M4A32: Vortex Dynamics

Problem Sheet 5 (Distributed vorticity)

1. Verify that the one-to-one conformal mapping $z(\zeta)$ from the unit ζ-disk given by

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta$$

where α and β are real parameters maps to the exterior of an ellipse.

If the ellipse has semi-major axis a and semi-minor axis b, find expressions for the parameters α and β in terms of a and b.

If the mapping $z(\zeta)$ is one-to-one, then the inverse function $\zeta(z)$ exists. Find $\zeta(z)$ explicitly.

2. Consider the classical rotating Kirchhoff ellipse solution with vorticity ω. It is known that if the semi-major axis is of length a and the semi-minor axis of length b then the angular velocity Ω of the ellipse is

$$\Omega = \frac{\omega ab}{(a+b)^2}$$

Verify that in the limit $\frac{a}{b} \to 1$, the rotating ellipse behaves like an infinitesimal perturbation to the Rankine vortex of the form

$$r = a + \epsilon \cos 2(\theta - \Omega t)$$

where $\epsilon \ll 1$ and θ is the angle between the major axis of the ellipse and the positive x-axis. Calculate the limit of Ω as $\frac{a}{b} \to 1$ (so that the ellipse becomes almost circular) and verify that this is consistent with the linear stability analysis of the Rankine vortex performed in lectures.

3. An elliptical vortex patch D with semi-major axis a (aligned with the x-axis) and semi-minor axis b (aligned with the y-axis) and vorticity ω sits in a straining flow which, at large distances from the patch, has the form

$$u \sim -\epsilon y, \quad v \sim -\epsilon x,$$

Let $\psi(z, \bar{z})$ be the instantaneous streamfunction associated with this flow. Show that

$$\psi_z = \begin{cases} -\frac{\omega}{4}\bar{z} - \frac{\omega}{4}C_i(z) & z \in D \\ -\frac{\omega}{4}C_o(z) + \frac{\epsilon}{2}z & z \notin D \end{cases}$$

where

$$C_i(z) = -\left(\frac{a-b}{a+b} + 2\frac{\epsilon}{\omega}\right) z$$

$$C_o(z) = \frac{2ab}{(a^2-b^2)}\left(z - \sqrt{z^2 - (a^2-b^2)}\right)$$

Verify that this expression for $C_i(z)$ is correct by using the alternative line-integral formula

$$C_i(z) = -\frac{1}{2\pi i}\oint_{\partial D}\left(\bar{z}' + \frac{2\epsilon}{\omega}z'\right)\frac{dz'}{z'-z}$$

where ∂D denotes the boundary of D.

4. At some instant, a uniform vortex patch D with vorticity ω sits in an otherwise irrotational flow decaying at large distances from the patch. The shape of the patch is given by a conformal map $z(\zeta)$ from the unit ζ-disc of the form

$$z(\zeta) = \frac{\alpha}{\zeta} + \beta\zeta + \gamma\zeta^2$$

where α, β and γ are real constants.

If $\psi(z, \bar{z})$ is the instantaneous streamfunction, show that

$$\psi_z = \begin{cases} -\frac{\omega}{4}\bar{z} - \frac{\omega}{4}C_i(z) & z \in D \\ -\frac{\omega}{4}C_o(z) & z \notin D \end{cases}$$

where

$$C_i(z) = \frac{2\beta\gamma}{\alpha} - \frac{\beta}{\alpha}z - \frac{\gamma}{\alpha^2}z^2$$

and

$$C_o(z(\zeta)) = \left(\alpha - \frac{\beta^2}{\alpha} - \frac{2\gamma^2}{\alpha}\right)\zeta - \left(\frac{\beta\gamma}{\alpha} + \frac{\beta^2\gamma}{\alpha^2}\right)\zeta^2 - \frac{2\beta\gamma^2}{\alpha^2}\zeta^3 - \frac{\gamma^3}{\alpha^2}\zeta^4$$

5. Consider a time-evolving ellipse of semi-major axis $a(t)$, semi-minor axis $b(t)$ where $\theta(t)$ is the angle which the semi-major axis makes with the x-axis. Suppose that the far field flow is a general time-evolving strain given by

$$U_\infty - iV_\infty = i\epsilon(t)z + \gamma(t)z$$

Show that, under the evolution of the Euler equations, the ellipse remains an ellipse with parameters a, b and θ satisfying the coupled ordinary differential equations given by

$$a\dot{a} - b\dot{b} = -\epsilon(a^2 + b^2)\sin 2\theta + \gamma(a^2 + b^2)\cos 2\theta$$

$$\dot{\theta} = \frac{\omega ab}{(a+b)^2} - \epsilon\frac{a^2 + b^2}{a^2 - b^2}\cos 2\theta - \gamma\frac{a^2 + b^2}{a^2 - b^2}\sin 2\theta$$

(the time-dependence of the parameters has been suppressed for brevity).

6. Consider the rotating Kirchhoff ellipse of uniform vorticity ω with semi-major axis a and semi-minor axis b. Consider now the double limit

$$b \to 0, \quad \omega \to \infty \quad \text{with } 2\omega b \to \kappa$$

where κ is a non-zero constant.

Show that this limit corresponds to a vortex sheet of length $2a$ along the x-axis with strength

$$\kappa\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

Using the well-known formula for the angular velocity of the Kirchhoff ellipse, calculate the angular velocity of the vortex sheet.

7. Verify that the streamfunction

$$\psi = \log\left(\cosh y - \epsilon\cos x\right)$$

satisfies Liouville's equation

$$\nabla^2\psi = (1-\epsilon^2)e^{-2\psi}$$

where ϵ is a constant and is therefore a solution of the steady Euler equation. Verify that when $\epsilon = \pm 1$ this solution represents an infinite row of identical point vortices.

Verify that the streamfunction

$$\psi = \log\left(\frac{\cosh\epsilon y - \epsilon\cos x}{\cosh\epsilon y + \epsilon\cos x}\right)$$

satisfies the equation

$$\nabla^2\psi = -\frac{(1-\epsilon^2)}{2}\sinh^{2\psi}$$

and is therefore a solution of the steady Euler equation. What does this solution correspond to when $\epsilon = \pm 1$? How about when $\epsilon = 0$?