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CONSTRUCTING MULTIPLY CONNECTED QUADRATURE DOMAINS*

DARREN CROWDY† AND JONATHAN MARSHALL†

Abstract. Multiply connected bounded quadrature domains, with finite connectivity, are reconstructed from their quadrature data using conformal mappings that are ratios of products of Schottky–Klein prime functions. This method provides the natural generalization of the conformal maps to simply and doubly connected quadrature domains constructed by the first author in a number of physical applications. The efficacy of the method is demonstrated by the explicit construction of a range of examples as well as by comparison with alternative constructive methods recently introduced by Crowdy [*R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457 (2001), pp. 2337–2359] and Richardson [*European J. Appl. Math.*, 12 (2001), pp. 571–599].

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1. Introduction.

The mathematical theory of quadrature domains is well developed (e.g., [1], [2], [3], [4], [5], [6]). The simplest example of a quadrature domain is a circular disc. Let $z = x + iy$ and suppose that the disc is centered at the origin $z = 0$ with radius r. The well-known "mean value theorem" says that, if $h(z)$ is any function analytic in the disc D and integrable over it, then

$$\int\int_D h(z)dxdy = \pi r^2 h(0). \tag{1}$$

Equation (1) is a simple example of a *quadrature identity*. The idea of quadrature domain theory is to consider more complicated domains satisfying more complicated quadrature identities. Consider a planar domain D and let $h(z)$ be any function that is analytic in D and integrable over it. Suppose that

$$\int\int_D h(z)dxdy = \sum_{k=1}^{N}\sum_{j=0}^{n_k-1} c_{jk}h^{(j)}(z_k), \tag{2}$$

where $\{z_k \in \mathbb{C}\}$ is a set of points strictly inside D, $\{c_{jk} \in \mathbb{C}\}$, and $h^{(j)}(z)$ denotes the jth derivative of h. Here, N and $\{n_k \geq 1\}$ are integers. Then D is known as a *quadrature domain*. The quadrature identity (2) generalizes (1).

While quadrature domains are mathematically interesting in their own right, perhaps more remarkable is the fact that they are relevant to the mathematical study of a wide range of physical problems. An important early paper of Richardson [7] was the first to illustrate the connection with the study of the free boundary problem involving singularity-driven flows in a Hele–Shaw cell. Richardson's paper involved

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†Department of Mathematics, Imperial College, 180 Queen's Gate, London, SW7 2AZ United Kingdom (d.crowdy@imperial.ac.uk, jonathan.marshall@imperial.ac.uk).

simply connected fluid domains. The essential result is that quadrature domains are preserved by the dynamics of the physical problem. Since then, quadrature domains have been found to be useful in a variety of different problems. For example, Entov, Etingof, and Kleinbock [8] have discussed a number of generalizations of the singularity-driven Hele–Shaw problem, including the dynamics of flows in a rotating Hele–Shaw cell and the problem of "squeeze flow" in a Hele–Shaw cell. Both problems also preserve quadrature domains [9], [10]. Crowdy [11] has pointed out the relevance of quadrature domains to a biharmonic-governed (as opposed to the harmonic-governed Hele–Shaw model) free boundary problem involving slow viscous flows driven by surface tension. Here, in certain circumstances, the dynamics is also such as to preserve quadrature domains. Outside the realm of free boundary problems, it has also been shown [12], [13] that quadrature domains have relevance to the study of multipolar vortical equilibria of the two-dimensional Euler equations governing the inviscid flow of an ideal fluid. The problem of finding equilibrium shapes of free boundaries involving irrotational Euler flows with surface tension (see, e.g., [14]) can also be interpreted in terms of quadrature domain theory.

This compendium of different physical applications suggests a need to be able to construct quadrature domains of various finite connectivities. Gustafsson [5] has shown that construction of an N-connected quadrature domain is equivalent to the construction of a conformal mapping, which is a meromorphic function on a Riemann surface of genus $N-1$. For $N = 1$ and $N = 2$, this is possible using the theory of rational functions and elliptic functions (or, equivalently, loxodromic functions, which are naturally related to elliptic functions [15]). Indeed, these two cases constitute most of the existing literature. Richardson used rational function conformal mappings in his original paper [7] and, more recently, elliptic function conformal mappings for singularity-driven Hele–Shaw flows of doubly connected fluid regions [16]. Crowdy [9] has used loxodromic functions to derive exact solutions for the evolution of doubly connected domains in a rotating Hele–Shaw cell, providing a mathematical model for some recent experimental results involving a fluid annulus [17]. Elliptic/loxodromic function theory has also been used to construct exact solutions to the problem of the surface tension–driven Stokes flow of doubly connected fluid regions [18], [19], [20].

For higher connectivities, the situation is much more challenging. The subject of constructing multiply connected quadrature domains (of connectivity greater than 2) has been the focus of much recent activity. Two new methods of construction have recently been proposed in the context of specific applications. The first author [13] has implemented a construction based on the fact that the boundaries of quadrature domains are algebraic curves. This method has been successfully applied, for example, to the construction of vortical equilibria of the Euler equations [13] and to the squeeze flow problem in a Hele–Shaw cell [10]. Meanwhile, in considering the related problem of singularity-driven flow of multiply connected fluid domains (with zero surface tension) in a Hele–Shaw cell, Richardson [23] has proposed a different method based on conformal mapping. In this paper we present a new method which, like Richardson's, is based on conformal maps. However, our approach is different. In light of all the recent work on this problem, we also discuss in detail how the new construction differs from other methods, and how it compares to them in terms of practical application.

To motivate the present work, we recall that it is a standard result [1] that simply connected quadrature domains can be constructed by rational function mappings from a unit ζ-disc to the domain. Any rational function with given zeros and poles can be written as a ratio of products of the fundamental function $P(\zeta) = 1 - \zeta$. For example, if the conformal map has poles $\{\alpha_k | k = 1, \ldots, N\}$ and zeros $\{\beta_k | k = 1, \ldots, N\}$, it can be written

$$z(\zeta) = R \frac{\prod_{k=1}^{N} P(\zeta \beta_k^{-1})}{\prod_{k=1}^{N} P(\zeta \alpha_k^{-1})}, \tag{3}$$

where R is a constant. Furthermore, it is also known that a representation of a general conformal mapping (again with poles $\{\alpha_k | k = 1, \ldots, N\}$ and zeros $\{\beta_k | k = 1, \ldots, N\}$) from the annulus $\rho < |\zeta| < 1$ to a doubly connected quadrature domain can be written exactly as in (3) but with the fundamental function $P(\zeta, \rho)$ defined differently as

$$P(\zeta, \rho) = (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^{2k}\zeta)(1 - \rho^{2k}\zeta^{-1}). \tag{4}$$

Note that when $\rho = 0$, (4) reduces to the function $P(\zeta) = (1 - \zeta)$ relevant for the construction of the rational functions in the simply connected case. In light of this, it is natural to ask whether the representation (3) can *also* be used for quadrature domains with connectivity greater than two but with suitably generalized "fundamental functions." This is the question addressed in this paper. The generalized "fundamental functions" needed are known as the Schottky–Klein prime functions [21].

The treatment in this paper is based on the presentation in Chapter 12 of a monograph by Baker [21]. Our aim here is to show how to apply these general results for the specific purpose of constructing multiply connected quadrature domains. For clarity, any general results needed are stated without proof and in modified form suited to present purposes. The interested reader is referred to [21] for more details.

Richardson's constructive method also employs the Schottky model and mappings from the circular domains used here, but his representation of the conformal maps is different. Richardson does not use, or define, the Schottky–Klein prime function. Instead, his conformal maps are constructed as ratios of Poincaré series—a method of constructing meromorphic functions on compact Riemann surfaces described, for example, by Beardon [22]. The present authors believe the new construction based on the Schottky–Klein prime function presented herein to be an important alternative to Richardson's method for two reasons. First, it is the natural generalization of the representation (4) used by Crowdy [20] and, moreover, it is closely related to a representation in terms of ratios of products of generalized theta functions defined on Riemann surfaces [21], [25]. Note that (4) *is* the Schottky–Klein prime function in the genus-1 case. Second, we have found the present method to be easier to implement than Richardson's method both conceptually and practically. The majority of the example domains in this paper have been constructed using both methods. In all cases considered, the boundaries of the domains are indistinguishable even at very low orders of truncation.

2. Quadrature domains. Let D denote a bounded $g+1$-connected quadrature domain. It is known [5] that the conformal mapping from a conformally equivalent region (in, say, a parametric ζ-plane) to D is given by a meromorphic function on a Riemann surface of genus g. This Riemann surface can be identified with the *Schottky double* of the region D [5]. These conformal mapping functions will be explicitly constructed here as ratios of products of Schottky–Klein prime functions [21]. Such functions are defined in section 4. In order to define them, it is necessary to introduce Schottky groups [22], [24]; these are discussed in section 3. In what follows, we first show how Schottky groups are relevant to multiply connected quadrature domains.

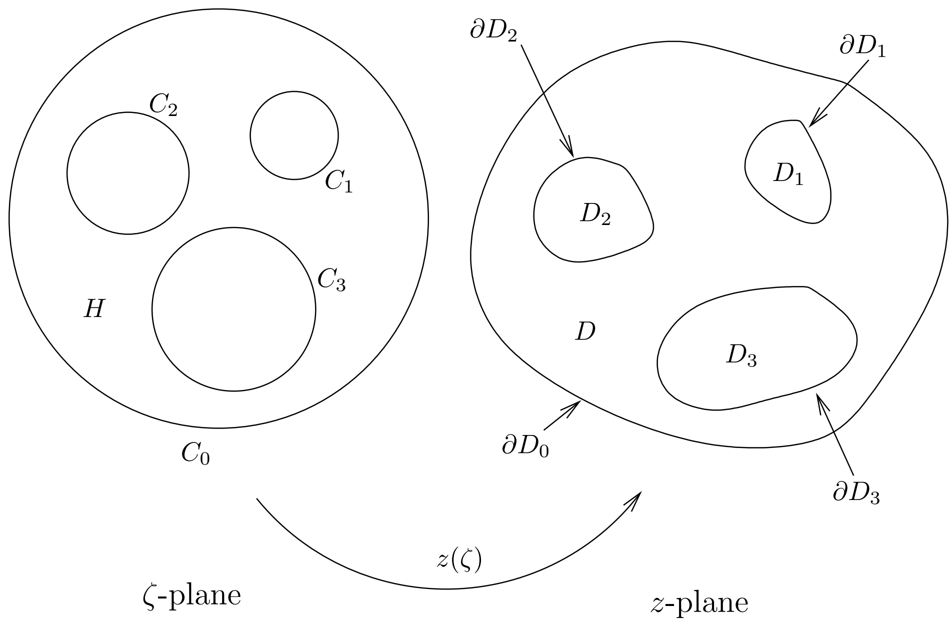

FIG. 1. *Schematic of conformal mapping from region H in the ζ-plane to region D in the physical z-plane.*

Consider the integral

$$\int\int_D h(z)d\bar{z} \wedge dz, \tag{5}$$

where $h(z)$ is analytic in D and $d\bar{z} \wedge dz = 2idxdy$ if $z = x + iy$. If D is a quadrature domain, then

$$\int\int_D h(z)d\bar{z} \wedge dz = \sum_{k=1}^{N} \sum_{j=0}^{N_k-1} c_{jk} h^{(j)}(z_k) \tag{6}$$

for some set of complex numbers $\{c_{jk}, z_k\}$, where the points $\{z_k\}$ are strictly inside D. $\{N_k \geq 1 | k = 1, \ldots, N\}$ are a set of integers, and $\sum_{k=1}^{N} N_k$ is known as the *order* of the quadrature identity. Using Green's theorem,

$$\int\int_D h(z)d\bar{z} \wedge dz = \oint_{\partial D_0} h(z)\bar{z}dz - \sum_{j=1}^{g} \oint_{\partial D_j} h(z)\bar{z}dz, \tag{7}$$

where ∂D_0 denotes the outer boundary of the bounded quadrature domain and $\partial D_i,\ i = 1, \ldots, g$, denotes the boundaries of the g enclosed holes.

