M4A32: Vortex dynamics

Problem Sheet 4: SOLUTIONS

(Point vortex dynamics on a sphere)

1. On use of the facts that

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{\phi} = \left.\frac{\partial\psi}{\partial\zeta}\right|_{\overline{\zeta}}\left.\frac{\partial\zeta}{\partial\theta}\right|_{\phi} + \left.\frac{\partial\psi}{\partial\overline{\zeta}}\right|_{\zeta}\left.\frac{\partial\overline{\zeta}}{\partial\theta}\right|_{\phi}, \qquad \left.\frac{\partial\psi}{\partial\phi}\right|_{\theta} = \left.\frac{\partial\psi}{\partial\zeta}\right|_{\overline{\zeta}}\left.\frac{\partial\zeta}{\partial\phi}\right|_{\theta} + \left.\frac{\partial\psi}{\partial\overline{\zeta}}\right|_{\zeta}\left.\frac{\partial\overline{\zeta}}{\partial\phi}\right|_{\theta}, \tag{1}$$

and the relation

$$\zeta = \cot(\theta/2)e^{\mathrm{i}\phi} \tag{2}$$

it follows that

$$\left.\frac{\partial\zeta}{\partial\theta}\right|_{\phi} = -\frac{\zeta}{\sin\theta}, \qquad \left.\frac{\partial\overline{\zeta}}{\partial\theta}\right|_{\phi} = -\frac{\overline{\zeta}}{\sin\theta}, \qquad \left.\frac{\partial\zeta}{\partial\phi}\right|_{\theta} = \mathrm{i}\zeta, \qquad \left.\frac{\partial\overline{\zeta}}{\partial\phi}\right|_{\theta} = -\mathrm{i}\overline{\zeta}. \tag{3}$$

Now, on use of the facts that

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{\phi} = \left.\frac{\partial\psi}{\partial\zeta}\right|_{\overline{\zeta}}\left.\frac{\partial\zeta}{\partial\theta}\right|_{\phi} + \left.\frac{\partial\psi}{\partial\overline{\zeta}}\right|_{\zeta}\left.\frac{\partial\overline{\zeta}}{\partial\theta}\right|_{\phi}, \qquad \left.\frac{\partial\psi}{\partial\phi}\right|_{\theta} = \left.\frac{\partial\psi}{\partial\zeta}\right|_{\overline{\zeta}}\left.\frac{\partial\zeta}{\partial\phi}\right|_{\theta} + \left.\frac{\partial\psi}{\partial\overline{\zeta}}\right|_{\zeta}\left.\frac{\partial\overline{\zeta}}{\partial\phi}\right|_{\theta}, \tag{4}$$

then

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{\phi} = -\frac{\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta} - \frac{\overline{\zeta}}{\sin\theta}\frac{\partial\psi}{\partial\overline{\zeta}} \tag{5}$$

so that

$$\sin\theta\left.\frac{\partial\psi}{\partial\theta}\right|_{\phi} = -\zeta\frac{\partial\psi}{\partial\zeta} - \overline{\zeta}\frac{\partial\psi}{\partial\overline{\zeta}}. \tag{6}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) &= -\frac{\zeta}{\sin\theta}\frac{\partial}{\partial\zeta}\left(-\zeta\frac{\partial\psi}{\partial\zeta} - \overline{\zeta}\frac{\partial\psi}{\partial\overline{\zeta}}\right) - \frac{\overline{\zeta}}{\sin\theta}\frac{\partial}{\partial\overline{\zeta}}\left(-\zeta\frac{\partial\psi}{\partial\zeta} - \overline{\zeta}\frac{\partial\psi}{\partial\overline{\zeta}}\right)\\ &= \frac{1}{\sin\theta}\left[\zeta\frac{\partial\psi}{\partial\zeta} + \zeta^2\frac{\partial^2\psi}{\partial\zeta^2} + 2\zeta\overline{\zeta}\frac{\partial^2\psi}{\partial\zeta\partial\overline{\zeta}} + \overline{\zeta}^2\frac{\partial^2\psi}{\partial\overline{\zeta}^2} + \overline{\zeta}\frac{\partial\psi}{\partial\overline{\zeta}}\right].\end{aligned} \tag{7}$$

