M4A32: Vortex dynamics

Problem Sheet 5: SOLUTIONS

(Distributed vorticity)

1.(a) Let

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta \tag{1}$$

with α and β real. Then, for any point z on the ellipse, the reflected point $\overline{z}$ (in the real axis) is *also* on the ellipse. Moreover, if z is the image of ζ then $\overline{z}$ is the image of $\overline{\zeta}$. To see this note that

$$\overline{z} = \overline{z(\zeta)} = \overline{z}(\overline{\zeta}) = z(\overline{\zeta}) \tag{2}$$

because

$$\overline{z}(\zeta) = \overline{z(\overline{\zeta})} = z(\zeta) \tag{3}$$

owing to the fact that α and β are real. These results mean that the ellipse is reflectionally symmetric about the x-axis.

The image of $\zeta = 1$ is

$$z(1) = \alpha + \beta \tag{4}$$

while the image of $-\mathrm{i}$ is

$$z(-\mathrm{i}) = -\frac{\alpha}{\mathrm{i}} - \beta\mathrm{i} = \mathrm{i}(\alpha - \beta). \tag{5}$$

Note that if $|\zeta| = 1$ is traversed anticlockwise then the boundary of the ellipse is traversed clockwise. Therefore we let

$$a = \alpha + \beta, \qquad b = \alpha - \beta. \tag{6}$$

Multiplying the mapping by ζ produces the following quadratic for ζ:

$$\beta\zeta^2 - z\zeta + \alpha = 0 \tag{7}$$

yielding

$$\zeta = \frac{z \pm \sqrt{z^2 - 4\alpha\beta}}{2\beta} \tag{8}$$

We must pick the $-$ sign because we require $\zeta \to 0$ as $z \to \infty$. Hence the inverse mapping is

$$\zeta(z) = \frac{z - \sqrt{z^2 - 4\alpha\beta}}{2\beta} \tag{9}$$

Or

$$ab = \alpha^2 - \beta^2, \qquad \alpha = \frac{a+b}{2}, \qquad \beta = \frac{a-b}{2} \tag{10}$$

Hence

$$\zeta(z) = \frac{z - \sqrt{z^2 - (a^2 - b^2)}}{a-b}. \tag{11}$$

2. The mapping from $|\zeta| < 1$ to the exterior of the Kirchhoff ellipse is

$$\frac{\alpha}{\zeta} + \beta\zeta \tag{12}$$

where $\alpha, \beta \in \mathbb{R}$. Take $\alpha = a$ and $\beta = \epsilon \ll 1$ then the ellipse is close to circular. Then

$$\begin{aligned} r^2 = z\bar{z} &= \left(\frac{\alpha}{\zeta} + \beta\zeta\right)\left(\alpha\zeta + \frac{\beta}{\zeta}\right) \\ &= \alpha^2 + \alpha\beta\zeta^2 + \frac{\alpha\beta}{\zeta^2} + \mathcal{O}(\epsilon^2) \\ &= a^2 + \alpha\beta(\zeta^2 + \zeta^{-2}) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{13}$$

Let $\zeta = e^{\mathrm{i}\phi}$ then

$$r^2 = a^2 + 2\alpha\beta\cos(2\phi) + \mathcal{O}(\epsilon^2) = a^2 + 2a\epsilon\cos(2\phi) + \mathcal{O}(\epsilon^2) \tag{14}$$

On taking a square root

$$r = a + \epsilon\cos(2\phi) + \mathcal{O}(\epsilon^2) \tag{15}$$

Let θ be the angle between the positive real axis and the major axis of the ellipse. The major axis of the ellipse corresponds to $\phi = 0$ in the ζ-plane and the ellipse rotates steadily with angular velocity Ω so

$$\theta = \phi + \Omega t \tag{16}$$

therefore

$$r = a + \epsilon\cos 2(\theta - \Omega t). \tag{17}$$

Now, as $a \to b$ (so that the Kirchhoff ellipse is close to circular), and

$$\Omega(a, b) \to \frac{\omega a^2}{4a^2} = \frac{\omega}{4} \tag{18}$$

so, from (17), the time dependence of the perturbation to the Rankine vortex is of the form $e^{\sigma t}$ where