Now let us introduce a conformal mapping $z(\zeta)$ to the domain D from a region H in a parametric ζ-plane bounded by the unit ζ-circle and a set of g smaller nonoverlapping circles totally contained inside $|\zeta| = 1$. Such a region shall be referred to as a *circular region*. Let the unit circle be denoted C_0, and let the g enclosed circles be labeled $C_i,\ i = 1, \ldots, g$, with centers δ_i and radii ρ_i. The circle C_0 will map to the outer boundary ∂D_0, while the circle C_i maps to the boundary ∂D_i. A schematic is shown in Figure 1. Note that, by the assumed reflectional symmetry about the real axis of the domains considered here, the conjugate conformal map, defined by $\bar{z}(\zeta) \equiv \overline{z(\bar{\zeta})}$, satisfies

$$\bar{z}(\zeta) = z(\zeta). \tag{8}$$

Using this conformal mapping function, the integral in (7) becomes

$$\int\int_D h(z)d\bar{z} \wedge dz = \oint_{C_0} h(z(\zeta))\bar{z}(\bar{\zeta})z_\zeta(\zeta)d\zeta - \sum_{j=1}^{g} \oint_{C_j} h(z(\zeta))\bar{z}(\bar{\zeta})z_\zeta(\zeta)d\zeta. \tag{9}$$

On C_0,

$$\bar{\zeta} = \zeta^{-1}, \tag{10}$$

so that, written as a function of ζ, the first integrand on the right-hand side of (9) is

$$h(z(\zeta))\bar{z}(\zeta^{-1})z_\zeta(\zeta). \tag{11}$$

Let us now consider what is necessary in order that the sum of *all* the integrals on the right-hand side of (9) reduces to a *single* integral of the *same* integrand (11) around the entire boundary of H. For this to happen, it is necessary that

$$\bar{z}(\phi_j(\zeta)) = \bar{z}(\zeta^{-1}), \qquad j = 1, \ldots, g, \tag{12}$$

where, on C_j,

$$\bar{\zeta} = \phi_j(\zeta) \equiv \bar{\delta}_j + \frac{\rho_j^2}{\zeta - \delta_j}. \tag{13}$$

Therefore, defining

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{\rho_j^2 \zeta}{1 - \bar{\delta}_j \zeta}, \tag{14}$$

we require that the conformal mapping $z(\zeta)$ satisfy

$$z(\zeta) = z(\theta_j(\zeta)), \qquad j = 1, \ldots, g. \tag{15}$$

The g maps $\{\theta_j\}$ are Mobius maps and generate a free group of transformations known as a Schottky group [21], [24]. See section 3 to follow. The mapping $z(\zeta)$ must be invariant with respect to the substitutions of this group.

For $j = 1, 2, \ldots, g$, let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$ (i.e., the circle obtained by the transformation $\zeta \mapsto 1/\bar{\zeta}$). C_j' lies in the region exterior to the unit ζ-circle. The image of the circle C_j' under the transformation θ_j is the circle C_j. Since the g circles $\{C_j\}$ are nonoverlapping, so are the g circles $\{C_j'\}$. Consider the region of the plane exterior to the $2g$ circles $\{C_j\}$ and $\{C_j'\}$; a schematic is shown in Figure 2. This region turns out to have a special significance. (It is known as the *fundamental region* associated with the Schottky group generated by the Mobius maps $\{\theta_j | j = 1, \ldots, g\}$ and their inverses—see the next section.) This is because the functional relations (15) allow the function $z(\zeta)$ to be analytically continued outside this fundamental region to any point of the plane which can be reached by a finite number of applications of the transformations $\{\theta_j\}$

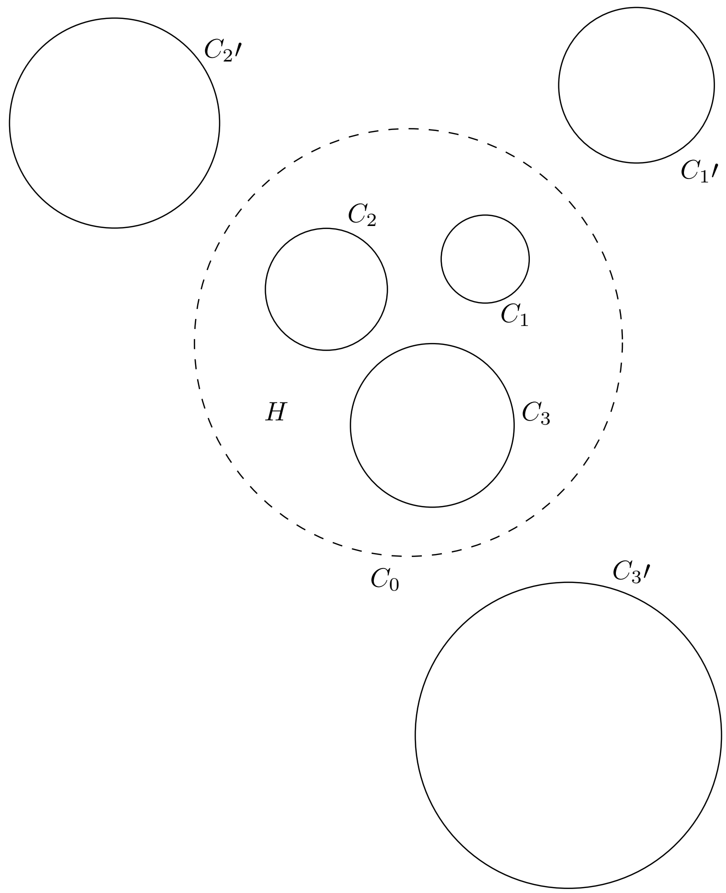

FIG. 2. *The fundamental region is the unbounded region exterior to all six Schottky circles* $C_1, C_1', C_2, C_2', C_3, C_3'$.

and their inverses to a point in this fundamental region. It is therefore enough to establish the singularity structure of $z(\zeta)$ within just this fundamental region.

Now, if the functional relations (15) hold, we have

$$\int\int_D h(z) d\bar{z} \wedge dz = \oint_{\partial H} h(z(\zeta))\bar{z}(\zeta^{-1}) z_\zeta(\zeta) d\zeta, \tag{16}$$

where ∂H denotes the whole boundary of H. Now let

$$z_k = z(\bar{\alpha}_k^{-1}), \qquad k = 1, \ldots, N, \tag{17}$$

for some points $\{\bar{\alpha}_k^{-1} | k = 1, \ldots, N\}$ contained in H. Now if $\bar{z}(\zeta^{-1})$ has poles in H only at the points $\{\bar{\alpha}_k^{-1}\}$, then the integral on the right-hand side of (16) produces the pure sum of residues (6). This means that $z(\zeta)$ will have poles in the fundamental region only at the points $\{\alpha_k\}$. That is, $z(\zeta)$ is meromorphic in the fundamental region.

3. Schottky groups. Consider the $2g$ Mobius maps given by

$$\theta_1, \theta_1^{-1}, \theta_2, \theta_2^{-1}, \ldots, \theta_g, \theta_g^{-1}. \tag{18}$$

The Schottky group of transformations will be denoted Θ and is the infinite free group formed by all possible compositions of the Mobius maps (18). These maps are also referred to as the primary substitutions. Associated with this group is a fundamental region mentioned in the previous section. Sometimes we shall refer to the *ordinary* and *singular* points of the group [22], [24]. If a point in the plane can be reached by a finite number of applications of any of the $2g$ primary substitutions to a point in the fundamental region, then it is called an ordinary point of the group. If it can be reached only by an infinite number of applications, then it is called a singular point. A very accessible discussion of Schottky groups and their applications can be found in a recent monograph by Mumford, Series, and Wright [24].

Some special infinite subsets of transformations in a given Schottky group will be needed in the construction of the conformal mapping functions taking the circular regions in the ζ-plane to the multiply connected quadrature domains. A special notation is now introduced. This notation is not standard but is introduced here to clarify the presentation.

Notation. The full Schottky group is denoted Θ. The notation ${}_i\Theta_j$ is used to denote all transformations of the full group which do not have a power of θ_i or θ_i^{-1} on the left-hand end or a power of θ_j or θ_j^{-1} on the right-hand end. As a special case of this, the notation Θ_j simply means all substitutions of the group which do not have any positive or negative power of θ_j at the right-hand end (but with no stipulation about what appears on the left-hand end). Similarly, ${}_j\Theta$ means all substitutions which do not have any positive or negative power of θ_j at the left-hand end (but with no stipulation about what appears on the right-hand end). In addition, the single prime notation will be used to denote a subset where the identity is excluded from the set; thus Θ_1' denotes all substitutions, excluding the identity, and all transformations with a positive or negative power of θ_1 at the right-hand end. The double prime notation will be used to denote a subset where the identity *and* all inverse substitutions are excluded from the set. This means, for example, that if $\theta_1\theta_2$ is included in the set, the transformation $\theta_2^{-1}\theta_1^{-1}$ must be excluded. Thus, Θ'' means all substitutions of the group excluding the identity and all inverses. Similarly, the notation ${}_1\Theta_2''$ denotes all substitutions of the group, excluding inverses and the identity, which do not have any power of θ_1 or θ_1^{-1} on the left-hand end or any power of θ_2 or θ_2^{-1} on the right-hand end. In the same way, Θ_j'' denotes all substitutions of the group, excluding the identity and all inverses, which do not have any positive or negative power of θ_j at the right-hand end.

