M4A32: Vortex Dynamics
Problem Sheet 2

1. The equations of motion, for $j = 1, ..., N$, are

$$\frac{d\bar{z}_j}{dt} = -\sum_k{}' \frac{i\Gamma_k}{z_j - z_k} \tag{1}$$

where the notation $\sum_k{}'$ denotes the sum for $k = 1, ..., N$ with $k \neq j$. Let

$$\begin{aligned} H &= -\frac{1}{4\pi}\sum_{j,k}{}' \,\Gamma_j\Gamma_k \log|z_j - z_k| \\ &= -\frac{1}{4\pi}\sum_{j,k}{}' \,\frac{\Gamma_j\Gamma_k}{2}\Big(\log(z_j - z_k) + \log(\overline{z_j} - \overline{z_k})\Big). \end{aligned} \tag{2}$$

It follows (using dots to denote time derivatives) that

$$\begin{aligned} \frac{dH}{dt} &= -\frac{1}{4\pi}\sum_{j,k}{}' \,\frac{\Gamma_j\Gamma_k}{2}\left(\frac{\dot{z}_j - \dot{z}_k}{z_j - z_k} + \frac{\dot{\overline{z_j}} - \dot{\overline{z_k}}}{\overline{z_j} - \overline{z_k}}\right) \\ &= -\frac{1}{4\pi}\sum_{j,k,m}{}' \frac{i\Gamma_j\Gamma_k\Gamma_m}{2}\Bigg[\frac{1}{(z_j - z_k)(\overline{z_j} - \overline{z_m})} - \frac{1}{(z_j - z_k)(\overline{z_k} - \overline{z_m})} \\ &\qquad - \frac{1}{(z_j - z_m)(\overline{z_j} - \overline{z_k})} + \frac{1}{(z_k - z_m)(\overline{z_j} - \overline{z_k})}\Bigg]. \end{aligned} \tag{3}$$

But the first and third terms in the square brackets cancel (after summation), as do the second and fourth terms. Therefore

$$\frac{dH}{dt} = 0. \tag{4}$$

Next note that

$$Q - iP = \sum_j \Gamma_j\,(x_j - iy_j) = \sum_j \Gamma_j \overline{z_j}. \tag{5}$$

Then

$$\frac{d(Q - iP)}{dt} = \sum_j \Gamma_j \dot{\overline{z_j}} = -\sum_{j,k}{}' \frac{i\Gamma_j\Gamma_k}{z_j - z_k} = 0 \tag{6}$$

where the last equality follows because on changing j to k in the summation, we get the negative of the same sum. The sum is therefore zero.

Next consider

$$I = \sum_j \Gamma_j(x_j^2 + y_j^2) = \sum_j \Gamma_j z_j \overline{z_j}. \tag{7}$$

Taking the time derivative yields

$$
\begin{aligned}
\frac{dI}{dt} &= \sum_j \Gamma_j \left(z_j \dot{\overline{z_j}} + \dot{z}_j \overline{z_j} \right) \\
&= \sum_{j,k}{}' - \frac{i\Gamma_j \Gamma_k z_j}{z_j - z_k} + \frac{i\Gamma_j \Gamma_k \overline{z_j}}{\overline{z_j} - \overline{z_k}} \\
&= \sum_{j,k}{}' - i\Gamma_j \Gamma_k \left[\frac{z_j(\overline{z_j} - \overline{z_k}) - \overline{z_j}(z_j - z_k)}{|z_j - z_k|^2} \right] \\
&= \sum_{j,k}{}' \frac{i\Gamma_j \Gamma_k}{|z_j - z_k|^2} \left(z_j \overline{z_k} - \overline{z_j} z_k \right) \\
&= 0
\end{aligned}
\tag{8}
$$

where the last equality follows because, on swapping indices j and k in the sum, we get the negative of the same sum implying that sum is zero.

