M4A32: Vortex Dynamics

Problem Sheet 3

(Point vortex dynamics with boundaries)

1. It is easy to show, directly from the equations of motion, that a single point vortex next to an infinite straight wall moves at constant speed parallel to the wall, maintaining a constant distance from it. This questions retrieves this result using the Hamiltonian perspective.

(a) Verify that the conformal map given by

$$z(\zeta) = \frac{1-\zeta}{1+\zeta}$$

maps the interior of the unit circle in the ζ-plane to the right-half z-plane.

(b) By making use of the fact that the Hamiltonian for the motion of a single vortex of circulation Γ at a point $\alpha(t)$ in the unit ζ-disk is

$$H^{(\zeta)}(\alpha, \bar{\alpha}) = \frac{\Gamma^2}{4\pi} \log(1 - \alpha\bar{\alpha})$$

and the transformation rule

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = H^{(\zeta)}(\alpha, \bar{\alpha}) + \frac{\Gamma^2}{4\pi} \log |z_\zeta(\alpha)|$$

where

$$z_\alpha = z(\alpha)$$

show that

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi} \log |z_\alpha + \bar{z}_\alpha|$$

(c) Hence verify that the trajectories are straight lines parallel to the wall $\text{Re}[z] = 0$.

2. In Question 1, the simple Hamiltonian for point vortex motion in a unit disk is used as a starting point. Now, the starting point is taken to be the Hamiltonian for the motion of a single point vortex in the upper-half ζ-plane.

(a) Let the position of the point vortex be at $\zeta = \alpha$. Define the Green's function $G(\zeta; \alpha, \bar{\alpha})$ associated with point vortex motion in the upper-half ζ-plane (that is, explain the boundary value problem satisfied by this Green's function).

(b) Verify that the function

$$G(\zeta; \alpha, \bar{\alpha}) = -\frac{1}{2\pi} \log \left| \frac{\zeta - \alpha}{\zeta - \bar{\alpha}} \right|$$

satisfies all the conditions required of the Green's function $G(\zeta; \alpha, \bar{\alpha})$ defined in part (a);

(c) Use the result of (b) to find the Hamiltonian $H^{(\zeta)}(\alpha, \bar{\alpha})$ governing the motion of the vortex in the upper-half ζ-plane.

(d) Find a simple conformal mapping $z(\zeta)$ that takes the upper-half ζ-plane to the right-half z-plane.

(e) Use the transformation between Hamiltonians, i.e.,

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = H^{(\zeta)}(\alpha, \bar{\alpha}) + \frac{\Gamma^2}{4\pi} \log |z_\zeta(\alpha)|$$

to verify the final Hamiltonian found in Question 1.

3. In lectures it was shown how to find the complex potential for a single point vortex in a wedge-region with angle $2\pi/3$. The idea is to use conformal mapping. With the complex potential obtained in this way, the equations of motion can be written down and the point vortex trajectories computed, in principle. In this question, we show how to calculate the point vortex trajectories using the Hamiltonian approach.

(a) Write down the Hamiltonian for the motion of a single point vortex in the upper-half ζ-plane.

(b) Find a simple conformal mapping $z(\zeta)$ from the upper-half ζ-plane to the wedge region in the z-plane.

(c) Use the transformation rule for Hamiltonians to find the Hamiltonian for the point vortex motion in the wedge region.

(d) Hence show that the point vortex trajectories are given by

$$r \sin\left(\frac{3\theta}{2}\right) = \text{constant}$$

where (r, θ) are the usual polar coordinates.

4. Consider the motion of a point vortex in the region of the first quadrant of a z-plane inside the unit z-disk. That is, the finite-area region bounded by the positive real z-axis, the positive imaginary z axis and the circular arc $|z| = 1$.

(a) Show that the sequence of conformal mappings given by

$$\begin{aligned} \zeta_1(z) &= z^2; \\ \zeta_2(\zeta_1) &= \frac{1 - \zeta_1}{1 + \zeta_1}; \\ \zeta(\zeta_2) &= \zeta_2^2 \end{aligned}$$

maps this region in the z-plane to the upper-half ζ-plane. Hence find the mapping $\zeta(z)$.

(b) Show that the Hamiltonian in the z-plane is

$$H^{(z)}(z_\alpha, \bar{z}_\alpha) = \frac{\Gamma^2}{4\pi} \log \left| \frac{(\alpha(z_\alpha) - \overline{\alpha(z_\alpha)})}{\zeta_z(z_\alpha)} \right|.$$

where z_α is the point vortex position in the z-plane, i.e., $\alpha = \zeta(z_\alpha)$ and where $\zeta_z(z)$ denotes the derivative $\frac{d\zeta}{dz}$.

(c) Hence show the trajectories in the z-plane are given by

$$(r^8 + 1 - 2r^4 \cos 4\theta) = Ar^2(r^4 - 1)^2 \sin^2(2\theta)$$

where A is a constant and (r, θ) are the usual polar coordinates.