$$\sigma = 2\mathrm{i}\Omega = \frac{\mathrm{i}\omega}{2} \tag{19}$$

This is consistent with the eigenvalue of the $\epsilon\zeta^2$ perturbation to the Rankine vortex as performed in the linear stability analysis in lectures; there it was shown that if the Rankine vortex is perturbed as follows:

$$z = \zeta + \epsilon\zeta^n e^{\sigma_n t}, \qquad n \in \mathbb{Z}^+ \tag{20}$$

then

$$\sigma_n = \frac{\mathrm{i}\omega(n-1)}{2}. \tag{21}$$

For $n = 2$ this gives $\mathrm{i}\omega/2$.

3. If $u \sim -\epsilon y$ and $v \sim -\epsilon x$ then

$$u - \mathrm{i}v \sim \mathrm{i}\epsilon(x + \mathrm{i}y) = 2\mathrm{i}\frac{\partial\psi}{\partial z}. \tag{22}$$

Let

$$\frac{\partial\psi}{\partial z} = \begin{cases} -(\omega/4)\overline{z} - (\omega/4)C_i(z) & z \in D \\ -(\omega/4)C_o(z) + (\epsilon/2)z & z \notin D. \end{cases} \tag{23}$$

Here $C_o(z)$ decays as $|z| \to \infty$. The continuity of the velocity on ∂D implies

$$\overline{z} + \frac{2\epsilon}{\omega}z = C_o(z) - C_i(z). \tag{24}$$

Let

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta \tag{25}$$

then, on $|\zeta| = 1$ (which corresponds to ∂D)

$$\overline{z} = \alpha\zeta + \frac{\beta}{\zeta} = \left(\alpha - \frac{\beta^2}{\alpha}\right)\zeta + \frac{\beta}{\alpha}z \tag{26}$$

where we have used (25) to substitute for $1/\zeta$. We can also write

$$\left(\alpha - \frac{\beta^2}{\alpha}\right)\zeta + \left(\frac{\beta}{\alpha} + \frac{2\epsilon}{\omega}\right)z = C_o(z) - C_i(z) \tag{27}$$

and recognize

$$C_i(z) = -\left(\frac{\beta}{\alpha} + \frac{2\epsilon}{\omega}\right) z, \qquad C_o(z) = \left(\alpha - \frac{\beta^2}{\alpha}\right) \zeta \tag{28}$$

Since

$$z\zeta = \alpha + \beta\zeta^2 \tag{29}$$

then

$$\zeta(z) = \frac{z - \sqrt{z^2 - 4\alpha\beta}}{2\beta} \tag{30}$$

and

$$C_o(z) = \frac{\alpha^2 - \beta^2}{2\alpha\beta}\left(z - \sqrt{z^2 - 4\alpha\beta}\right). \tag{31}$$

But from Q1,

$$\alpha = \frac{a+b}{2}, \qquad \beta = \frac{a-b}{2}. \tag{32}$$

Therefore

$$C_i(z) = -\left(\frac{a-b}{a+b} + \frac{2\epsilon}{\omega}\right) z \tag{33}$$

$$C_o(z) = \frac{2ab}{a^2 - b^2}\left(z - \sqrt{z^2 - 4\alpha\beta}\right). \tag{34}$$

As verification, note that

$$C_i(z) = -\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \left(\overline{z}' + \frac{2\epsilon}{\omega} z'\right) \frac{dz'}{z' - z} \tag{35}$$

First consider

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{2\epsilon}{\omega} \frac{z' dz'}{z' - z} = \frac{2\epsilon z}{\omega} \tag{36}$$

by the residue theorem.