2. The equations of point vortex motion, in Hamiltonian form, are

$$
\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{9}
$$

where $q_j = \sqrt{\Gamma_j} x_j$ and $p_j = \sqrt{\Gamma_j} y_j$. Now

$$
\begin{aligned}
[x_k, H] &= \sum_j \frac{1}{\Gamma_j} \left[\frac{\partial x_k}{\partial x_j} \frac{\partial H}{\partial y_j} - \frac{\partial x_k}{\partial y_j} \frac{\partial H}{\partial x_j} \right] \\
&= \frac{1}{\Gamma_k} \frac{\partial H}{\partial y_k} \\
&= \frac{1}{\sqrt{\Gamma_k}} \frac{\partial H}{\partial p_k} \\
&= \frac{1}{\sqrt{\Gamma_k}} \dot{q}_k \\
&= \dot{x}_k.
\end{aligned}
\tag{10}
$$

The equation $\dot{y}_k = [y_k, H]$ follows analogously.

Next, note that

$$
\begin{aligned}
[Q, H] &= \sum_j \frac{1}{\Gamma_j} \left[\frac{\partial Q}{\partial x_j} \frac{\partial H}{\partial y_j} - \frac{\partial Q}{\partial y_j} \frac{\partial H}{\partial x_j} \right] \\
&= \sum_j \frac{\partial H}{\partial y_j} \\
&= -\sum_j 2\mathrm{Im} \left[\frac{\partial H}{\partial z_j} \right].
\end{aligned}
\tag{11}
$$

But

$$\begin{aligned}\frac{\partial H}{\partial z_j} &= \frac{\partial}{\partial z_j}\left[-\frac{1}{8\pi}\sum_{k,l}{}'\Gamma_k\Gamma_l\Big(\log(z_k - z_l) + \log(\overline{z_k} - \overline{z_l})\Big)\right] \\ &= -\frac{1}{8\pi}\sum_k{}'\frac{\Gamma_j\Gamma_k}{z_j - z_k}.\end{aligned} \tag{12}$$

Therefore

$$[Q, H] = \frac{1}{4\pi}\sum_{j,k}{}'\mathrm{Im}\left[\frac{\Gamma_j\Gamma_k}{z_j - z_k}\right] = 0 \tag{13}$$

where the last equality follows by swapping j and k in the sum.

The proof that $[P, H] = 0$ is similar.

Now

$$[I, H] = \sum_j \frac{1}{\Gamma_j}\left[\frac{\partial I}{\partial x_j}\frac{\partial H}{\partial y_j} - \frac{\partial I}{\partial y_j}\frac{\partial H}{\partial x_j}\right]. \tag{14}$$

But

$$\frac{\partial I}{\partial x_j} = 2\Gamma_j x_j, \quad \frac{\partial I}{\partial y_j} = 2\Gamma_j y_j. \tag{15}$$

Therefore

$$[I, H] = 2\sum_j\left(x_j\frac{\partial H}{\partial y_j} - y_j\frac{\partial H}{\partial x_j}\right). \tag{16}$$

But

$$(x_j + iy_j)\frac{\partial H}{\partial z_j} = \frac{1}{2}(x_j + iy_j)\left[\frac{\partial H}{\partial x_j} - i\frac{\partial H}{\partial y_j}\right] \tag{17}$$

so that

$$x_j\frac{\partial H}{\partial y_j} - y_j\frac{\partial H}{\partial x_j} = -2\mathrm{Im}\left[z_j\frac{\partial H}{\partial z_j}\right]. \tag{18}$$

However, we already know that

$$\frac{\partial H}{\partial z_j} = -\frac{1}{8\pi}\sum_k{}'\frac{\Gamma_j\Gamma_k}{z_j - z_k}. \tag{19}$$