Also

$$\left.\frac{\partial\psi}{\partial\phi}\right|_{\theta} = \mathrm{i}\zeta\frac{\partial\psi}{\partial\zeta} - \mathrm{i}\overline{\zeta}\frac{\partial\psi}{\partial\overline{\zeta}}, \tag{8}$$

so that

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial\phi^2} &= \mathrm{i}\zeta\frac{\partial}{\partial\zeta}\left(\mathrm{i}\zeta\frac{\partial\psi}{\partial\zeta}-\mathrm{i}\bar{\zeta}\frac{\partial\psi}{\partial\bar{\zeta}}\right)-\mathrm{i}\bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}\left(\mathrm{i}\zeta\frac{\partial\psi}{\partial\zeta}-\mathrm{i}\bar{\zeta}\frac{\partial\psi}{\partial\bar{\zeta}}\right)\\
&= -\zeta\frac{\partial\psi}{\partial\zeta}-\zeta^2\frac{\partial^2\psi}{\partial\zeta^2}+2\zeta\bar{\zeta}\frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}}-\bar{\zeta}^2\frac{\partial^2\psi}{\partial\bar{\zeta}^2}-\bar{\zeta}\frac{\partial\psi}{\partial\bar{\zeta}}.
\end{aligned}
\tag{9}
$$

By adding appropriate multiples of (7) and (9), we arrive at

$$
\nabla^2_\Sigma\psi = \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right) = \frac{4\zeta\bar{\zeta}}{\sin^2\theta}\frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}}. \tag{10}
$$

But

$$
\sin^2\theta = \frac{4\zeta\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} \tag{11}
$$

so

$$
\nabla^2_\Sigma\psi = (1+\zeta\bar{\zeta})^2\frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}}. \tag{12}
$$

2. In spherical polar coordinates

$$
\nabla.\mathbf{u} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0. \tag{13}
$$

So we define

$$
u_\phi = -\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}. \tag{14}
$$

Therefore

$$
u_\phi - \mathrm{i}u_\theta = -\frac{\partial\psi}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{15}
$$

But we know, from Q1, that

$$
\begin{aligned}
\frac{\partial\psi}{\partial\theta} &= -\frac{\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}-\frac{\bar{\zeta}}{\sin\theta}\frac{\partial\psi}{\partial\bar{\zeta}},\\
\frac{\partial\psi}{\partial\phi} &= \mathrm{i}\zeta\frac{\partial\psi}{\partial\zeta}-\mathrm{i}\bar{\zeta}\frac{\partial\psi}{\partial\bar{\zeta}},
\end{aligned}
\tag{16}
$$

Therefore

$$
u_\phi - \mathrm{i}u_\theta = \frac{\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}+\frac{\bar{\zeta}}{\sin\theta}\frac{\partial\psi}{\partial\bar{\zeta}}+\frac{\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}-\frac{\bar{\zeta}}{\sin\theta}\frac{\partial\psi}{\partial\bar{\zeta}} = \frac{2\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}. \tag{17}
$$

3. Consider

$$\zeta = \cot(\theta/2)e^{\mathrm{i}\phi}, \qquad \zeta' = \cot(\theta'/2)e^{\mathrm{i}\phi'}. \tag{18}$$

and

$$\frac{2(\zeta-\zeta')(\overline{\zeta}-\overline{\zeta'})}{(1+\zeta\overline{\zeta})(1+\zeta'\overline{\zeta'})}. \tag{19}$$

Now

$$1+\zeta\overline{\zeta} = 1+\cot^2(\theta/2) = \mathrm{cosec}^2(\theta/2), \qquad 1+\zeta'\overline{\zeta'} = 1+\cot^2(\theta'/2) = \mathrm{cosec}^2(\theta'/2) \tag{20}$$

while

$$\begin{aligned}(\zeta-\zeta')(\overline{\zeta}-\overline{\zeta'}) &= (\cot(\theta/2)e^{\mathrm{i}\phi} - \cot(\theta'/2)e^{\mathrm{i}\phi'})(\cot(\theta/2)e^{-\mathrm{i}\phi} - \cot(\theta'/2)e^{-\mathrm{i}\phi'}) \\ &= \cot^2(\theta/2) + \cot^2(\theta'/2) - \cot(\theta/2)\cot(\theta'/2)\left[e^{\mathrm{i}(\phi'-\phi)} + e^{-\mathrm{i}(\phi'-\phi)}\right]\end{aligned} \tag{21}$$