Next consider

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{\overline{z}' dz'}{z' - z} = -\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\overline{z}(\zeta^{-1})}{z(\zeta) - z} \frac{dz(\zeta)}{d\zeta} d\zeta. \tag{37}$$

Now with

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta, \qquad \frac{dz(\zeta)}{d\zeta} = -\frac{\alpha}{\zeta^2} + \beta, \qquad \overline{z}(\zeta^{-1}) = \alpha\zeta + \frac{\beta}{\zeta} \tag{38}$$

then

$$-\frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\left(\alpha\zeta+\frac{\beta}{\zeta}\right)\left(-\frac{\alpha}{\zeta^2}+\beta\right)\frac{1}{z(\zeta)-z}d\zeta. \tag{39}$$

The term $1/(z(\zeta)-z)$ has no singularity inside $|\zeta|<1$ because $z\in D$ and $|\zeta|<1$ corresponds to the exterior of D. Hence the only pole is at $\zeta=0$. Computing the residue leads to

$$\begin{aligned}
&-\frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\left[-\frac{\alpha\beta}{\zeta^3}-\frac{\alpha^2}{\zeta}+\alpha\beta\zeta+\frac{\beta^2}{\zeta}\right]\frac{\zeta d\zeta}{\alpha-z\zeta+\beta\zeta^2}\\
&=-\frac{1}{2\pi \mathrm{i}}\oint_{|\zeta|=1}\left[(\beta^2-\alpha^2)+\alpha\beta\zeta^2-\frac{\alpha\beta}{\zeta^2}\right]\left(\frac{1}{\alpha}+\frac{(z\zeta-\beta\zeta^2)}{\alpha^2}+\mathcal{O}(\zeta^2)\right)d\zeta\\
&=\frac{\beta z}{\alpha}
\end{aligned} \tag{40}$$

Addition of (36) and (40) as in (35) yields

$$C_i(z)=-\left(\frac{\beta}{\alpha}+\frac{2\epsilon}{\omega}\right)z=-\left(\frac{a-b}{a+b}+\frac{2\epsilon}{\omega}\right)z \tag{41}$$

4. Let

$$z(\zeta)=\frac{\alpha}{\zeta}+\beta\zeta+\gamma\zeta^2 \tag{42}$$

Now we must have

$$\frac{\partial\psi}{\partial z}=\begin{cases}-(\omega/4)\overline{z}-(\omega/4)C_i(z) & z\in D\\ -(\omega/4)C_o(z) & z\notin D\end{cases} \tag{43}$$

Continuity of velocity implies

$$-\overline{z}=C_i(z)-C_o(z) \tag{44}$$

On ∂D

$$\overline{z}=\alpha\zeta+\frac{\beta}{\zeta}+\frac{\gamma}{\zeta^2} \tag{45}$$

But

$$\frac{1}{\zeta}=\frac{z}{\alpha}-\frac{\beta}{\alpha}\zeta-\frac{\gamma}{\alpha}\zeta^2 \tag{46}$$

so

$$
\begin{aligned}
\overline{z} &= \alpha\zeta + \beta\left[\frac{z}{\alpha} - \frac{\beta}{\alpha}\zeta - \frac{\gamma}{\alpha}\zeta^2\right] + \gamma\left[\frac{z}{\alpha} - \frac{\beta}{\alpha}\zeta - \frac{\gamma}{\alpha}\zeta^2\right]^2 \\
&= \alpha\zeta + \frac{\beta z}{\alpha} - \frac{\beta^2\zeta}{\alpha} - \frac{\gamma\beta\zeta^2}{\alpha} + \frac{\gamma z^2}{\alpha^2} + \frac{\gamma\beta^2\zeta^2}{\alpha^2} \\
&+ \frac{\gamma^3\zeta^4}{\alpha^2} - \frac{2\beta\gamma(z\zeta)}{\alpha^2} - \frac{2\gamma^2(z\zeta)\zeta}{\alpha^2} + \frac{2\beta\gamma^2\zeta^3}{\alpha^2}
\end{aligned}
\tag{47}
$$