Therefore

$$\begin{aligned}[I, H] &= -4\sum_{j,k}{}'\ \mathrm{Im}\left[-\frac{1}{8\pi}\frac{\Gamma_j\Gamma_k z_j}{z_j - z_k}\right] \\ &= \frac{1}{2\pi}\sum_{j,k}{}'\frac{\Gamma_j\Gamma_k}{2i}\left[\frac{z_j}{z_j - z_k} - \frac{\overline{z_j}}{\overline{z_j} - \overline{z_k}}\right] \\ &= \frac{1}{4\pi i}\sum_{j,k}{}'\frac{\Gamma_j\Gamma_k}{|z_j - z_k|^2}\left[-z_j\overline{z_k} + z_k\overline{z_j}\right] \\ &= 0\end{aligned} \tag{20}$$

where, again, the last equality follows by swapping indices in the sum.

(c)

$$[I, P^2 + Q^2] = \sum_j \frac{1}{\Gamma_j} \left[\frac{\partial I}{\partial x_j} \frac{\partial (P^2 + Q^2)}{\partial y_j} - \frac{\partial (P^2 + Q^2)}{\partial x_j} \frac{\partial I}{\partial y_j} \right]. \tag{21}$$

But

$$\frac{\partial (P^2 + Q^2)}{\partial y_j} = 2P\Gamma_j, \quad \frac{\partial (P^2 + Q^2)}{\partial x_j} = 2Q\Gamma_j \tag{22}$$

therefore

$$\begin{aligned} [I, P^2 + Q^2] &= 2 \sum_j 2\Gamma_j P x_j - 2\Gamma_j Q y_j \\ &= 4 \sum_{j,k} \Gamma_j \Gamma_k \left(x_j y_k - x_k y_j \right) \\ &= 0 \end{aligned} \tag{23}$$

where the last equality follows by swapping indices in the sum.

3. Let n line vortices be at points

$$z_j = \tilde{z}(t) e^{2\pi i j/n} \tag{24}$$

where $j = 0, 1, ..., n-1$ and where $\tilde{z}(t)$ is a complex parameter. We will show that $\tilde{z}(t) = ae^{i\omega t}$ for some constant ω.

Consider only $\tilde{z}_0 = \tilde{z}(t)$. The complex potential is

$$\begin{aligned} w(z,t) &= \sum_{j=0}^{n-1} -\frac{i\Gamma_j}{2\pi} \log(z - z_j) \\ &= -\frac{i\Gamma}{2\pi} \log \left[\prod_{j=0}^{n-1} (z - z_j) \right] = -\frac{i\Gamma}{2\pi} \log(z^n - \tilde{z}^n) \end{aligned} \tag{25}$$

The motion of $\tilde{z}_0$ is given by the non-self-induced velocity, that is

$$\frac{d\tilde{z}_0(t)}{dt} = \left. \frac{d\hat{w}(z,t)}{dz} \right|_{z=\tilde{z}_0=\tilde{z}(t)} \tag{26}$$

where

$$\hat{w}(z) = -\frac{i\Gamma}{2\pi} \log \left(\sum_{j=0}^{n-1} z^{(n-1-j)} \tilde{z}(t)^j \right) \tag{27}$$

where we have used the factorization

$$z^n - \tilde{z}(t)^n = (z - \tilde{z}(t))(z^{n-1} + \tilde{z}(t) z^{n-2} + ... + \tilde{z}(t)^{n-2} z + \tilde{z}(t)^{n-1}). \tag{28}$$

However, it is easy to check that

$$\frac{d\hat{w}(z,t)}{dz} = -\frac{i\Gamma}{2\pi}\frac{1}{\tilde{z}(t)n}\sum_{j=0}^{n-1} j \tag{29}$$

