M4A32: Vortex Dynamics

Problem Sheet 3 SOLUTIONS

(Kirchhoff-Routh theory)

1.(a) Consider the conformal map

$$z(\zeta) = \frac{1-\zeta}{1+\zeta}.$$

Since it is a Möbius map, it maps cirles/straight lines to circles/straight lines. Note that $\zeta = -1$ maps to $z = \infty$ and $\zeta = 1$ maps to $z = 0$. On $|\zeta| = 1$, note that

$$\bar{z} = \frac{1-\bar{\zeta}}{1+\bar{\zeta}} = \frac{1-\zeta^{-1}}{1+\zeta^{-1}} = \frac{\zeta-1}{\zeta+1} = -z$$

where we have used the fact that $\bar{\zeta} = \zeta^{-1}$ for ζ on the unit circle. Hence $|\zeta| = 1$ maps to the imaginary z-axis. Finally, since $\zeta = 0$ maps to $z = 1$, it follows that $|\zeta| < 1$ maps to $\mathrm{Re}[z] > 0$.

(b) Note that

$$z(\zeta) = -1 + \frac{2}{\zeta+1}$$

so that

$$z_\zeta(\zeta) = -\frac{2}{(\zeta+1)^2}.$$

On use of the transformation rule,

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi}\log\left|\frac{2(1-\alpha\bar{\alpha})}{(\alpha+1)^2}\right|.$$

But some simple algebra reveals that if

$$z_\alpha = \frac{1-\alpha}{1+\alpha}$$

then

$$\alpha = \frac{1-z_\alpha}{1+z_\alpha}$$

and

$$1-\alpha\bar{\alpha} = \frac{2(z_\alpha+\bar{z}_\alpha)}{(1+z_\alpha)(1+\bar{z}_\alpha)}.$$

Also,

$$|\alpha+1|^2 = (\alpha+1)(\bar{\alpha}+1) = \frac{4}{(1+z_\alpha)(1+\bar{z}_\alpha)}.$$

Therefore

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi} \log \left| \left(\frac{4(z_\alpha + \bar{z}_\alpha)}{(1+z_\alpha)(1+\bar{z}_\alpha)} \right) \left(\frac{(1+z_\alpha)(1+\bar{z}_\alpha)}{4} \right) \right|$$

which gives the required answer after cancellations.

(c) Trajectories of a single vortex are contours of $H^{(z)}$, i.e.,

$$z_\alpha + \bar{z}_\alpha = \text{constant}$$

which are straight lines parallel to the imaginary axis.

2.(a) The Green's function $G(\zeta; \alpha, \bar{\alpha})$ must satisfy the requirements that (a) it is a harmonic function (i.e. satisfies Laplace's equation) everywhere in the upper-half ζ-plane except that, near $\zeta = \alpha$, it behaves like

$$G = -\frac{1}{2\pi} \log |\zeta - \alpha| + \text{regular terms.}$$

It must also satisfy the boundary condition that G is equal to zero everywhere on the real ζ-axis.

(b) Consider

$$G = -\frac{1}{2\pi} \log \left| \frac{\zeta - \alpha}{\zeta - \bar{\alpha}} \right|.$$

Notice that

$$G = \mathrm{Im}\left[-\frac{i}{2\pi} \log\left(\frac{\zeta - \alpha}{\zeta - \bar{\alpha}} \right) \right]$$

so it is the imaginary part of an analytic function of ζ; hence it is harmonic everywhere except at $\zeta = \alpha, \bar{\alpha}$ (which are the singularities of the analytic function). Only $\zeta = \alpha$ is in the upper-half ζ-plane so G only has one singularity in the upper-half plane as required. Note also that

$$G = -\frac{1}{2\pi} \log |\zeta - \alpha| + \frac{1}{2\pi} \log |\zeta - \bar{\alpha}|$$

so G also has the correct local behaviour at $\zeta = \alpha$ since the function

$$+\frac{1}{2\pi} \log |\zeta - \bar{\alpha}|$$

is regular at $\zeta = \alpha$. Also, let

$$R(\zeta; \alpha, \bar{\alpha}) = \frac{\zeta - \alpha}{\zeta - \bar{\alpha}}.$$

