mapping!**

Example 5: A wall with two gaps



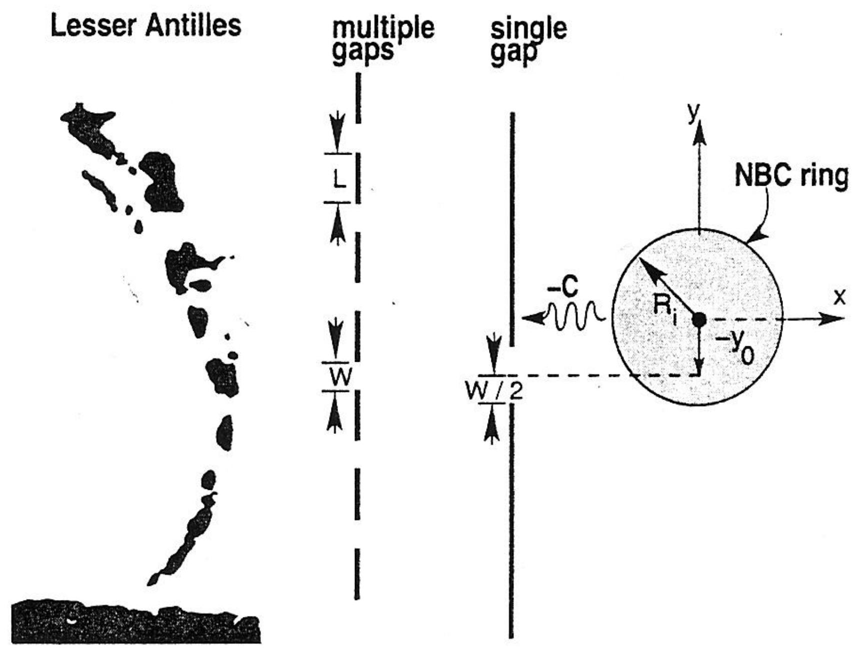
Critical vortex trajectories for a wall with two gaps.

Example 6: A wall with three gaps



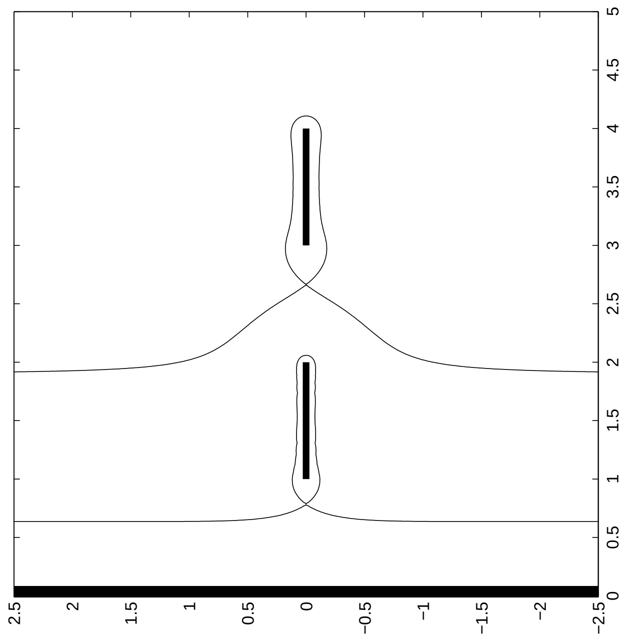
Critical vortex trajectories for a wall with three gaps.

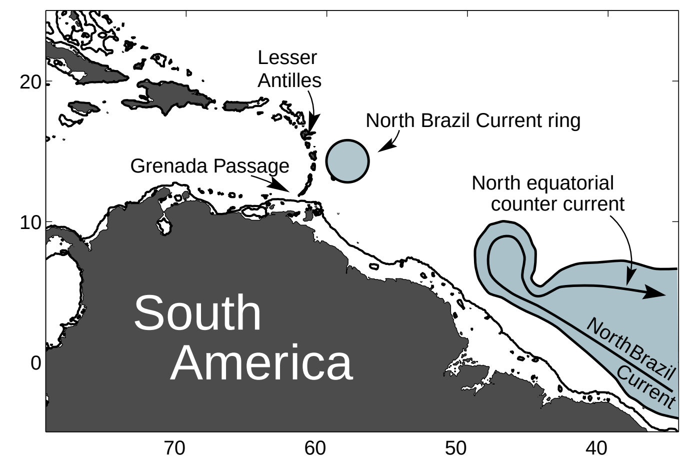
Modelling geophysical flows



Simmons & Nof, “The squeezing of eddies through gaps”, J. Phys. Ocean., (2002).