3.1. The loxodromic group. Consider a mapping to a doubly connected quadrature domain from an annular region $\rho_1 < |\zeta| < 1$ in the ζ-plane. In this case, the mapping must satisfy

$$z(\zeta) = z(\rho_1^2\zeta). \tag{19}$$

Meromorphic functions satisfying (19) are known as loxodromic functions [15]. They are automorphic with respect to the transformations of the *loxodromic group* generated by a single map of the form $\theta_1(\zeta) = \rho_1^2\zeta$. It should be noted that the fundamental region in this case can be taken to be the annulus $\rho_1 < |\zeta| < \rho_1^{-1}$, which does not include the point at infinity. The usual definition of the fundamental region associated with a classical Schottky group [22] *does* include the point at infinity. However, we adopt the convention of considering the loxodromic group to be a special case of a general Schottky group. Richardson [23] adopts the same convention.

4. The Schottky–Klein prime function. Following Baker [21], if the ith substitution of the group Θ acts on some point ζ, then the image point will be denoted ζ_i for brevity. Using this notation, the Schottky–Klein prime function is defined as

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\prod_{i\in\Theta''}\left\{\zeta,\frac{\gamma}{\gamma_i},\zeta_i\right\}, \tag{20}$$

where the product is over all substitutions in the set Θ''. The curly bracket notation denotes the cross-ratio defined in the standard way as

$$\left\{\zeta,\frac{\gamma}{\gamma_i},\zeta_i\right\} \equiv \frac{(\zeta_i-\gamma)(\gamma_i-\zeta)}{(\zeta_i-\zeta)(\gamma_i-\gamma)}. \tag{21}$$

The function $\omega(\zeta,\gamma)$ is single-valued on the whole ζ-plane, has a zero at γ and all points equivalent to γ under the substitutions of the group Θ, and, excepting the singular points of the group, is infinite only at $\zeta=\infty$.

The Schottky–Klein prime function can be regarded as fundamental and is the generalization to Riemann surfaces of genus g of the irreducible factor $(\zeta-\gamma)$ used in the construction of meromorphic functions on a genus-0 Riemann surface (i.e., the rational functions) and the function $P(\zeta/\gamma,\rho)$ (see (4)) used in the construction of meromorphic functions on a genus-1 Riemann surface (i.e., the loxodromic functions).

4.1. Trivial group. When the Schottky group is just the trivial group, the definition (20) reduces to $\omega(\zeta,\gamma)=(\zeta-\gamma)$. It is well known that any rational function with poles at the N points $\{\alpha_k|k=1,\dots,N\}$ and zeros at $\{\beta_k|k=1,\dots,N\}$ admits the representation

$$R\frac{(\zeta-\beta_1)(\zeta-\beta_2)\cdots(\zeta-\beta_N)}{(\zeta-\alpha_1)(\zeta-\alpha_2)\cdots(\zeta-\alpha_N)}, \tag{22}$$

where R is a multiplicative constant. Note that in this case there is no restriction on the locations of the poles and zeros of the function.

4.2. Loxodromic group. When the relevant Schottky group is the loxodromic group generated by the single substitution $\theta_1(\zeta)=\rho_1^2\zeta$, the definition (20) reduces to

$$\begin{aligned}\omega(\zeta,\gamma) &= (\zeta-\gamma)\prod_{k=1}^{\infty}\frac{(\rho_1^{2k}\zeta-\gamma)(\rho_1^{2k}\gamma-\zeta)}{(\rho_1^{2k}\zeta-\zeta)(\rho_1^{2k}\gamma-\gamma)}\\ &= (\zeta-\gamma)\prod_{k=1}^{\infty}\frac{(\rho_1^{2k}\zeta/\gamma-1)(\rho_1^{2k}\gamma/\zeta-1)}{(\rho_1^{2k}-1)(\rho_1^{2k}-1)}\\ &= \left(\frac{-\gamma}{\prod_{k=1}^{\infty}(\rho_1^{2k}-1)^2}\right)P(\zeta/\gamma,\rho_1),\end{aligned}$$

so that the relevant Schottky–Klein prime function $\omega(\zeta,\gamma)$ in this case is simply proportional to the function $P(\zeta/\gamma,\rho_1)$ given in the introduction. It is well known [15] that one representation for a loxodromic function with poles at $\{\alpha_k|k=1,\dots,N\}$ and zeros at $\{\beta_k|k=1,\dots,N\}$ is

$$R\frac{P(\zeta/\beta_1,\rho_1)P(\zeta/\beta_2,\rho_1)\cdots P(\zeta/\beta_N,\rho_1)}{P(\zeta/\alpha_1,\rho_1)P(\zeta/\alpha_2,\rho_1)\cdots P(\zeta/\alpha_N,\rho_1)}, \tag{23}$$

provided the poles and zeros satisfy the condition

$$\prod_{k=1}^{N} \alpha_k = \prod_{k=1}^{N} \beta_k, \tag{24}$$

i.e., there is a single condition on the poles and zeros of the function. It is important to point out that another representation of a loxodromic function with the same poles and zeros is given by

$$R\zeta \frac{P(\zeta/\beta_1, \rho_1)P(\zeta/\beta_2, \rho_1)\cdots P(\zeta/\beta_N, \rho_1)}{P(\zeta/\alpha_1, \rho_1)P(\zeta/\alpha_2, \rho_1)\cdots P(\zeta/\alpha_N, \rho_1)}, \tag{25}$$

where we emphasize the appearance of an additional prefactor of ζ in front of the ratio of products of the $P(\zeta, \rho)$-functions. In this case, the poles and zeros must satisfy the modified condition

$$\prod_{k=1}^{N} \alpha_k = \rho_1^2 \prod_{k=1}^{N} \beta_k. \tag{26}$$

Crowdy [20] has explicitly constructed quadrature domains corresponding to annular arrays of near-touching cylindrical particles using the second representation (25).

4.3. More general Schottky groups. By a natural extension of the familiar special cases of sections 4.1 and 4.2, it can be shown [21] that one representation of a meromorphic function on a Riemann surface of genus g with the poles $\{\alpha_k | k = 1, \ldots, N\}$ and zeros $\{\beta_k | k = 1, \ldots, N\}$ is

$$R\frac{\omega(\zeta, \beta_1)\omega(\zeta, \beta_2)\cdots\omega(\zeta, \beta_N)}{\omega(\zeta, \alpha_1)\omega(\zeta, \alpha_2)\cdots\omega(\zeta, \alpha_N)}. \tag{27}$$

It is natural that in the genus-g case there exist g conditions on the poles and zeros. These are the generalizations of the single condition (24) or (26) in the $g = 1$ case. To ascertain these conditions, introduce A_k and B_k as the two fixed points of the generating substitution θ_k defined as

$$A_k = \theta_k^{-\infty}\zeta, \quad B_k = \theta_k^{\infty}\zeta, \tag{28}$$

where ζ is any given point. Note that A_k and B_k are simply the roots of $\zeta = \theta_k(\zeta)$, which is just a quadratic because $\theta_k(\zeta)$ is a Mobius transformation. Letting $\theta_k = \zeta'$, it is possible to write

$$\frac{\zeta' - B_k}{\zeta' - A_k} = \mu_k e^{i\kappa_k} \frac{\zeta - B_k}{\zeta - A_k}, \tag{29}$$

where $\mu_k, \kappa_k \in \mathbb{R}$. The two roots A_k and B_k are then distinguished by the fact that $|\mu_k| < 1$ in (29). Now the function (27) is the required meromorphic function, provided the following g conditions hold:

$$\prod_{j=1}^{N} \prod_{\theta_i \in \Theta_k} \frac{(\alpha_j - \theta_i(B_k))}{(\alpha_j - \theta_i(A_k))} \bigg/ \frac{(\beta_j - \theta_i(B_k))}{(\beta_j - \theta_i(A_k))} = 1, \qquad k = 1, \ldots, g. \tag{30}$$

Note that the substitutions in the second product are taken from the subset Θ_k. The g conditions (30) will be referred to henceforth as the *automorphicity conditions*.

In the same way that both (23) and (25) are two different representations of a loxodromic function with the same distribution of poles and zeros, there are a number of distinct representations of meromorphic functions on a Riemann surface of genus $g > 1$ as shown in Baker [21]. In constructing a particular quadrature domain, it is necessary to ascertain which representation is the appropriate one needed to construct the required conformal mapping. One alternative representation (used later in the case studies) is

$$\left(R\zeta \prod_{i\in\Theta_1'} \frac{\zeta - \theta_i(B_1)}{\zeta - \theta_i(A_1)} \right) \frac{\omega(\zeta,\beta_1)\omega(\zeta,\beta_2)\cdots\omega(\zeta,\beta_N)}{\omega(\zeta,\alpha_1)\omega(\zeta,\alpha_2)\cdots\omega(\zeta,\alpha_N)}, \tag{31}$$

where θ_1 denotes the loxodromic transformation given as

$$\theta_1(\zeta) = \rho_1^2\zeta. \tag{32}$$

The poles and zeros also satisfy g automorphicity conditions, one of which is given by

(33)
$$\prod_{i=1}^{N}\prod_{j\in\Theta_1}\left(\frac{\beta_i - \theta_j(B_1)}{\beta_i - \theta_j(A_1)}\Big/\frac{\alpha_i - \theta_j(B_1)}{\alpha_i - \theta_j(A_1)}\right) = \frac{1}{\mu_1 e^{i\kappa_1}} \prod_{s\in_1\Theta_1''}\left(\frac{B_1 - \theta_s(A_1)}{A_1 - \theta_s(A_1)}\Big/\frac{B_1 - \theta_s(B_1)}{A_1 - \theta_s(B_1)}\right)^2,$$

while the remaining $g - 1$ conditions are given by

(34)
$$\prod_{i=1}^{N}\prod_{j\in\Theta_b}\left(\frac{\beta_i - \theta_j(B_b)}{\beta_i - \theta_j(A_b)}\Big/\frac{\alpha_i - \theta_j(B_b)}{\alpha_i - \theta_j(A_b)}\right) = \prod_{s\in_1\Theta_b}\left(\frac{\theta_s^{-1}(B_1) - A_b}{\theta_s^{-1}(A_1) - A_b}\Big/\frac{\theta_s^{-1}(B_1) - B_b}{\theta_s^{-1}(A_1) - B_b}\right)$$

for $b = 2, \ldots, g$, where A_b and B_b denote the fixed points of the mapping θ_b.