On $\zeta \in \mathbb{R}$,

$$\overline{R} = \frac{\zeta - \bar{\alpha}}{\zeta - \alpha} = \frac{1}{R}$$

which means that $R\bar{R} = 1$ so that, on $\zeta \in \mathbb{R}$,

$$G = -\frac{1}{2\pi} \log |R| = -\frac{1}{2\pi} \log 1 = 0$$

as required.

(c) For a single vortex, the Hamiltonian is

$$H^{(\zeta)}(\alpha, \bar{\alpha}) = \frac{\Gamma^2}{2} g(\alpha; \alpha, \bar{\alpha})$$

where g is defined by

$$G(\zeta; \alpha, \bar{\alpha}) = -\frac{1}{2\pi} \log |\zeta - \alpha| + g(\zeta; \alpha, \bar{\alpha})$$

Hence, from above, it follows that in this case

$$g(\zeta; \alpha, \bar{\alpha}) = \frac{1}{2\pi} \log |\zeta - \bar{\alpha}|$$

so

$$H^{(\zeta)}(\alpha, \bar{\alpha}) = \frac{\Gamma^2}{2\pi} \log |\alpha - \bar{\alpha}|.$$

(d) The conformal mapping from the ζ-plane to the z-plane given by

$$z = -i\zeta$$

clearly takes the upper-half ζ-plane to the right half z-plane since multiplication by $-i$ rotates everything clockwise by $\pi/2$.

(e) It follows from the transformation property of the Hamiltonians that

$$\begin{aligned}H^{(z)}(z_\alpha, \bar{z}_\alpha) &= \frac{\Gamma^2}{4\pi}\log|\alpha(z_\alpha) - \overline{\alpha(z_\alpha)}| + \frac{\Gamma^2}{4\pi}\log 1 \\ &= \frac{\Gamma^2}{4\pi}\log|z_\alpha + \bar{z}_\alpha|\end{aligned}$$

where we have used the fact that $\alpha = iz_\alpha$.

3.(a) From Q2,

$$H^{(\zeta)}(\alpha, \bar{\alpha}) = \frac{\Gamma^2}{4\pi}\log|\alpha - \bar{\alpha}|$$

(b) The required conformal mapping is

$$z(\zeta) = \zeta^{2/3}$$

since the positive real ζ-axis maps to the positive real z-axis while the line $\arg[\zeta] = \pi$ maps to $\arg[z] = 2\pi/3$.

(c) On use of the transformation rule for Hamiltonians

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi}\log|(\alpha(z_\alpha) - \overline{\alpha(z_\alpha)})z_\zeta(\alpha)|.$$

But $z_\alpha = \alpha^{2/3}$ and $\alpha = z_\alpha^{3/2}$ while

$$z_\zeta(\zeta) = \frac{2}{3\zeta^{1/3}}.$$

Combining all this yields

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi}\log\left|(z_\alpha^{3/2} - \bar{z}_\alpha^{3/2})\frac{2}{3z_\alpha^{1/2}}\right|.$$

(d) The trajectories are given by

$$\left|(z_\alpha^{3/2} - \bar{z}_\alpha^{3/2})\frac{2}{3z_\alpha^{1/2}}\right| = \text{constant}.$$

Let

$$z_\alpha = re^{i\theta}.$$

Then equation for trajectories becomes

$$
\begin{aligned}
\left|\frac{(z_\alpha^{3/2} - \bar{z}_\alpha^{3/2})(\bar{z}_\alpha^{3/2} - z_\alpha^{3/2})}{r}\right| &= \text{constant}, \\
\text{or} \quad \left|r^2(2 - 2\cos 3\theta)\right| &= \text{constant}, \\
\text{or} \qquad r\sin(3\theta/2) &= \text{constant}.
\end{aligned}
$$