Therefore

$$\begin{aligned}\frac{2(\zeta-\zeta')(\overline{\zeta}-\overline{\zeta'})}{(1+\zeta\overline{\zeta})(1+\zeta'\overline{\zeta'})} = 2\sin^2(\theta/2)\sin^2(\theta'/2)\Bigg[&\frac{\cos^2(\theta/2)}{\sin^2(\theta/2)} + \frac{\cos^2(\theta'/2)}{\sin^2(\theta'/2)} \\ &- \frac{2\cos(\theta/2)\cos(\theta'/2)}{\sin(\theta/2)\sin(\theta'/2)}\cos(\phi-\phi')\Bigg].\end{aligned} \tag{22}$$

This can be written

$$\begin{aligned}\frac{2(\zeta-\zeta')(\overline{\zeta}-\overline{\zeta'})}{(1+\zeta\overline{\zeta})(1+\zeta'\overline{\zeta'})} = 2\Bigg[&\cos^2(\theta/2)\sin^2(\theta'/2) + \cos^2(\theta'/2)\sin^2(\theta/2) \\ &- 2[\sin(\theta/2)\cos(\theta/2)][\sin(\theta'/2)\cos(\theta'/2)]\cos(\phi-\phi')\Bigg].\end{aligned} \tag{23}$$

But, from the half-angle formulae,

$$\begin{aligned}1-\cos\theta\cos\theta' &= (\cos^2(\theta/2)+\sin^2(\theta/2))(\cos^2(\theta'/2)+\sin^2(\theta'/2)) \\ &\quad - (\cos^2(\theta/2)-\sin^2(\theta/2))(\cos^2(\theta'/2)-\sin^2(\theta'/2)) \\ &= 2\left[\cos^2(\theta'/2)\sin^2(\theta/2) + \cos^2(\theta/2)\sin^2(\theta'/2)\right].\end{aligned} \tag{24}$$

By substituting for this in (23) we arrive at

$$\frac{2(\zeta-\zeta')(\overline{\zeta}-\overline{\zeta'})}{(1+\zeta\overline{\zeta})(1+\zeta'\overline{\zeta'})} = 1-\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi-\phi') = 1-\cos\gamma. \quad (25)$$

4. Suppose there are N equal strength vortices at positions $a, a\omega, a\omega^2, ..., a\omega^{N-1}$ where

$$\omega^N = 1. \quad (26)$$

The streamfunction in a frame of reference corotating with some angular velocity Ω is

$$\psi = -\frac{\kappa}{2}\log\left[\frac{(\zeta^N-a^N)(\overline{\zeta}^N-a^N)}{(1+\zeta\overline{\zeta})^N(1+a^2)^N}\right] + \frac{2\Omega}{(1+\zeta\overline{\zeta})}. \quad (27)$$

The velocity field is

$$u_\phi - \mathrm{i}u_\theta = \frac{2\zeta}{\sin\theta}\left[-\frac{N\kappa\zeta^{N-1}}{2(\zeta^N-a^N)} + \frac{N\kappa\overline{\zeta}}{2(1+\zeta\overline{\zeta})} - \frac{2\Omega\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}\right] \quad (28)$$

For the configuration to be steady we must subtract off the velocity due to the vortex at $\zeta = a$ and set the remaining terms equal to zero. The velocity due to the point vortex at $\zeta = a$ is

$$\frac{2\zeta}{\sin\theta}\left(-\frac{\kappa}{2}\left[\frac{1}{\zeta-a} - \frac{\overline{\zeta}}{1+\zeta\overline{\zeta}}\right]\right). \quad (29)$$

so the condition for steadiness reduces to

$$\left[-\frac{N\kappa\zeta^{N-1}}{2(\zeta^N-a^N)} + \frac{\kappa}{2(\zeta-a)} + \frac{(N-1)\kappa\overline{\zeta}}{2(1+\zeta\overline{\zeta})} - \frac{2\Omega\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}\right]\Bigg|_{\zeta=a} = 0. \quad (30)$$