Finally, it is instructive to see how the general condition (30) reduces to (24) in the $g = 1$ case, where the Schottky group is the loxodromic group. In this case the group is generated by the single substitution,

$$\theta_1(\zeta) = \rho_1^2\zeta. \tag{35}$$

The subset Θ_1 then contains only the identity. It is also clear that

$$A_1 = \infty, \quad B_1 = 0. \tag{36}$$

With these identifications, it is easy to show that (30) is precisely equivalent to (24). Indeed, it is also straightforward to show that the factor

$$\zeta \prod_{i\in\Theta_1'} \frac{\zeta - \theta_i(B_1)}{\zeta - \theta_i(A_1)} \tag{37}$$

in (31) reduces simply to ζ in the case where the Schottky group is precisely the loxodromic group, so that (31) reduces to (25). At the same time, the automorphicity condition (33) reduces to (26).

5. Equations for the mapping parameters. Only quadrature domains satisfying quadrature identities of the form

$$\int\int_D h(z)d\bar{z} \wedge dz = 2i \sum_{k=1}^{N} a_k h(z_k) \tag{38}$$

will be considered here. The equations to be satisfied by the conformal mapping parameters come from the specified quadrature identity together with any assumptions made regarding the areas of the enclosed holes. Intuitively, it is useful to think of specifying the real parameter ρ_i as equivalent to specifying the area of the ith hole.

From (17) recall that we require

$$z(\bar{\alpha}_k^{-1}) = z_k, \quad k = 1, \ldots, N. \tag{39}$$

Also, recall that we require $\bar{z}(\zeta^{-1})$ to have poles in H only at the points $\bar{\alpha}_k^{-1}$. In this case, where the quadrature identity is of the form (38), these poles are simple. Thus near $\zeta = \bar{\alpha}_k^{-1}$, $\bar{z}(\zeta^{-1})$ has the form

$$\bar{z}(\zeta^{-1}) = \frac{P_k}{\zeta - \bar{\alpha}_k^{-1}} + \text{regular}, \tag{40}$$

where $P_k \in \mathbb{C}$. We therefore require that

$$a_k = \pi P_k z_\zeta(\bar{\alpha}_k^{-1}), \quad k = 1, \ldots, N. \tag{41}$$

It is useful to think of the N conditions (39) as being equations for the N poles $\{\alpha_k | k = 1, \ldots, N\}$, while (41) provides equations for the N zeros $\{\beta_k | k = 1, \ldots, N\}$.

This leaves only the set $\{\delta_k | k = 1, \ldots, g\}$ to be determined. However, equations for these can be understood as being given by the g automorphicity conditions (30). The equation count is therefore very natural, as indicated by the following schematic encapsulating the correspondence between parameters:

$$\begin{aligned} \{z_k \in \mathbb{C} | k = 1, \ldots, N\} &\to \{\alpha_k \in \mathbb{C} | k = 1, \ldots, N\}, \\ \{a_k \in \mathbb{C} | k = 1, \ldots, N\} &\to \{\beta_k \in \mathbb{C} | k = 1, \ldots, N\}, \\ \{\text{specifying the area of g holes}\} &\to \{\rho_k \in \mathbb{R} | k = 1, \ldots, g\}, \\ \{\text{g automorphicity conditions}\} &\to \{\delta_k \in \mathbb{C} | k = 1, \ldots, g\}. \end{aligned} \tag{42}$$

A minor modification of the prime function (20) is needed when the mapping is required to have a zero or pole at the point at infinity. In this case formula (20) must be replaced by

$$\omega(\zeta, \infty) = \prod_{i \in \Theta''} \frac{(\infty_i - \zeta)}{(\zeta_i - \zeta)}, \tag{43}$$

where ∞_i denotes the images of the point at infinity under the ith substitution of the set Θ''.

Many of the examples to follow possess various rotational symmetries in the distribution of the poles and zeros of the relevant conformal mapping function. It is therefore convenient to define $\omega_n(\zeta, \gamma)$ as

$$\omega_n(\zeta, \gamma) \equiv \prod_{k=0}^{n-1} \omega(\zeta, e^{2\pi i k/n}\gamma). \tag{44}$$

It should be noted that the Schottky–Klein prime function depends implicitly on the parameters $\{\delta_k, \rho_k | k = 1, \ldots, g\}$ from which the primary substitutions are constructed; however, the notation $\omega(\zeta, \gamma)$ suppresses this dependence.

6. Examples. In computing explicit cases it is necessary to truncate the number of maps used from any of the relevant infinite sets. This is done in a natural way by picking a *level* of composition of the primary substitutions up to which all composed substitutions are included. (Mumford, Series, and Wright [24] discuss various other methods of truncation.) For example, if a Schottky group has two primary substitutions θ_1 and θ_2 and all maps up to and including level 2 are used, the following maps would be included in the definition of $\omega(\zeta, \gamma)$:

$$\text{level } 1: \ \theta_1, \theta_2; \qquad \text{level } 2: \ \theta_1^2, \theta_2^2, \theta_1\theta_2, \theta_2\theta_1, \theta_1^{-1}\theta_2, \theta_1\theta_2^{-1}. \tag{45}$$

Note that the identity is the only map at level 0, and this is excluded in defining the Schottky–Klein prime function.

Since the zeros and poles in the Schottky–Klein prime function representation are explicit, construction of a given quadrature domain requires only consideration of the distribution of the poles and zeros of the conformal mapping in the ζ preimage plane. Often, the quadrature identity combined with symmetry considerations can be used to deduce the positions of these poles and zeros. The functional form of the relevant conformal mapping can then be written down immediately.

6.1. A triply connected quadrature domain. Consider four circular discs of equal radius r initially less than 1, with centers at $\pm\sqrt{3}$ and $\pm i$. For $r < 1$ the circular discs are disconnected. If we increase r to 1, then the discs touch. If $r \leq 1$, such a configuration is a disconnected quadrature domain satisfying the quadrature identity (38) with $N = 4$, $a_1 = a_2 = a_3 = a_4 = \pi r^2$, and $z_1 = \sqrt{3} = -z_3$, $z_2 = i = -z_4$. When r increases above 1, the domain satisfying the quadrature identity (38) with quadrature data given by $a_1 = a_2 = a_3 = a_4 = \pi r^2$ and $z_1 = \sqrt{3} = -z_3, z_2 = i = -z_4$ can be expected to form a triply connected quadrature domain.

We shall construct a triply connected domain which is close to the case of touching circular discs. In particular, we take $a_1 = a_2 = a_3 = a_4 = 1.0010\pi$ and $z_1 = 1.6966, z_2 = 0.9969i$.

Note that the quadrature domain is symmetric with respect to reflection in both the real and imaginary axes, and its two holes have their centers on the real axis. It is natural to expect the same structure in the associated circular region H in the ζ-plane. If C_1 and C_2 are the circles mapping to the boundaries of these two holes, we expect them to have equal radii with centers at $\delta_1, \delta_2 \in \mathbb{R}$, where $\delta_1 = -\delta_2$. The conformal mapping will have four poles corresponding to each of the z_k for $k = 1, 2, 3, 4$. We label these α_k for $k = 1, 2, 3, 4$. It will also have four zeros, which we label β_k $(k = 1, 2, 3, 4)$. Again, it is natural to expect the distribution of the poles of the conformal map in the ζ-plane to mirror the distribution of the points z_k $(k = 1, 2, 3, 4)$ in the physical plane. We therefore expect $\alpha_1 = -\alpha_3$ purely real and $\alpha_2 = -\alpha_4$ purely imaginary. Thus, the combination $\omega_2(\zeta, \alpha_1)\omega_2(\zeta, \alpha_2)$ will appear in the denominator of the conformal map. Note that the compact notation $\omega_2(\zeta, \alpha_1)$ (defined in (44)) automatically captures both the pole at α_1 and that at $-\alpha_1$. As for the zeros, because we choose $\zeta = 0$ to map to $z = 0$, one of the zeros (say β_3) is at the origin. Thus $\omega(\zeta, 0)$ appears in the numerator of the conformal map. By symmetry, one of the remaining three zeros (say β_4) must be at ∞, while the other two, β_1 and β_2, should be either purely real or purely imaginary with $\beta_1 = -\beta_2$. Thus, the combination $\omega(\zeta, \infty)\omega_2(\zeta, \beta_1)$ also appears in the numerator. In fact, it is found that β_1 is purely real. Figure 3 shows a schematic illustrating the ζ preimage plane and the distribution of poles and zeros in the fundamental region.

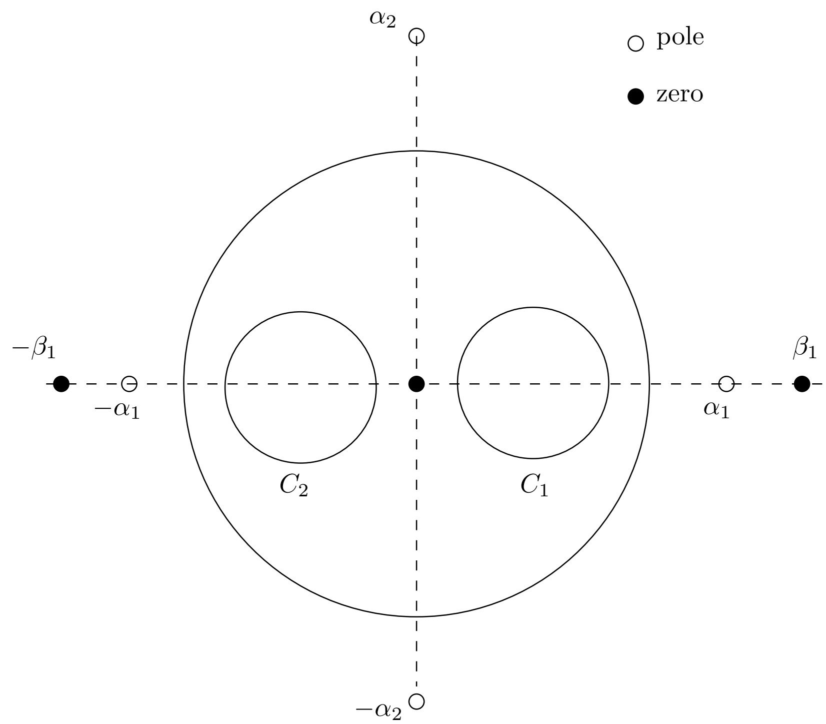

FIG. 3. *Schematic illustrating the ζ preimage plane with distribution of poles and zeros of the conformal mapping to the triply connected quadrature domain in Figure* 4.