Now use

$$
z\zeta = \alpha + \beta\zeta^2 + \gamma\zeta^3 \tag{48}
$$

to substitute for $z\zeta$:

$$
\begin{aligned}
\overline{z} = \frac{\beta z}{\alpha} + \frac{\gamma z^2}{\alpha^2} + \zeta\left(\alpha - \frac{\beta^2}{\alpha}\right) + \left(\frac{\gamma\beta^2}{\alpha^2} - \frac{\gamma\beta}{\alpha}\right)\zeta^2 + \frac{\gamma^3\zeta^4}{\alpha^2} + \frac{2\beta\gamma^2\zeta}{\alpha^2} \\
- \frac{2\beta\gamma}{\alpha^2}\left(\alpha + \beta\zeta^2 + \gamma\zeta^3\right) - \frac{2\gamma^2}{\alpha^2}\left(\alpha\zeta + \beta\zeta^3 + \gamma\zeta^4\right)
\end{aligned}
\tag{49}
$$

We recognize

$$
\begin{aligned}
C_i(z) &= \frac{2\beta\gamma}{\alpha} - \frac{\beta z}{\alpha} - \frac{\gamma z^2}{\alpha^2}, \\
C_o(z) &= \left(\alpha - \frac{\beta^2}{\alpha} - \frac{2\gamma^2}{\alpha}\right)\zeta - \left(\frac{\beta\gamma}{\alpha} + \frac{\beta^2\gamma}{\alpha^2}\right)\zeta^2 - \frac{2\beta\gamma^2\zeta^3}{\alpha^2} - \frac{\gamma^3\zeta^4}{\alpha^2}.
\end{aligned}
\tag{50}
$$

5. Consider a conformal map from a unit disc $|\zeta| < 1$ to the exterior of a time-evolving ellipse. Assuming its centroid remains fixed at the origin the map will have the form

$$
z(\zeta, t) = \frac{\alpha(t)}{\zeta} + \beta(t)\zeta \tag{51}
$$

To use the rotational degree of freedom in the Riemann mapping theorem, instead of insisting the α is real (as we do in the case of a mapping to a steady ellipse in a corotating frame) we will insist instead that $\zeta = 1$ always maps to the same point (call it A) at the end of one of the principal axes of the ellipse with distance a from the origin. Then

$$
\alpha + \beta = ae^{\mathrm{i}\theta}, \qquad \alpha - \beta = be^{\mathrm{i}\theta} \tag{52}
$$

where θ is the angle made by this principal axis to the real axis. Since the ellipse is a uniform vortex patch and since $2\mathrm{i}\partial\psi/\partial z \to (\mathrm{i}\epsilon + \gamma)z$ as $|z| \to \infty$ we can write

$$\frac{\partial\psi}{\partial z} = \begin{cases} -(\omega/4)\overline{z} - (\omega/4)C_i(z) & z \in D, \\ -(\omega/4)C_o(z) + (\epsilon - \mathrm{i}\gamma)/2z & z \notin D, \end{cases} \tag{53}$$

where $C_i(z)$ is analytic inside the patch, $C_o(z)$ is analytic outside the patch and decaying in the far-field. Continuity of velocity on the boundary of the patch implies that on ∂D

$$\overline{z} + \left(\frac{2\epsilon}{\omega} - \frac{2\mathrm{i}\gamma}{\omega}\right) z = C_o(z) - C_i(z) \tag{54}$$

Solving in the usual way gives

$$\begin{aligned} C_i(z) &= -\left(\frac{\overline{\beta}}{\alpha} + \left(\frac{2\epsilon}{\omega} - \frac{2\mathrm{i}\gamma}{\omega}\right)\right) z, \\ C_o(z) &= \left(\overline{\alpha} - \frac{|\beta|^2}{\alpha}\right)\zeta(z). \end{aligned} \tag{55}$$

This means that, on ∂D,

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial\psi}{\partial z} = -\frac{\mathrm{i}\omega}{2}\left(\overline{\alpha} - \frac{|\beta|^2}{\alpha}\right)\zeta(z) + (\mathrm{i}\epsilon + \gamma)\left(\frac{\alpha}{\zeta} + \beta\zeta\right) \tag{56}$$