But

$$\sum_{j=0}^{n-1} j = \frac{(n-1)n}{2}. \tag{30}$$

Therefore

$$\frac{d\tilde{z}(t)}{dt} = -\frac{i\Gamma(n-1)}{4\pi\tilde{z}(t)}. \tag{31}$$

Letting $\tilde{z}(t) = a(t)e^{i\theta(t)}$ and substituting yields

$$\dot{a}e^{-i\theta} - i\dot{\theta}ae^{-i\theta} = -\frac{i\Gamma(n-1)}{4\pi a}e^{-i\theta} \tag{32}$$

from which, on equating real and imaginary parts, it follows that

$$\dot{a} = 0, \quad \dot{\theta} = \frac{\Gamma(n-1)}{4\pi a^2}. \tag{33}$$

4. Let there be point vortices of circulation Γ at $z = 0, \pm 1$ at some instant. The instantaneous complex potential is

$$w(z) = -\frac{i\Gamma}{2\pi}\Big[\log z + \log(z-1) + \log(z+1)\Big]. \tag{34}$$

Assume the configuration is in solid body rotation with angular velocity Ω. Consider the point vortex at $z = 1$ (call it z_1).

$$\frac{d\overline{z_1}}{dt} = \frac{d\hat{w}}{dz}\bigg|_{z_1} \tag{35}$$

where

$$\hat{w}(z) = -\frac{i\Gamma}{2\pi}\Big[\log z + \log(z+1)\Big]. \tag{36}$$

This simplifies to

$$\frac{d\overline{z_1}}{dt} = -\frac{i\Gamma}{2\pi}\left[\frac{1}{z} + \frac{1}{z+1}\right]_{z=1} = -\frac{3i\Gamma}{4\pi}. \tag{37}$$

Let $z_1 = x_i + iy_1$, so that

$$\dot{\overline{z_1}} = \dot{x}_1 - i\dot{y}_1 = -\frac{3i\Gamma}{4\pi} \tag{38}$$

then $\dot{x}_1 = 0$ and $\dot{y}_1 = 3\Gamma/4\pi$ so the angular velocity is $\Omega = 3\Gamma/4\pi$.

To examine the linear stability, move to a frame of reference co-rotating with angular velocity Ω. The complex velocity in the co-rorating frame is

$$u - iv = i\Omega\bar{z} - \frac{i\Gamma}{2\pi}\left[\frac{1}{z_c(t)} + \frac{1}{z - z_+(t)} + \frac{1}{z - z_-(t)}\right] \tag{39}$$

where the point vortices are supposed to be at $z_c(t), z_\pm(t)$. For small departures from equilibrium, suppose

$$\begin{aligned} z_c(t) &= \epsilon\hat{z}_c e^{\sigma t}, \quad z_+(t) = 1 + \epsilon\hat{z}_+ e^{\sigma t}, \quad z_-(t) = -1 + \epsilon\hat{z}_- e^{\sigma t} \\ \overline{z_c(t)} &= \epsilon\hat{z}_c^* e^{\sigma t}, \quad \overline{z_+(t)} = 1 + \epsilon\hat{z}_+^* e^{\sigma t}, \quad \overline{z_-(t)} = -1 + \epsilon\hat{z}_-^* e^{\sigma t} \end{aligned} \tag{40}$$

where $z_c, z_c^*, z_+, z_+^*, z_-$ and z_-^* are all taken to be independent quantities.

Consider the evolution of $z_+(t)$:

$$\frac{d\overline{z_+(t)}}{dt} = \sigma\epsilon\hat{z}_+^* e^{\sigma t} = i\Omega(1 + \epsilon\hat{z}_+^* e^{\sigma t}) - \frac{i\Gamma}{2\pi}\left(\frac{1}{(1 + \epsilon\hat{z}_+ e^{\sigma t} - \epsilon\hat{z}_c e^{\sigma t})} + \frac{1}{(2 + \epsilon\hat{z}_+ e^{\sigma t} - \epsilon\hat{z}_- e^{\sigma t})}\right). \tag{41}$$