Combining the above considerations, the form of the conformal map is deduced to be

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega(\zeta,\infty)\omega_2(\zeta,\beta_1)}{\omega_2(\zeta,\alpha_1)\omega_2(\zeta,\alpha_2)}. \tag{46}$$

The map contains six parameters: $R, \beta_1, \alpha_1, \alpha_2, \delta_1, \rho_1$. We can specify ρ_1, which corresponds to fixing the area of each of the two holes. Then the equations to solve for the remaining five unknowns come from (30), (39), (41). Note that, due to symmetry, the two equations given by (30) are actually the same, and (39), (41) each give only two independent equations. Thus we have five equations for five unknowns. Explicitly, these are

$$\prod_{j=1}^{4}\prod_{\theta_i\in\Theta_1}\frac{(\alpha_j-\theta_i(B_1))}{(\alpha_j-\theta_i(A_1))}\Bigg/\frac{(\beta_j-\theta_i(B_1))}{(\beta_j-\theta_i(A_1))}=1, \tag{47}$$

$$z_1 = z(\bar{\alpha}_1^{-1}), \tag{48}$$

$$z_2 = z(\bar{\alpha}_2^{-1}), \tag{49}$$

$$a_1 = \pi P_1 z_\zeta(\bar{\alpha}_1^{-1}), \tag{50}$$

$$a_2 = \pi P_2 z_\zeta(\bar{\alpha}_2^{-1}), \tag{51}$$

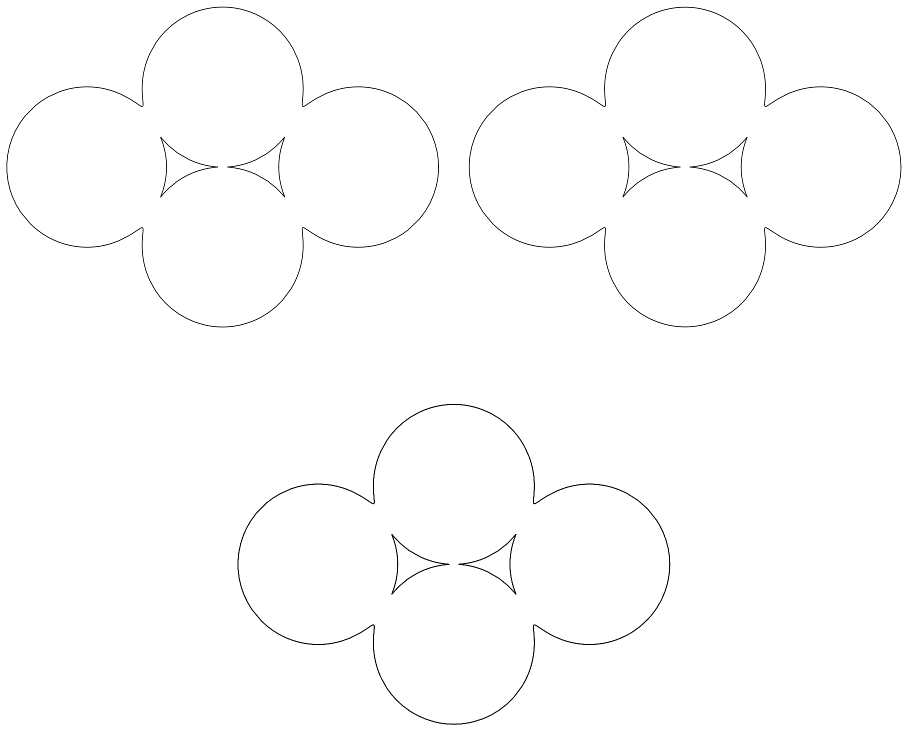

FIG. 4. *Triply connected domain constructed using the Schottky–Klein prime function (top left) and Poincaré series (top right) both at level* 3. *Here* $R = 2.5215$, $\beta_1 = 1.4776$, $\alpha_1 = 1.1520$, $\alpha_2 = 1.5969i$, $\delta_1 = 0.3160$, $\rho_1 = 0.2000$. *For comparison, the lower diagram shows a superposition of the upper two diagrams.*

where analytical formulae for P_1 and P_2 can easily be deduced. For example,

$$P_1 = -\frac{R}{\alpha_1^2}\left(\frac{\omega(\alpha_1, 0)\omega(\alpha_1, \infty)\omega_2(\alpha_1, \beta_1)}{\hat{\omega}(\alpha_1, \alpha_1)\omega(\alpha_1, -\alpha_1)\omega_2(\alpha_1, \alpha_2)}\right), \tag{52}$$

where $\hat{\omega}(\zeta, \gamma)$ is defined as

$$\hat{\omega}(\zeta, \gamma) \equiv \prod_{i\in\Theta''}\left\{\zeta, \frac{\gamma}{\gamma_i}, \zeta_i\right\}. \tag{53}$$

These five equations are solved for the unknown parameters using Newton's method. The image of the conformal map is shown in the left-most diagram in Figure 4.

For purposes of comparison with the approach to constructing quadrature domains expounded recently by Richardson [23] we constructed the *same* quadrature domain using a conformal map based on the use of Poincaré series as opposed to the Schottky–Klein prime function. The image of this map is shown in the right-most diagram in Figure 4. The images of the respective conformal maps are indistinguishable, as can be seen from their superposition in the lower diagram in Figure 4. A brief overview of Richardson's general method, and details of how it was used to construct the above triply connected domain, are given in the appendix.

6.2. A quadruply connected quadrature domain. A second example is to consider three circular discs in an annular array surrounding a smaller circular disc.

The quadrature identity associated with such a domain is of the form (38) with $N = 4$ and z_1 purely real, $z_2 = z_1 e^{2\pi i/3}, z_3 = z_1 e^{4\pi i/3}$, and $z_4 = 0$. In the case where the circular discs are touching, we have $a_1 = a_2 = a_3 = \pi$, $a_4 = \pi(2/\sqrt{3} - 1)^2$ and $z_1 = 2/\sqrt{3}, z_4 = 0$. We shall construct a quadruply connected domain which is close to the case of touching discs. In particular, we have taken the quadrature data to be $a_1 = a_2 = a_3 = 1.0010\pi, a_4 = 0.0250\pi$ and $z_1 = 1.1488, z_4 = 0$.

The domain has three holes. Since the holes in the physical plane are rotations of each other through $\frac{2\pi}{3}$, we expect the circles in the preimage plane to share these symmetries. Let C_1, C_2, and C_3 be the circles inside the unit ζ-circle mapping to the boundaries of the holes. Then C_1 will be centered on the ray $\arg[\zeta] = \frac{\pi}{3}$, and C_2 and C_3 will be rotations of this circle through $\frac{2\pi}{3}$.

If we fix the physical origin to be the image of $\zeta = 0$, then we require the factor $\omega(\zeta, 0)$ to appear in the numerator of the conformal map. Note that one of the z_k is zero, namely z_4. Thus from (17), we see that we require the pole α_4 (corresponding to the point z_4) to be at ∞. Thus we must include the factor $\omega(\zeta, \infty)$ in the denominator of the map. There will also be three other poles, symmetrically disposed about $\zeta = 0$, corresponding to the symmetrically disposed points z_1, z_2, and z_3. One of these, denoted α_1, is on the real ζ-axis. There will also be three additional zeros of the conformal map in the fundamental region, which are also expected to be symmetrically disposed about $\zeta = 0$. One of these, β_1 say, is found to be real.

Using these considerations, the conformal map is deduced to have the form

$$z(\zeta) = R\frac{\omega(\zeta, 0)\omega_3(\zeta, \beta_1)}{\omega(\zeta, \infty)\omega_3(\zeta, \alpha_1)}. \tag{54}$$

The image of this map, constructed to level-3 accuracy, is shown in the left-most diagram in Figure 5 along with the image of the conformal map constructed using the Poincaré series method of Richardson [23] to its right (again, to level-3 accuracy). Their superposition is also shown in Figure 5. The boundaries are indistinguishable.

6.3. A quintuply connected quadrature domain. It is straightforward to generalize the previous example to a quadrature domain which is close to the case of four circular discs in an annular array surrounding a smaller circular disc. The quadrature identity associated with such a domain is of the form (38) with $N = 5$ and z_1 purely real, $z_k = z_1 e^{(k-1)\pi i/2}, k = 2, 3, 4$, and $z_5 = 0$. In the case where the discs are touching, we have $a_1 = a_2 = a_3 = a_4 = \pi$, $a_5 = \pi(\sqrt{2} - 1)^2$ and $z_1 = \sqrt{2}, z_5 = 0$. We shall construct a quintuply connected domain which is close to the case of touching circular discs. In particular, we choose quadrature data given by $a_1 = a_2 = a_3 = a_4 = 0.9980\pi, a_5 = 0.1716\pi$ and $z_1 = 1.4029, z_5 = 0$.

Considerations similar to the previous example can be used to deduce that the associated map has the form

$$z(\zeta) = R\frac{\omega(\zeta, 0)\omega_4(\zeta, \beta_1)}{\omega(\zeta, \infty)\omega_4(\zeta, \alpha_1)}. \tag{55}$$

The image of the conformal map is shown in the left-most diagram in Figure 6 along with the image of the conformal map constructed using Poincaré series. Both are constructed to level-3 accuracy. Their superposition is also shown in Figure 6 and, again, the quadrature domain boundaries are indistinguishable. Crowdy [13] has considered this class of domains from the point of view of algebraic curves in the context of constructing multipolar equilibria of the Euler equations, and this will be considered again later in section 8.

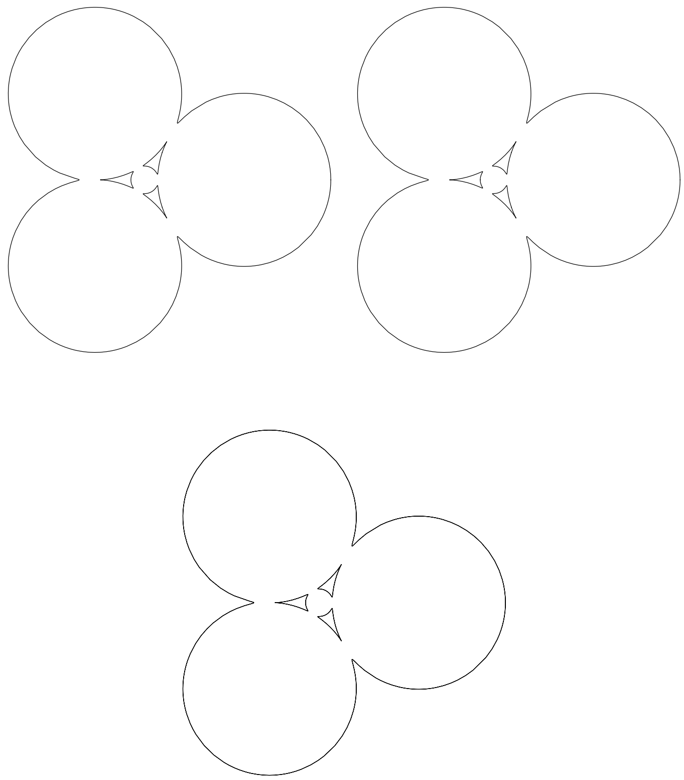

FIG. 5. *Quadruply connected domain constructed using the Schottky–Klein prime function (top left) and Poincaré series (top right) at level* 3*, with superposition (lower). Here* $R = 0.0536,\ \beta_1 = 3.4038,\ \alpha_1 = 1.2500,\ \delta_1 = 0.2608e^{i\pi/3},\ \rho_1 = 0.1275$.