Use of this in the kinematic condition that the normal velocity of the patch equals the normal fluid velocity leads to

$$\mathrm{Re}\left[\frac{\partial\overline{z}}{\partial t}\zeta z'(\zeta)\right] = \mathrm{Re}[(u - \mathrm{i}v)\zeta z'(\zeta)]. \tag{57}$$

But

$$\frac{\partial z}{\partial t} = \frac{\dot{\alpha}}{\zeta} + \dot{\beta}\zeta, \qquad \zeta z'(\zeta) = -\frac{\alpha}{\zeta} + \beta\zeta \tag{58}$$

so the boundary condition is

$$\mathrm{Re}\left[\left(\frac{\dot{\alpha}}{\zeta} + \dot{\beta}\zeta\right)\left(-\frac{\alpha}{\zeta} + \beta\zeta\right)\right] = \mathrm{Re}\left[\left(\frac{\alpha X}{\zeta} + Y\zeta\right)\left(-\frac{\alpha}{\zeta} + \beta\zeta\right)\right] \tag{59}$$

where

$$X = \mathrm{i}\epsilon + \gamma, \qquad Y = \beta X - \frac{\mathrm{i}\omega}{2}\left(\overline{\alpha} - \beta^2/\alpha\right). \tag{60}$$

This becomes

$$
\begin{aligned}
-\alpha\overline{\dot\alpha} + \beta\overline{\dot\beta} - \frac{\alpha\overline{\dot\beta}}{\zeta^2} - \overline{\dot\alpha}\beta\zeta^2 - \overline{\alpha}\dot\alpha + \overline{\beta}\dot\beta - \overline{\alpha}\dot\beta\zeta^2 + \frac{\dot\alpha\overline{\beta}}{\zeta^2} \\
= -\frac{\alpha^2 X}{\zeta^2} - \alpha Y + \beta Y\zeta^2 + \alpha\beta X - \overline{\alpha}^2\overline{X}\zeta^2 - \overline{\alpha}\overline{Y} + \frac{\overline{\beta Y}}{\zeta^2} + \overline{\alpha}\overline{\beta X}.
\end{aligned}
\tag{61}
$$

The constant term gives

$$
\frac{d}{dt}\left(|\beta|^2 - |\alpha|^2\right) = 0 \tag{62}
$$

which is a statement of conservation of area of the patch. The coefficients of ζ^2 gives

$$
\beta Y - \overline{\alpha^2 X} = -\overline{\alpha}\dot\beta + \overline{\dot\alpha}\beta = \beta^2 \frac{d}{dt}\left(\frac{\overline{\alpha}}{\beta}\right). \tag{63}
$$

This implies

$$
-\overline{\alpha}^2\overline{X} + \beta^2 X - \frac{\mathrm{i}\omega\beta}{2}\left(\overline{\alpha} - \frac{|\beta|^2}{\alpha}\right) = \beta^2 \frac{d}{dt}\left(\frac{\overline{\alpha}}{\beta}\right). \tag{64}
$$

Now let

$$
\frac{\overline{\alpha}}{\beta} = \left(\frac{a+b}{a-b}\right) e^{-2\mathrm{i}\theta} = Re^{\mathrm{i}\phi} \tag{65}
$$

so $R = (a+b)/(a-b)$ and $\phi = -2\theta$. Then

$$
\frac{d}{dt}\left(Re^{\mathrm{i}\phi}\right) = X - \frac{\overline{\alpha}^2}{\beta^2}\overline{X} - \frac{\mathrm{i}\omega}{2}\left(\frac{\overline{\alpha}}{\beta} - \frac{\overline{\beta}}{\alpha}\right) \tag{66}
$$

which implies

$$
\dot{R}e^{\mathrm{i}\phi} + \mathrm{i}R\dot\phi e^{\mathrm{i}\phi} = X - R^2\overline{X}e^{2\mathrm{i}\phi} - \frac{\mathrm{i}\omega}{2}\left(R - \frac{1}{R}\right)e^{\mathrm{i}\phi}. \tag{67}
$$