Linearizing in ϵ gives

$$\sigma\epsilon\hat{z}_+^* e^{\sigma t} \approx i\Omega(1+\epsilon\hat{z}_+^* e^{\sigma t}) - \frac{i\Gamma}{2\pi}\left(1 + \epsilon(\hat{z}_c - \hat{z}_+)e^{\sigma t} + \frac{1}{2} + \frac{1}{4}(\hat{z}_- - \hat{z}_+)e^{\sigma t}\right). \tag{42}$$

First, note that the $\mathcal{O}(\epsilon^0)$ terms are consistent. At $\mathcal{O}(\epsilon)$:

$$\sigma\hat{z}_+^* = i\Omega\hat{z}_+^* - \frac{i\Gamma}{2\pi}(\hat{z}_c - \hat{z}_+) - \frac{i\Gamma}{8\pi}(\hat{z}_- - \hat{z}_+). \tag{43}$$

The complex conjugate equation, at $\mathcal{O}(\epsilon)$, yields

$$\sigma\hat{z}_+ = -i\Omega\hat{z}_+ + \frac{i\Gamma}{2\pi}(\hat{z}_c^* - \hat{z}_+^*) + \frac{i\Gamma}{8\pi}(\hat{z}_-^* - \hat{z}_+^*). \tag{44}$$

By symmetry, the linearized evolution equations for $z_-(t)$ follow by swapping $+$ and $-$ in the above equations:

$$\sigma\hat{z}_-^* = i\Omega\hat{z}_-^* - \frac{i\Gamma}{2\pi}(\hat{z}_c - \hat{z}_-) - \frac{i\Gamma}{8\pi}(\hat{z}_+ - \hat{z}_-) \tag{45}$$

and

$$\sigma\hat{z}_- = -i\Omega\hat{z}_- + \frac{i\Gamma}{2\pi}(\hat{z}_c^* - \hat{z}_-^*) + \frac{i\Gamma}{8\pi}(\hat{z}_+^* - \hat{z}_-^*). \tag{46}$$

The evolution equation for $z_c(t)$ is

$$\epsilon\sigma\hat{z}_c^* e^{\sigma t} = i\Omega\epsilon\hat{z}_c^* e^{\sigma t} - \frac{i\Gamma}{2\pi}\left(\frac{1}{(\epsilon z_c e^{\sigma t} - 1 - \epsilon\hat{z}_+ e^{\sigma t})} + \frac{1}{(\epsilon z_c e^{\sigma t} + 1 - \epsilon\hat{z}_- e^{\sigma t})}\right). \tag{47}$$

Linearizing in ϵ gives

$$\epsilon\sigma\hat{z}_c^* e^{\sigma t} = i\Omega\epsilon\hat{z}_c^* e^{\sigma t} - \frac{i\Gamma}{2\pi}\left(-1 - \epsilon(\hat{z}_c - \hat{z}_+)e^{\sigma t} + 1 + \epsilon(\hat{z}_- - \hat{z}_c)e^{\sigma t}\right) \tag{48}$$

or

$$\sigma\hat{z}_c^* = i\Omega\hat{z}_c^* + \frac{i\Gamma}{2\pi}\left(2\hat{z}_c - \hat{z}_+ - \hat{z}_-\right). \tag{49}$$

The conjugate equation gives rise to

$$\epsilon\sigma\hat{z}_c = -i\Omega\hat{z}_c - \frac{i\Gamma}{2\pi}\left(2\hat{z}_c^* - \hat{z}_+^* - \hat{z}_-^*\right). \tag{50}$$