6.4. A septuply connected quadrature domain. Richardson [23] has considered the case of *six* circular discs in an annular array containing a disc of equal radius in the center. Such a case is a trivial extension of the examples in sections 6.2 and 6.3. The preceding two examples have conformal mappings of the general functional form

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega_n(\zeta,\beta_1)}{\omega(\zeta,\infty)\omega_n(\zeta,\alpha_1)}, \tag{56}$$

where section 6.2 deals with $n = 3$ while section 6.3 treats the case $n = 4$. The case of six circular discs surrounding a central one will have a conformal map of the form

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega_6(\zeta,\beta_1)}{\omega(\zeta,\infty)\omega_6(\zeta,\alpha_1)}, \tag{57}$$

i.e., it is given by a mapping of the form (56) with $n = 6$. The associated quadrature identity is of the form (38) with $N = 7$ and z_1 purely real, $z_k = z_1 e^{(k-1)\pi i/3}, k = 2, \dots, 6$, and $z_7 = 0$. In the case of touching circular discs, we have $a_1 = \cdots = a_7 = \pi$ and $z_1 = 2, z_7 = 0$. For illustration, we construct a septuply connected domain which is close to the case of touching circular discs with $a_1 = \cdots = a_6 = 1.0266\pi, a_7 = 1.0010\pi$ and $z_1 = 2.0002, z_7 = 0$. Figure 7 shows the results constructed using both methods to level-2 accuracy.

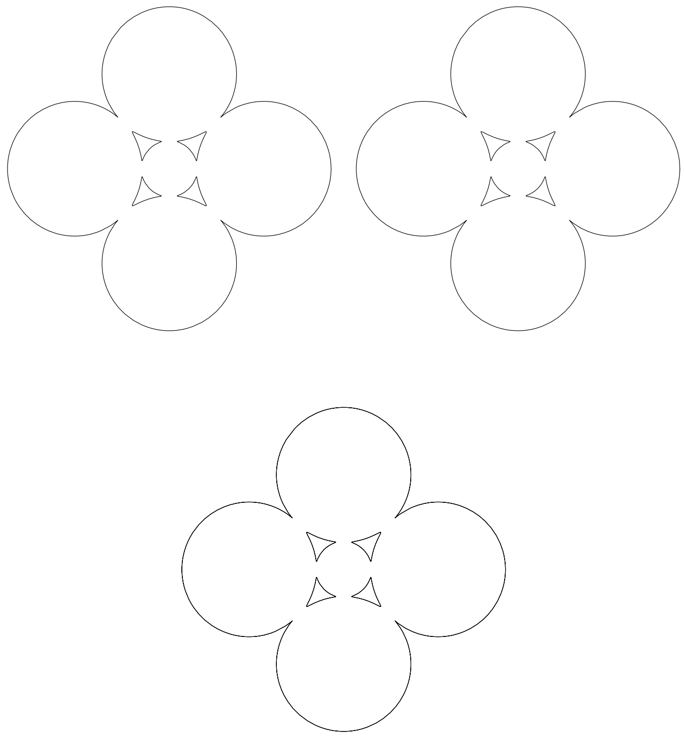

FIG. 6. *Quintuply connected domain constructed using the Schottky–Klein prime function (top left) and Poincaré series (top right) at level* 3, *with superposition (lower). Here* $R = 0.1849$, $\beta_1 = 2.0789$, $\alpha_1 = 1.2329$, $\delta_1 = 0.3591 e^{i\pi/4}$, $\rho_1 = 0.1290$.

6.5. 3-by-3 square array. More complicated domains are also easy to construct using the Schottky–Klein prime function representation. Consider, for example, nine circles in a 3-by-3 square array. The associated quadrature domain is of the form (38) with $N = 9$ and z_1 real, $z_2 = z_1 i, z_3 = -z_1, z_4 = -z_1 i$, and z_5 on the $\pi/4$ ray, with $z_6 = z_5 i, z_7 = -z_5, z_8 = -z_5 i$, and $z_9 = 0$. In the case where the circular discs are touching, we have $z_1 = 2, z_5 = 2\sqrt{2} e^{\pi i/4}$, and $a_1 = \cdots = a_9 = \pi$. We shall construct a quintuply connected domain which is close to the case of touching circular discs with $a_1 = \cdots = a_9 = 1.0010\pi$ and $z_1 = 1.9533, z_5 = 2.7696 e^{\pi i/4}, z_9 = 0$.

There will be four circles $C_1, \ldots, C_4$ inside the unit circle in the preimage ζ-plane. Since the holes in the physical plane are centered on the rays $\arg[z] = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, we also expect the centers of the circles $C_1, \ldots, C_4$ to be on these rays in the ζ-plane. Let $\zeta = 0$ map to $z = 0$. This means that $\omega(\zeta, 0)$ must appear in the numerator of the conformal map. Furthermore, because $z_9 = 0$, there must be a corresponding pole of the conformal map at infinity in the ζ-plane. Therefore, $\omega(\zeta, \infty)$ must appear in the denominator. We expect four symmetrically disposed poles in the ζ-plane corresponding to $z_1, \ldots, z_4$. Let one of these be α_1 on the real axis. Similarly, let α_2 (taken on the ray $\arg[\zeta] = \frac{\pi}{4}$) and its rotations through $\frac{\pi}{2}$ correspond to $z_5, \ldots, z_8$. Thus, the combination $\omega_4(\zeta, \alpha_1)\omega_4(\zeta, \alpha_2)$ will also appear in the denominator. The zeros are expected to be similarly distributed in the ζ-plane. Therefore we include the combination $\omega_4(\zeta, \beta_1)\omega_4(\zeta, \beta_2)$ in the numerator so that the zeros of the map are β_1 (and its three rotations through $\frac{\pi}{2}$) and β_2 (along with its three rotations through $\frac{\pi}{2}$). It is found that β_1 is real while β_2 is on the ray $\arg[\zeta] = \frac{\pi}{4}$.

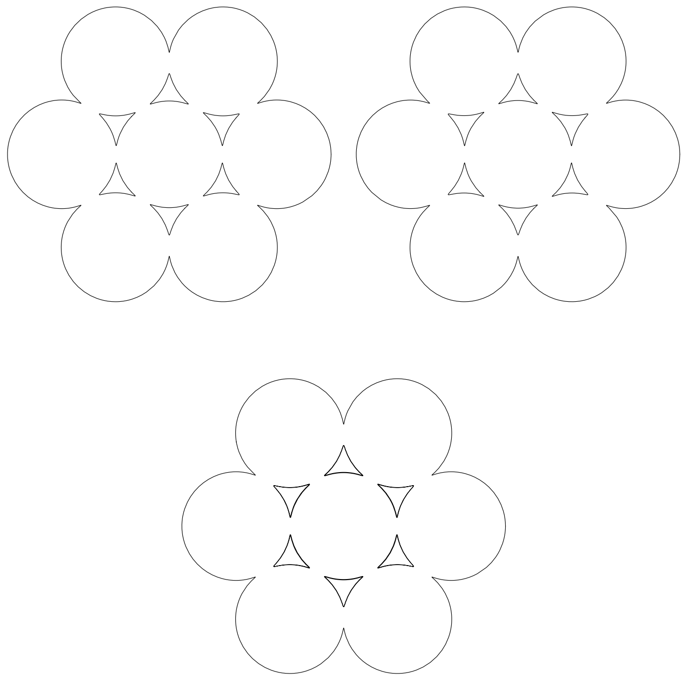

FIG. 7. *Septuply connected domain constructed using the Schottky–Klein prime function (top left) and Poincaré series (top right) at level* 2*, with superposition (lower). Here* $R = 0.6089$, $\beta_1 = 1.5358$, $\alpha_1 = 1.2195$, $\delta_1 = 0.4900e^{i\pi/6}$, $\rho_1 = 0.1260$.

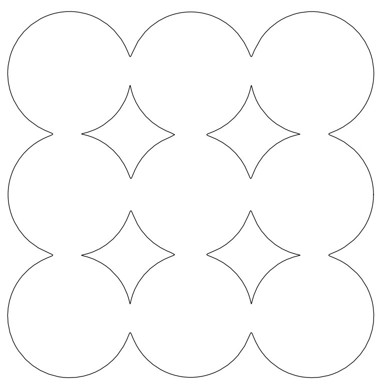

FIG. 8. 3*-by-*3 *square array constructed using the Schottky–Klein prime function at level* 2*. Here* $R = 0.7887$, $\beta_1 = 1.8298$, $\beta_2 = 1.1828e^{i\pi/4}$, $\alpha_1 = 1.3899$, $\alpha_2 = 1.0989e^{i\pi/4}$, $\delta_1 = 0.5093e^{i\pi/4}$, $\rho_1 = 0.2100$.

The conformal map therefore has the form

$$z(\zeta) = R\frac{\omega(\zeta,0)\omega_4(\zeta,\beta_1)\omega_4(\zeta,\beta_2)}{\omega(\zeta,\infty)\omega_4(\zeta,\alpha_1)\omega_4(\zeta,\alpha_2)}. \tag{58}$$

The image of the conformal map is shown in Figure 8.

6.6. Examples with a loxodromic subgroup. If we map H onto a multiply connected quadrature domain which is rotationally symmetric about the origin and which has the origin inside one of its holes, then the boundary ∂D_1 of this hole will be centered at the origin, and thus the circle C_1 in the associated circular region in the ζ-plane will be centered at $\zeta = 0$. So, referring to (14), we see that the associated Schottky group will contain a loxodromic subgroup.

If the quadrature domain is in fact just doubly connected and ∂D_1 is its only inner boundary, then the associated Schottky group will be precisely the loxodromic group. In this case, the form (23) does not map the circular region to the required image. However, Crowdy [20] has shown that the appropriate loxodromic function is given by (25) where the poles and zeros satisfy (26). It is similarly found that if a more general Schottky group contains the loxodromic group as a subgroup, it is necessary to use a suitably generalized representation of the required conformal mapping.

We now present an example where the Schottky group has a loxodromic subgroup. The example chosen is one suggested by Richardson [23]. Consider six circular discs arranged in a triangular array. The quadrature identity associated with such a domain is of the form (38) with $N = 6$ and z_1 purely real, $z_2 = z_1 e^{2\pi i/3}, z_3 = z_1 e^{4\pi i/3}$, and z_4 on the $\pi/3$ ray, $z_5 = z_4 e^{2\pi i/3}, z_6 = z_4 e^{4\pi i/3}$. In the case where the circular discs are touching we have $a_1 = \cdots = a_6 = \pi$ and $z_1 = \frac{2}{\sqrt{3}}, z_4 = \frac{4}{\sqrt{3}} e^{\pi i/3}$. We shall construct a quintuply connected domain which is close to the case of touching circular discs with $a_1 = \cdots = a_6 = 1.0500\pi$ and $z_1 = 1.1737, z_4 = 2.3536 e^{\pi i/3}$.