4. (a) First note that $\zeta_1(z)$ maps the first quadrant of the z-plane within the unit circle to the interior of the upper-half semi-disc in the ζ_1-plane. This is because squaring a complex number doubles its argument. Next, from Q1, the Möbius mapping $\zeta_2(\zeta_1)$ maps the interior of the unit ζ_1-disc to the right-half ζ_2-plane. It is easy to verify that the upper-half semi-disk in the ζ_1-plane maps to the first quadrant of the ζ_2-plane while the lower-half semi-disc in the ζ_1-plane maps to the fourth quadrant in the ζ_2-plane. Finally, squaring again, maps the first quadrant of the ζ_2-plane to the whole of the upper-half ζ-plane.

(b) It follows from the transformation property of Hamiltonians that

$$
H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi}\log|(\alpha - \bar{\alpha})z_\zeta(\alpha)|.
$$

But

$$
z_\zeta(\alpha) = \frac{1}{\zeta_z(z_\alpha)},
$$

hence the result follows.

(c) First note that

$$
\zeta(z) = \left(\frac{1 - z^2}{1 + z^2}\right)^2 = \left(1 - \frac{2}{z^2 + 1}\right)^2 = 1 - \frac{4}{z^2 + 1} + \frac{4}{(z^2 + 1)^2}.
$$

Hence, on differentiation with respect to z,

$$
\zeta_z = \frac{8z}{(z^2 + 1)^2} - \frac{16z}{(z^2 + 1)^3} = \frac{8z(z^2 - 1)}{(z^2 + 1)^3}.
$$

Now let

$$
\alpha = \left(\frac{1 - z_\alpha^2}{1 + z_\alpha^2}\right)^2 = 1 - \frac{4}{z_\alpha^2 + 1} + \frac{4}{(z_\alpha^2 + 1)^2}
$$

so that

$$\alpha - \bar{\alpha} = -\frac{4}{z_\alpha^2 + 1} + \frac{4}{(z_\alpha^2 + 1)^2} + \frac{4}{\bar{z}_\alpha^2 + 1} - \frac{4}{(\bar{z}_\alpha^2 + 1)^2}.$$

Simplifying this, we get

$$\alpha - \bar{\alpha} = \frac{4(z_\alpha^2 - \bar{z}_\alpha^2)(z_\alpha^2 \bar{z}_\alpha^2 - 1)}{(z_\alpha^2 + 1)^2(\bar{z}_\alpha^2 + 1)^2}.$$

The trajectories are given by

$$\frac{(\alpha - \bar{\alpha})(\bar{\alpha} - \alpha)}{\zeta_z(z_\alpha)\overline{\zeta_z(z_\alpha)}} = \text{constant}.$$

Let $z_\alpha = re^{i\theta}$. This becomes

$$\frac{(r^4 - 1)^2(z_\alpha^2 - \bar{z}_\alpha^2)(\bar{z}_\alpha^2 - z_\alpha^2)}{r^2(z_\alpha^2 + 1)(\bar{z}_\alpha^2 + 1)(z_\alpha^2 - 1)(\bar{z}_\alpha^2 - 1)} = \text{constant}$$

or

$$\frac{(r^4 - 1)^2(z_\alpha^2 - \bar{z}_\alpha^2)(\bar{z}_\alpha^2 - z_\alpha^2)}{r^2(z_\alpha^4 - 1)(\bar{z}_\alpha^4 - 1)} = \text{constant}.$$

This can be written in the form

$$\frac{(r^4 - 1)^2 r^2 (2 - 2\cos 4\theta)}{(r^8 - 2r^4 \cos 4\theta + 1)} = \text{constant}$$

which, on use of a trigonometric identity, gives the required result.