Now divide by $e^{\mathrm{i}\phi}$:

$$
\dot{R} + \mathrm{i}R\dot\phi = Xe^{-\mathrm{i}\phi} - R^2\overline{X}e^{\mathrm{i}\phi} - \frac{\mathrm{i}\omega}{2}\left(R - \frac{1}{R}\right). \tag{68}
$$

Thus

$$
\dot{R} - 2\mathrm{i}R\dot\theta = (\gamma + \mathrm{i}\epsilon)(\cos 2\theta + \mathrm{i}\sin 2\theta) - R^2(\gamma - \mathrm{i}\epsilon)(\cos 2\theta - \mathrm{i}\sin 2\theta) - \frac{\mathrm{i}\omega}{2}\left(R - \frac{1}{R}\right). \tag{69}
$$

The imaginary part of this equation gives

$$-2R\dot{\theta} = \epsilon(1+R^2)\cos 2\theta + \gamma(1+R^2)\sin 2\theta - \frac{\omega}{2}\left(R - \frac{1}{R}\right). \tag{70}$$

Now using the facts that

$$1 + R^2 = \frac{2(a^2+b^2)}{(a-b)^2}, \qquad 1 - \frac{1}{R^2} = \frac{4ab}{(a+b)^2} \tag{71}$$

then

$$\dot{\theta} = \frac{\omega ab}{(a+b)^2} - \epsilon\left(\frac{a^2+b^2}{a^2-b^2}\right)\cos 2\theta - \gamma\left(\frac{a^2+b^2}{a^2-b^2}\right)\sin 2\theta. \tag{72}$$

This is as required.

The real part of (69) gives

$$\dot{R} = \gamma(1-R^2)\cos 2\theta - \epsilon(1-R^2)\sin 2\theta. \tag{73}$$

But

$$\dot{R} = \frac{2(a\dot{b} - b\dot{a})}{(a-b)^2} \tag{74}$$

thus

$$\dot{a}b - \dot{b}a = 2ab\left(\gamma\cos 2\theta - \epsilon\sin 2\theta\right) \tag{75}$$

But $\pi ab =$ constant so

$$\dot{a}b + a\dot{b} = 0 \tag{76}$$

hence

$$2\dot{a}b = 2ab\left(\gamma\cos 2\theta - \epsilon\sin 2\theta\right) \tag{77}$$

or

$$\dot{a} = a\left(\gamma\cos 2\theta - \epsilon\sin 2\theta\right) \tag{78}$$

Similarly

$$\dot{b} = -b\left(\gamma\cos 2\theta - \epsilon\sin 2\theta\right) \tag{79}$$

This means that

$$a\dot{a} - b\dot{b} = (a^2+b^2)\left(\gamma\cos 2\theta - \epsilon\sin 2\theta\right), \tag{80}$$

as required.

6. From the lecture notes, the velocity field for the Kirchhoff ellipse in a fixed frame is $u - \mathrm{i}v = 2\mathrm{i}\partial\psi/\partial z$ where

$$\frac{\partial\psi}{\partial z} = \begin{cases} -(\omega/4)\overline{z} + (\omega/4)\beta z/\alpha, & z \in D, \\ -(\omega/4)[\alpha - \beta^2/\alpha]\zeta(z), & z \notin D. \end{cases} \tag{81}$$

From Q1,

$$\alpha = \frac{a+b}{2}, \qquad \beta = \frac{a-b}{2} \tag{82}$$

and the limit $b \to 0$ corresponds to $\beta \to \alpha$. Thus

$$\frac{\partial\psi}{\partial z} = \begin{cases} -(\omega/4)\overline{z} + (\omega/4)(a-b)z/(a+b), & z \in D, \\ -(\omega/4)[2ab/(a+b)]\zeta(z), & z \notin D. \end{cases} \tag{83}$$

The angular velocity is

$$\Omega = \frac{\omega ab}{(a+b)^2} \to \frac{\kappa}{2a} \tag{84}$$

in the limit. Therefore the velocity on the boundary of the patch in a corotating frame is

$$u - \mathrm{i}v = 2\mathrm{i}\frac{\partial\psi}{\partial z} = \frac{\mathrm{i}k\overline{z}}{2a} - \frac{\mathrm{i}\omega ab}{a+b}\zeta \tag{85}$$