In this case, there will be a total of four enclosed holes: one centered at the origin and three others at symmetrically disposed positions about the origin. Let C_1 be the circle in the ζ-plane mapping to the central hole, and let C_2, C_3, C_4 map to the other three holes. C_2 is a circle centered at some point δ_2 on the ray $\arg[\zeta] = \frac{\pi}{3}$, while C_3 and C_4 are the rotations of this circle through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively. Corresponding to z_1, z_2, and z_3 we expect three symmetrically disposed poles in the ζ-plane. Let one of these be α_1 on the real axis. Similarly, let α_2 (on the ray $\arg[\zeta] = \frac{\pi}{3}$) and its two rotations through $\frac{2\pi}{3}$ correspond to z_4, z_5, and z_6. The combination $\omega_3(\zeta, \alpha_1)\omega_3(\zeta, \alpha_2)$ will therefore appear in the denominator of the conformal map. Again, the distribution of zeros is expected to be similar. Thus, we put $\omega_3(\zeta, \beta_1)\omega_3(\zeta, \beta_2)$ in the numerator so that β_1 and β_2 (along with their respective rotations through $\frac{2\pi}{3}$) will be the zeros of the conformal map in the fundamental region. It is found that β_1 is real while β_2 is on the ray $\arg[\zeta] = \frac{\pi}{3}$.

A natural choice to make for the mapping is therefore

$$z(\zeta) = R \frac{\omega_3(\zeta, \beta_1)\omega_3(\zeta, \beta_2)}{\omega_3(\zeta, \alpha_1)\omega_3(\zeta, \alpha_2)}. \tag{59}$$

However, no univalent conformal maps to a quadrature domain with the given quadrature data could be found for a map of this form. Therefore, a modified representation of a meromorphic function on the same Riemann surface (and with the same poles and zeros) is required. Such a representation is given by (31). Thus, it is natural to propose that the conformal mapping has the generalized form

$$z(\zeta) = \left(R\zeta \prod_{i \in \Theta_1'} \frac{\zeta - \theta_i(B_1)}{\zeta - \theta_i(A_1)} \right) \frac{\omega_3(\zeta, \beta_1)\omega_3(\zeta, \beta_2)}{\omega_3(\zeta, \alpha_1)\omega_3(\zeta, \alpha_2)}, \tag{60}$$

where θ_1 denotes the loxodromic transformation

$$\theta_1(\zeta) = \rho_1^2 \zeta \tag{61}$$

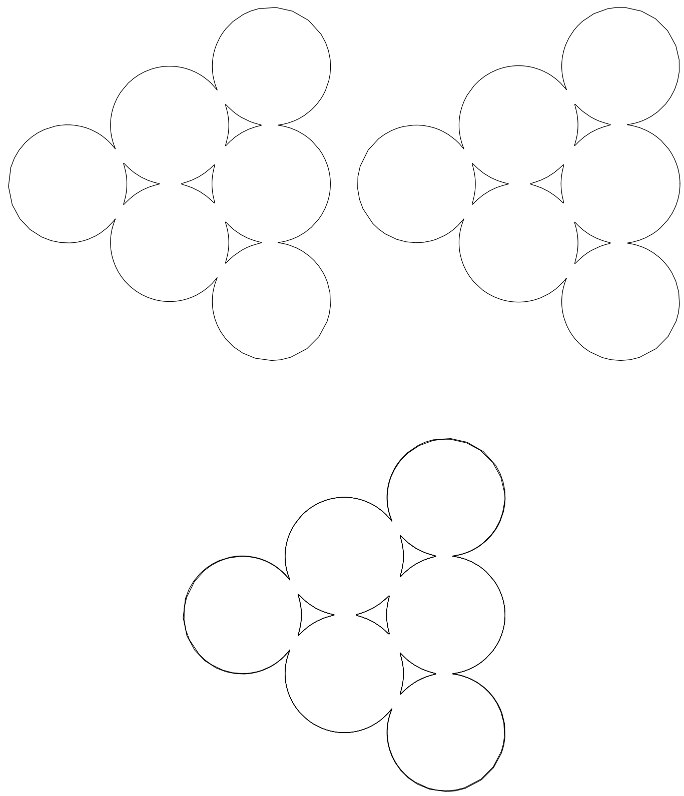

FIG. 9. *Quintuply connected domain constructed using the Schottky–Klein prime function (top left) and Poincaré series (top right) at level* 2*, with superposition (lower). Here* $R = 0.0302$*,* $\beta_1 = 5.3203$*,* $\beta_2 = 1.1228e^{i\pi/3}$*,* $\alpha_1 = 1.6393$*,* $\alpha_2 = 1.0526e^{i\pi/3}$*,* $\delta_1 = 0$*,* $\delta_2 = 0.6054e^{i\pi/3}$*,* $\rho_1 = 0.1450$*,* $\rho_2 = 0.1450$*.*

associated with the circle $C_1 = \{|\zeta| = \rho_1\}$. Note that, accordingly, the poles and zeros must now satisfy g modified automorphicity conditions given (in the general case) by (33) and (34). The map (60) is indeed found to provide the required univalent map to a quadrature domain satisfying the given quadrature identity. It is emphasized that the additional prefactor in (60) relative to (59) is precisely the generalization of the additional ζ-prefactor in (25) relative to (23).

The image of the conformal map constructed using both conformal mapping methods is shown in Figure 9 along with their superposition. Although this complicated domain is only constructed to level-2 accuracy (in both methods), the plots are again virtually indistinguishable.

7. Nonsymmetric domains. All the examples considered so far have certain degrees of spatial symmetry. However, the general method also applies to domains devoid of any such symmetry. Figure 10 shows two typical quadrature domains, plotted using conformal maps based on Schottky–Klein prime functions, possessing less symmetry than those in Figure 4. The constructive method is essentially the same, with only minor differences. For example, with no symmetry, there are now *two* independent automorphicity conditions, whereas in the symmetric case there was just one.

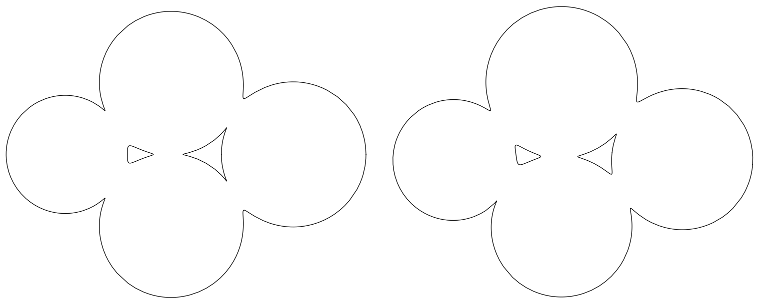

FIG. 10. *More general triply connected quadrature domains.*

8. Algebraic curves and uniformization. In the context of steady vortical equilibria of the Euler equation, Crowdy [13] has recently presented an alternative construction of multiply connected quadrature domains from their quadrature data. The method makes use of the result that the boundaries of quadrature domains are algebraic curves [2]. For completeness, and purposes of comparison, we now use conformal maps to reconstruct one of the domains of [13].

The quintuply connected quadrature domains constructed in [13] satisfy the identity

$$\int\int_D h(z)dxdy = \pi r^2 h(z_1) + \pi r^2 h(z_2) + \pi r^2 h(z_3) + \pi r^2 h(z_4) + \pi p^2 h(0). \tag{62}$$

To within a finite set of *special points* [6] (which turn out to be useful in the construction; see [13]), the boundaries of the domains corresponding to (62) are given by the algebraic curve

$$\mathcal{P}(z, \bar{z}) = 0, \tag{63}$$

where

$$\mathcal{P}(z, w) = \sum_{k,j=0}^{5} a_{kj} z^k w^j. \tag{64}$$

The set of coefficients $\{a_{kj}\}$ form a Hermitian matrix $\mathbf{A}$, where $\mathbf{A}_{kj} = a_{kj}$ and

$$\mathbf{A} = \begin{pmatrix} k & 0 & 0 & 0 & 4p^2 & 0 \\ 0 & g & 0 & 0 & 0 & -4 \\ 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \\ 4p^2 & 0 & 0 & 0 & -(4r^2 + p^2) & 0 \\ 0 & -4 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{65}$$

The top-left diagram of Figure 11 features a reproduction of the quadrature domain in Figure 8 of Crowdy [13] (this reference contains all the information required to derive the matrix $\mathbf{A}$).

The relevant conformal map will have the form (55). In addition to the quadrature data, to determine this map we also need to specify the area of the holes. This can be computed using the algebraic curve, but we employ an alternative (equivalent)

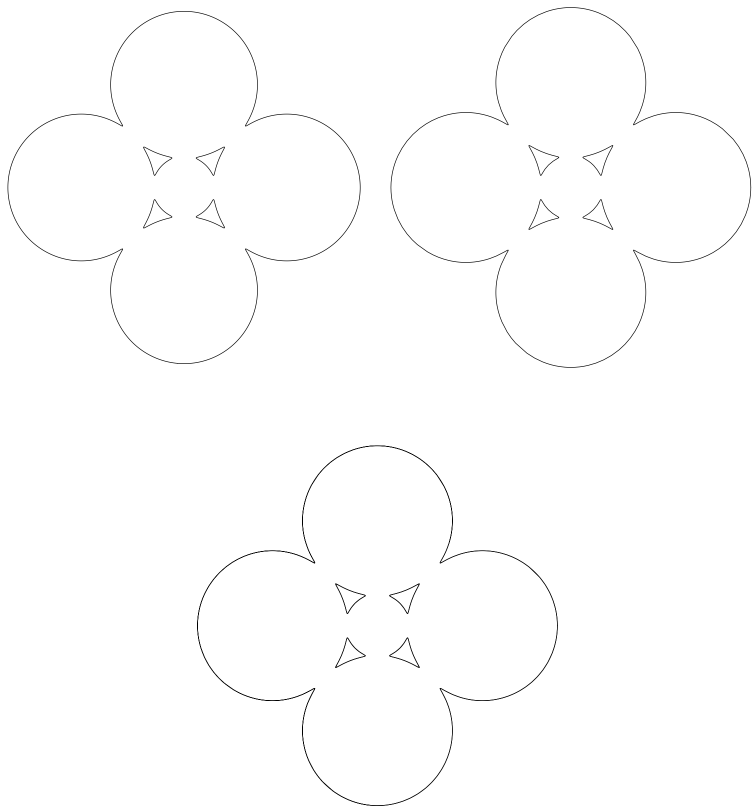

FIG. 11. *A quintuply connected quadrature domain from Crowdy (see Figure* 8 *of* [13]*) constructed using algebraic curves (top left); the same domain constructed using conformal maps based on the Schottky–Klein prime function (top right). The superposition is shown in the lower diagram.*