Let

$$\mathbf{x} = (\hat{z}_c, \hat{z}_c^*, \hat{z}_+, \hat{z}_+^*, \hat{z}_-, \hat{z}_-^*), \tag{51}$$

then the linearized equations (43), (44), (45), (46), (49) and (50) can be written as the matrix eigenvalue problem

$$A\mathbf{x} = \sigma\mathbf{x} \tag{52}$$

where

$$A = \frac{i\Gamma}{2\pi}\begin{pmatrix} -3/2 & -2 & 0 & 1 & 0 & 1 \\ 2 & 3/2 & -1 & 0 & -1 & 0 \\ 0 & 1 & -3/2 & -5/4 & 0 & 1/4 \\ -1 & 0 & 5/4 & 3/2 & -1/4 & 0 \\ 0 & 1 & 0 & 1/4 & -3/2 & -5/4 \\ -1 & 0 & -1/4 & 0 & 5/4 & 3/2 \end{pmatrix}. \tag{53}$$

A numerical calculation of the eigenvalues (e.g. using `MATLAB`) gives them as

$$\frac{i\Gamma}{2\pi}\left(0, 0, 3/2, -3/2, 2.598i, -2.598i\right). \tag{54}$$

The configuration is therefore linearly unstable because there is a positive real eigenvalue.

5. A first guess at the complex potential is

$$w(z) = Uz + \frac{i\Gamma}{2\pi}\log(z - z_0) - \frac{i\Gamma}{2\pi}\log(z - \overline{z_0}) \tag{55}$$

since this has the correct singularity structure in $|z| > a$. However, it does not satisfy the streamline condition on $|z| = a$ where it is required that $\mathrm{Im}[w(z)] = 0$. Consider instead

$$W(z, \bar{z}) = w(z) + \overline{w(z)}. \tag{56}$$

This is real everywhere, but it is not analytic. However, it is only required to be real on $|z| = a$ where $\bar{z} = a^2/z$. Therefore consider the analytic function

$$W(z) = w(z) + \overline{w}(a^2/z) \tag{57}$$

It can be verified that this satisfies all the required conditions and is therefore the correct complex potential. To within an unimportant constant, it is given by

$$\begin{aligned} w(z) = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\log(z - z_0) - \frac{i\Gamma}{2\pi}\log(z - a^2/\overline{z_0}) \\ - \frac{i\Gamma}{2\pi}\log(z - \overline{z_0}) + \frac{i\Gamma}{2\pi}\log(z - a^2/z_0). \end{aligned} \tag{58}$$

If z_0 is to be in steady equilibrium, we require

$$0 = U\left(1 - \frac{a^2}{z_0^2}\right) - \frac{i\Gamma}{2\pi}\left(\frac{1}{z_0 - a^2/\overline{z_0}} + \frac{1}{z_0 - \overline{z_0}} - \frac{1}{z_0 - a^2/z_0}\right), \tag{59}$$

or

$$\left(1 - \frac{a^2}{z_0^2}\right) = \frac{i\Gamma}{2\pi U}\left[\frac{\overline{z_0}}{|z_0|^2 - a^2} + \frac{1}{z_0 - \overline{z_0}} - \frac{z_0}{z_0^2 - a^2}\right]. \tag{60}$$

The complex conjugate of this equation is

$$\left(1 - \frac{a^2}{\overline{z_0}^2}\right) = -\frac{i\Gamma}{2\pi U}\left[\frac{z_0}{|z_0|^2 - a^2} + \frac{1}{\overline{z_0} - z_0} - \frac{\overline{z_0}}{\overline{z_0}^2 - a^2}\right]. \tag{61}$$

Dividing (60) and (61) to eliminate $i\Gamma/2\pi U$ yields

$$\begin{aligned} -\left(1 - \frac{a^2}{z_0^2}\right)\left(\frac{z_0}{|z_0|^2 - a^2} + \frac{1}{\overline{z_0} - z_0} - \frac{\overline{z_0}}{\overline{z_0}^2 - a^2}\right) \\ = \left(1 - \frac{a^2}{\overline{z_0}^2}\right)\left(\frac{\overline{z_0}}{|z_0|^2 - a^2} + \frac{1}{z_0 - \overline{z_0}} - \frac{z_0}{z_0^2 - a^2}\right) \end{aligned} \tag{62}$$