In the limit $\beta \to \alpha$ the conformal map tends to the slit mapping

$$z(\zeta) = \frac{a}{2}\left(\zeta + \zeta^{-1}\right). \tag{86}$$

This means that, if x is real,

$$\zeta = \frac{x}{a} \pm \sqrt{(x/a)^2 - 1} \tag{87}$$

where the two signs correspond to the top and bottom of the slit. Substituting this into the velocity field on the boundary gives

$$u - \mathrm{i}v = \mp\frac{\mathrm{i}\kappa}{2}\sqrt{(x/a)^2 - 1} \tag{88}$$

But $x^2/a^2 < 1$ so

$$u - \mathrm{i}v = \mp\frac{\kappa}{2}\sqrt{1 - (x/a)^2}. \tag{89}$$

Therefore $v = 0$ everywhere on the slit (as expected) and the jump in tangential velocity (i.e., the vortex sheet strength) is

$$\kappa\sqrt{1 - (x/a)^2}. \tag{90}$$

The angular velocity of the sheet is $\Omega = \kappa/(2a)$.

7. If

$$\psi = \log(\text{cosh}y - \epsilon \cos x) \tag{91}$$

Then

$$\frac{\partial \psi}{\partial x} = \frac{\epsilon \sin x}{\text{cosh}y - \epsilon \cos x} \tag{92}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\epsilon \cos x}{\text{cosh}y - \epsilon \cos x} - \frac{\epsilon^2 \sin^2 x}{(\text{cosh}y - \epsilon \cos x)^2} \tag{93}$$

This can be simplified to

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\epsilon \cos x \text{cosh}y - \epsilon^2}{(\text{cosh}y - \epsilon \cos x)^2} \tag{94}$$

Similarly it can be shown that

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{1 - \epsilon \text{cosh}y \cos x}{(\text{cosh}y - \epsilon \cos x)^2} \tag{95}$$

Therefore

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1 - \epsilon^2}{(\text{cosh}y - \epsilon \cos x)^2} = (1 - \epsilon^2) e^{-2\psi}. \tag{96}$$

When $\epsilon = 1$

$$\begin{aligned} \psi = \log\left(\cosh y - \cos x\right) &= \log\left[\cosh\left(\frac{z - \overline{z}}{2\mathrm{i}}\right) - \cos\left(\frac{z + \overline{z}}{2}\right)\right]. \\ &= \log(\sin(z/2)) + \log(\sin \overline{z}/2) + \text{cst} \\ &= \operatorname{Im}\left[w(z)\right] \end{aligned} \tag{97}$$

where

$$w(z) = 2\mathrm{i} \log(\sin(z/2)) \tag{98}$$

Since the complex potential for a singly periodic row of circulation Γ point vortices at $z = na$ is known to be

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi} \log(\sin(\pi z/a)) \tag{99}$$

then we recognize the $\epsilon = 1$ solution as a row of point vortices all of circulation $\Gamma = -4\pi$ at $z = 2n\pi$.

Verification that ψ satisfies

$$\nabla^2\psi = -\left(\frac{1-\epsilon^2}{2}\right)\sinh(2\psi) \tag{100}$$

is by direct differentiation as above.

When $\epsilon = 1$

$$\begin{aligned}\psi &= \log(\cosh y - \epsilon\cos x) - \log(\cosh y + \epsilon\cos x)\\ &= \log\left(2\sin(z/2)\sin(\overline{z}/2)\right) - \log\left(2\cos(z/2)\cos(\overline{z}/2)\right)\\ &= \mathrm{Im}[\hat{w}(z)]\end{aligned} \tag{101}$$

where

$$\hat{w}(z) = 2\mathrm{i}\log(\sin(z/2)) - 2\mathrm{i}\log(\sin((z+\pi)/2)) \tag{102}$$

which corresponds to an alternating street of vortices of strengths $\pm 4\pi$ separated by distance π.