Using the facts that

$$\begin{aligned}\zeta^N &= a^N + Na^{N-1}(\zeta-a) + \frac{N(N-1)a^{N-2}}{2!}(\zeta-a)^2 + \dots \\ \zeta^{N-1} &= a^{N-1} + (N-1)a^{N-2}(\zeta-a) + \dots\end{aligned} \quad (31)$$

the condition (30) becomes

$$-\frac{\kappa(N-1)}{4a}+\frac{\kappa a(N-1)}{2(1+a^2)}-\frac{2\Omega a}{(1+a^2)^2}=0. \tag{32}$$

On rearrangement, we deduce that

$$\Omega=\frac{\kappa(N-1)(a^4-1)}{8a^2}. \tag{33}$$

This can also be written as

$$\Omega=\frac{\kappa(N-1)\cos\theta_a}{2\sin^2\theta_a} \tag{34}$$

where $a=\cot(\theta_a/2)$.

If the vortices are on the equator then $\theta_a=\pi/2$ and $\Omega=0$ so the configuration is stationary.

5. If fluid is in solid body rotation about an axis through the north pole with angular velocity Ω then

$$u_\phi=\Omega\sin\theta, \qquad u_\theta=0 \tag{35}$$

so that

$$u_\phi-\mathrm{i}u_\theta=\Omega\sin\theta=\frac{2\zeta}{\sin\theta}\frac{\partial\psi}{\partial\zeta}. \tag{36}$$

This means that

$$\frac{\partial\psi}{\partial\zeta}=\frac{2\Omega\overline{\zeta}}{(1+\zeta\overline{\zeta})^2}. \tag{37}$$

But

$$\omega=-\nabla^2_\Sigma\psi=-(1+\zeta\overline{\zeta})^2\frac{\partial^2\psi}{\partial\zeta\partial\overline{\zeta}}, \tag{38}$$

so since

$$\frac{\partial^2\psi}{\partial\zeta\partial\overline{\zeta}}=\frac{2\Omega}{(1+\zeta\overline{\zeta})^2}-\frac{4\Omega\zeta\overline{\zeta}}{(1+\zeta\overline{\zeta})^3}=\frac{2\Omega(1-\zeta\overline{\zeta})}{(1+\zeta\overline{\zeta})^3} \tag{39}$$

then

$$\omega=-(1+\zeta\overline{\zeta})^2\frac{\partial^2\psi}{\partial\zeta\partial\overline{\zeta}}=-\frac{2\Omega(1-\zeta\overline{\zeta})}{(1+\zeta\overline{\zeta})}=2\Omega\cos\theta. \tag{40}$$

6. The required streamfunction is

$$\psi(\zeta,\overline{\zeta}) = \begin{cases} -\omega_S \log(1+\zeta\overline{\zeta}), & |\zeta|<1, \\ -\omega_N \log[(1+\zeta\overline{\zeta})/(\zeta\overline{\zeta})], & |\zeta|>1. \end{cases}$$

To verify this, on differentiation we find

$$\frac{\partial\psi}{\partial\zeta} = \begin{cases} -\omega_S\overline{\zeta}/(1+\zeta\overline{\zeta}), & |\zeta|<1, \\ -\omega_N\overline{\zeta}/(1+\zeta\overline{\zeta}) + \omega_N/\zeta, & |\zeta|>1, \end{cases}$$

and

$$\frac{\partial^2\psi}{\partial\zeta\partial\overline{\zeta}} = \begin{cases} -\omega_S/(1+\zeta\overline{\zeta})^2, & |\zeta|<1, \\ -\omega_N/(1+\zeta\overline{\zeta})^2, & |\zeta|>1, \end{cases}$$

so that

$$\nabla^2_\Sigma\psi = (1+\zeta\overline{\zeta})^2\frac{\partial^2\psi}{\partial\zeta\partial\overline{\zeta}} = \begin{cases} -\omega_S, & |\zeta|<1, \\ -\omega_N, & |\zeta|>1. \end{cases}$$

Continuity of velocity at $|\zeta|=1$ requires continuity of $\partial\psi/\partial\zeta$ there which implies

$$-\omega_S\overline{\zeta}/(1+\zeta\overline{\zeta}) = -\omega_N\overline{\zeta}/(1+\zeta\overline{\zeta}) + \omega_N/\zeta$$

which, on the equator $|\zeta|=1$, implies

$$\omega_N = -\omega_S.$$

This is consistent with the Gauss constraint that the total vorticity over the sphere is zero.