After some algebra, this reduces to

$$r^2 - a^2 = 2ry \tag{63}$$

where $z_0 = x + iy$ and $|z_0| = r$. Using (63) in (60) gives the second result

$$\Gamma = 4\pi U y\left(1 - \frac{a^4}{r^4}\right). \tag{64}$$

6. Let a circulation $-\Gamma$ point vortex be at $z_1 = d(-1+i)$ and a point vortex of circulation Γ be at $z_2 = d(1+i)$. If

$$u = -\alpha x, \quad v = \alpha y \tag{65}$$

then

$$\frac{dw}{dz} = u - iv = -\alpha x - i\alpha y = -\alpha z \tag{66}$$

so

$$w(z) = -\frac{\alpha z^2}{2}. \tag{67}$$

Using the method of images,

$$w(z) = -\frac{\alpha z^2}{2} + \frac{i\Gamma}{2\pi}\log(z - z_1) - \frac{i\Gamma}{2\pi}\log(z - z_2) + \frac{i\Gamma}{2\pi}\log(z - \overline{z_2}) - \frac{i\Gamma}{2\pi}\log(z - \overline{z_1}) \tag{68}$$

The equation of motion for z_1 is

$$\frac{d\overline{z_1}}{dt} = -\alpha z_1 - \frac{i\Gamma}{2\pi}\left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - \overline{z_2}} + \frac{1}{z_1 - \overline{z_1}}\right). \tag{69}$$

The equation of motion for z_2 follows by swapping 1 and 2 and letting $\Gamma \mapsto -\Gamma$, i.e.,

$$\frac{d\overline{z_2}}{dt} = -\alpha z_2 + \frac{i\Gamma}{2\pi}\left(\frac{1}{z_2 - z_1} - \frac{1}{z_2 - \overline{z_1}} + \frac{1}{z_2 - \overline{z_2}}\right). \tag{70}$$

For equilibrium, we require $\dot{z_1} = \dot{z_2} = 0$ so that

$$\frac{i\Gamma}{2\pi}\left(\frac{1}{z_{10} - z_{20}} - \frac{1}{z_{10} - \overline{z_{20}}} + \frac{1}{z_{10} - \overline{z_{10}}}\right) = -\alpha z_{10} \tag{71}$$

and

$$\frac{i\Gamma}{2\pi}\left(\frac{1}{z_{20} - z_{10}} - \frac{1}{z_{20} - \overline{z_{10}}} + \frac{1}{z_{20} - \overline{z_{20}}}\right) = \alpha z_{20} \tag{72}$$

for the equilibrium positions z_{10} and z_{20}. It can be verified by direct substitution into the previous two equations that the solutions are given by

$$z_{10} = d(-1 + i), \quad z_{20} = d(1 + i). \tag{73}$$

where $d^2 = \Gamma/(8\pi\alpha)$.

To examine the linear stability, let

$$\begin{aligned} z_1 &= z_{10} + \epsilon\hat{z}_1 e^{\sigma t}, \\ \overline{z_1} &= \overline{z_{10}} + \epsilon\hat{z}_1^* e^{\sigma t}, \\ z_2 &= z_{20} + \epsilon\hat{z}_2 e^{\sigma t}, \\ \overline{z_2} &= \overline{z_{20}} + \epsilon\hat{z}_2^* e^{\sigma t} \end{aligned} \tag{74}$$

where $\hat{z}_1, \hat{z}_1^*, \hat{z}_2$ and $\hat{z}_2^*$ are considered to be independent quantities. Substitution into (69) yields

$$\begin{aligned} \epsilon\sigma\hat{z}_1^* e^{\sigma t} = -\alpha\left(z_{10} + \epsilon\hat{z}_1 e^{\sigma t}\right) - \frac{i\Gamma}{2\pi}\Bigg[&\frac{1}{(z_{10} - z_{20} + \epsilon\hat{z}_1 e^{\sigma t} - \epsilon\hat{z}_2 e^{\sigma t})} \\ &- \frac{1}{(z_{10} - \overline{z_{20}}) + \epsilon\hat{z}_1 e^{\sigma t} - \epsilon\hat{z}_2^* e^{\sigma t}} + \frac{1}{(z_{10} - \overline{z_{10}}) + \epsilon\hat{z}_1 e^{\sigma t} - \epsilon\hat{z}_1^* e^{\sigma t}}\Bigg]. \end{aligned} \tag{75}$$