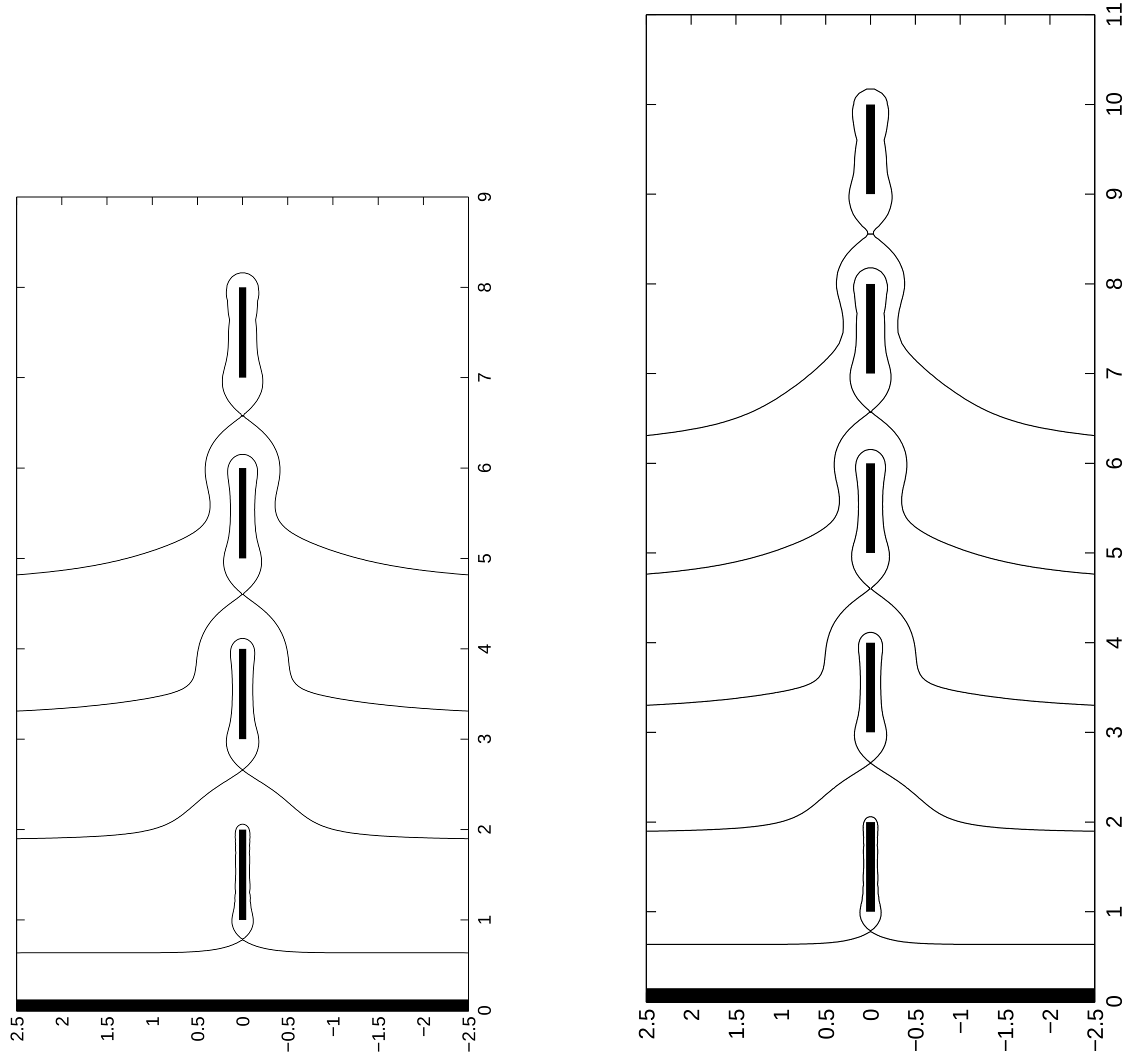
Example 7: Offshore islands





Critical vortex trajectories for two offshore islands

Example 8: any number of offshore islands....



4 islands

5 islands

Summary

- new formulas automatically place image vortices in correct place – new “method of images”;
- given a conformal map, can generate Hamiltonians for vortex motion in any other conformally-equivalent domains;
- have constructed new conformal mappings to a class of “slit” domains of geophysical significance.

Website

General information, references + this talk

`www.ma.ic.ac.uk/~dgcrowdy`

Email: `d.crowdy@imperial.ac.uk`