method. Following Crowdy [13], it is known that there exists a so-called special point at some point $z_s = se^{i\pi/4}$. At such a point it is known [13] that

$$\bar{z}_s = S(z_s), \tag{66}$$

where $S(z)$ is known as the Schwarz function of the quadrature domain boundary [13]. It is related to the conformal mapping function by the relation

$$S(z(\zeta)) = \bar{z}(\zeta^{-1}). \tag{67}$$

Crowdy [13] gives the explicit value $s = 1.008$ for the domain shown in Figure 11 (on the left). In terms of the conformal map, this point must correspond to the image of some point $\hat{\delta}e^{i\pi/4}$ in the ζ preimage plane, i.e.,

$$z_s = se^{i\pi/4} = z(\hat{\delta}e^{i\pi/4}). \tag{68}$$

At the same time, by the property (66), we must have

$$\bar{z}_s = se^{-i\pi/4} = \bar{z}(\hat{\delta}^{-1}e^{-i\pi/4}). \tag{69}$$

Therefore, instead of considering the area of the holes, (68) and (69) provide two equations relating the conformal mapping parameters, one determining the newly introduced $\hat{\delta}$ and the other effectively specifying the area of the hole. The top-right diagram in Figure 11 shows the domain constructed using the conformal map. The lower figure shows a superposition with the domain constructed using algebraic curves. Again, the boundaries are indistinguishable.

From a theoretical viewpoint, the conformal map just constructed essentially provides the *uniformization* of the algebraic curve (63). That is, given the matrix $\mathbf{A}$, the conformal map is such that

$$\mathcal{P}(z(\zeta), \bar{z}(\zeta^{-1})) = 0. \tag{70}$$

This relation holds everywhere on the boundary of the quadrature domain, but it also holds globally by analytic continuation. In this sense, the conformal map has uniformized the algebraic curve.

9. Discussion. There are a variety of ways in which multiply connected quadrature domains can be constructed from their quadrature data (and information regarding the area of any holes). The algebraic curve method of Crowdy [13] has many conceptual advantages and requires the least analytical overhead. Using this method, an implicit description of the boundary is obtained. The idea is to iterate on the algebraic curve coefficients until equations deriving from the quadrature identity are satisfied. When the domains have symmetry, the consideration of the special points of the domain can greatly facilitate the construction by providing explicit sets of equations to be satisfied by the coefficients of the curve. The special points can also have physical significance; in Crowdy [13] they corresponded to stagnation points of the vortical flow.

In this paper, a conceptually different method has been used based on conformal mapping from a canonical region in a parametric plane. This leads to an explicit representation of the boundary curve. The Schottky model has been employed and the mappings written as ratios of products of Schottky–Klein prime functions. These functions are the natural generalizations of the well-known prime functions in a simply and doubly connected case, as discussed in the introduction. The conformal mappings are essentially "uniformizing functions" of the algebraic curves considered in [13]. Richardson [23] has presented an alternative conformal mapping method based on the use of Poincaré series to represent the mapping functions.

From a mathematical point of view, it is natural to ask questions about the convergence properties of the infinite products used in defining the Schottky–Klein prime functions. We have not studied such questions in detail. However, the boundaries of the quadrature domains obtained in the explicit examples of this paper have been found to be indistinguishable from those obtained using either the algebraic curve method of Crowdy [13] or the conformal mapping method based on Poincaré series introduced by Richardson [23]. We consider this to be direct evidence that convergence issues do not necessarily constitute an impediment to the practical use of the Schottky–Klein prime function in the reconstruction of quadrature domains from their quadrature data.

Appendix. The method of Richardson [23]. We shall now briefly describe an alternative construction of the quadrature domains via an approach using Poincaré series as expounded recently by Richardson [23]. This method also produces maps from circular regions of a parametric ζ-plane and requires the machinery of the Schottky groups associated with these circular regions. The method differs in the functional form, and representation, of the conformal mapping functions; Richardson constructs his maps as a ratio of two automorphic forms which are each constructed as Poincaré series.

DEFINITION A.1. *A Poincaré series associated with a given Schottky group is of the form*

$$T(\zeta) = \sum_{i=0}^{\infty} \frac{H(\theta_i(\zeta))}{(c_i\zeta + d_i)^{2m}}, \tag{71}$$

where

$$\theta_i(\zeta) \equiv \frac{a_i\zeta + b_i}{c_i\zeta + d_i}, \quad a_i d_i - b_i c_i = 1 \tag{72}$$

denotes the ith Mobius map of the Schottky group, $H(\zeta)$ is some rational function of which none of the poles is at a singular point of the Schottky group, and m is an integer. Provided $\zeta = \infty$ is not a singular point of the Schottky group, this series converges for all $m \geq 2$.

DEFINITION A.2. *A form $\phi(\zeta)$ is called an automorphic form with respect to the Schottky group if it has the property*

$$\phi(\theta_i(\zeta)) = (c_i\zeta + d_i)^{2m}\phi(\zeta) \tag{73}$$

for all maps θ_i of the Schottky group, where m is some integer.

If the Schottky group Θ is generated by g basic maps, and $\phi(\zeta)$ is an automorphic form with Z zeros and P poles in the fundamental region, then it is known that

$$Z - P = 2mg. \tag{74}$$

If $\phi(\zeta)$ is in fact an automorphic function, then we see $Z = P$.

Richardson's construction is to use two different choices of the rational functions $H_n(\zeta), H_d(\zeta)$ to form the respective Poincaré series for two automorphic forms $T_n(\zeta), T_d(\zeta)$ corresponding to the same value of $m \geq 2$. Then the ratio

$$\frac{T_n(\zeta)}{T_d(\zeta)} \tag{75}$$

and any constant multiple of this give the required automorphic function. Richardson's strategy is precisely the one described by Beardon [22] for the construction of meromorphic functions on compact Riemann surfaces.

Given a quadrature domain, there are a number of constraints on the relevant choices for $H_n(\zeta)$ and $H_d(\zeta)$. These are discussed in the context of a number of specific examples in Richardson [23]. Here we give very brief details of the construction for the triply connected example of section 6.1. Following Richardson, we choose $m = 2$. Recall that, in this example, there are poles at α_1 and α_3, where $\alpha_1 = -\alpha_3$ (purely real), and two at α_2 and α_4, where $\alpha_2 = -\alpha_4$ (purely imaginary). Also, $g = 2$. Following Richardson [23], we take $H_d(\zeta)$ to be 1. From (74) it follows that $T_d(\zeta)$ has eight zeros in the fundamental region. Due to the symmetry of the quadrature domain, we expect these zeros to be arranged in a pattern that is symmetric with respect to reflection in both axes. Thus we include in $H_n(\zeta)$ the polynomial factor $(\zeta^8 + a\zeta^6 + b\zeta^4 + c\zeta^2 + d)$, where the four real parameters a, b, c, d are to be chosen so that $T_n(\zeta)$ has the same zeros as $T_d(\zeta)$ in the fundamental region. Also due to the symmetry of the quadrature domain, we include a factor of ζ in the numerator of $H_n(\zeta)$. Finally, because none of the z_k in the associated quadrature identity are zero, the map must be bounded as $\zeta \to \infty$; in fact, we require it to behave like $1/\zeta$ at ∞. So the denominator of $H_n(\zeta)$ must be of degree 10. Since we require simple poles at $\alpha_1, \alpha_2, \alpha_3$, and α_4, we include the factors $(\zeta - \alpha_j)$ for $j = 1, \ldots, 4$ in the denominator of $H_n(\zeta)$. We must then choose the remaining factors such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are still the only poles of the map in the fundamental region. This can be done by choosing to include $(\zeta - \theta_i(\alpha_j))$ for $i = 1, 2$ and $j = 1, 2, 3$ as the extra factors. Finally, the form for the map is the ratio (75) multiplied by some constant R to be determined.

In this representation there are *nine* unknowns, namely $R, \alpha_1, \alpha_2, \delta_1, \rho_1$ as well as a, b, c, d. We can specify a value for ρ_1, thus leaving eight unknowns. The eight equations for the remaining eight unknowns are (39) and (41) plus the four from the requirement that $T_n(\zeta)$ be zero at the zeros of $T_d(\zeta)$ in the fundamental region. Note that these zeros are not known explicitly and must therefore be found (numerically) as part of the solution.

There are a number of comments to be made concerning the two methods:

(i) The zeros of the map are not explicit in the Poincaré series representation, but are explicit in the Schottky–Klein prime function representation. The explicitness of the poles and zeros means that the general form of the required mapping can be written down immediately.

(ii) Once ρ is specified, the Poincaré series representation depends on *eight* parameters compared to only *five* parameters when the Schottky–Klein prime function representation is used. Moreover, the determination of the eight parameters in the Poincaré series representation in fact requires the solution of *twelve* nonlinear equations, owing to the fact that the four (distinct) zeros of $T_d(\zeta)$ (in the fundamental region) must be found numerically during the solution process. In the prime function representation, exactly five equations are solved for exactly five unknowns.

(iii) Two of the equations to be solved in either method are the residue equations (41). With the prime function representation, explicit formulae for the residues P_1, P_2 are straightforward to compute (cf. the formula for P_1 in (52)). However, care has to be taken when finding the analogous equations with the Poincaré series representation because the inclusion of factors such as $(\zeta - \theta_i(\alpha_j))$ in the denominator of $H_n(\zeta)$ can mean that more than one term in the sum $T_n(\zeta)$ contributes to the residue at each of the poles.

(iv) As discussed in detail by Richardson [23], the most convenient choice is $H_d(\zeta) = 1$. However, if the Schottky group has a loxodromic subgroup, then the Poincaré series $T_d(\zeta)$ with $H_d(\zeta) = 1$ does not converge. Richardson therefore proposes three possible remedial measures in this case, two of which are not implemented for various reasons. Such complications do not arise when using the Schottky–Klein prime function representation. In the latter case, it is simply necessary to pick the appropriate representation for the mapping, which can involve additional prefactors of the ratio of products of prime functions, as illustrated explicitly in the context of the example in section 6.6.

(v) A particular advantage of using the Schottky–Klein prime function representations concerns changes of topology, particularly in cases where the connectivity of the domain decreases. In the conformal mappings constructed in this paper, the functional form of the mappings as ratios of products of prime functions is the same; the only change is the definition of the relevant Schottky group.

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