But

$$z_{10} - z_{20} = -2d, \quad z_{10} - \overline{z_{20}} = -2d + 2id, \quad z_{10} - \overline{z_{10}} = 2id \tag{76}$$

so, the linearized equation at $\mathcal{O}(\epsilon)$ is

$$\sigma\hat{z}_1^* = -\frac{i\Gamma}{2\pi}\left[\frac{\hat{z}_2-\hat{z}_1}{4d^2}+\frac{\hat{z}_2^*-\hat{z}_1}{8d^2i}-\frac{\hat{z}_1^*-\hat{z}_1}{4d^2}\right]-\alpha\hat{z}_1 \tag{77}$$

or, using the relationship between α and Γ for equilibrium,

$$\sigma\hat{z}_1^* = -i\alpha\left(\hat{z}_2-\frac{i}{2}(\hat{z}_2^*-\hat{z}_1)-\hat{z}_1^*\right)-\alpha\hat{z}_1. \tag{78}$$

The conjugate equation similarly leads to

$$\sigma\hat{z}_1 = i\alpha\left(\hat{z}_2^*+\frac{i}{2}(\hat{z}_2-\hat{z}_1^*)-\hat{z}_1\right)-\alpha z_1^*. \tag{79}$$

By similar manipulations, the linearized equation for z_2 is

$$\sigma\hat{z}_2^* = i\alpha\left(\hat{z}_1-\hat{z}_2^*+\frac{i}{2}\left(\hat{z}_1^*-\hat{z}_2\right)\right)-\alpha\hat{z}_2 \tag{80}$$

and its conjugate equation is

$$\sigma\hat{z}_2 = -i\alpha\left(\hat{z}_1^*-\hat{z}_2-\frac{i}{2}(\hat{z}_1-\hat{z}_2^*)\right)-\alpha z_2^*. \tag{81}$$

Now let

$$\mathbf{x} = (\hat{z}_1,\hat{z}_1^*,\hat{z}_2,\hat{z}_2^*) \tag{82}$$

then

$$\sigma\mathbf{x} = A\mathbf{x} \tag{83}$$

where

$$A = \alpha\begin{pmatrix} -i & -1/2 & -1/2 & i \\ -1/2 & i & -i & -1/2 \\ -1/2 & -i & i & -1/2 \\ i & -1/2 & -1/2 & -i \end{pmatrix}. \tag{84}$$

Using `MATLAB`, eigenvalues are found to be $\pm\alpha, \pm 2i\alpha$ so configuration is linearly unstable.

7. Let a street of vortices of circulation Γ be positioned at $z_n = na$ and a street of vortices of circulation $-\Gamma$ be positioned at $z = na + ib$. This is the symmetric double vortex street. The effect on the vortex at $z_0 = ib$ of the vortices in the same street is zero. The lower street has complex potential

$$w(z) = -\frac{i\Gamma}{2\pi}\log\sin\left(\frac{\pi z}{a}\right). \tag{85}$$

The effect of the lower street on z_0 is therefore

$$\frac{d\overline{z_0}}{dt} = -\frac{i\Gamma}{2a}\cot\left(\frac{\pi z}{a}\right)\bigg|_{z=ib} = -\frac{\Gamma}{2a}\coth\left(\frac{\pi b}{a}\right) \tag{86}$$

so z_0 moves to the left with speed $[\Gamma/(2a)]\coth(\pi b/a)$.