M4A32: Vortex Dynamics

Problem Sheet 2 (Point vortex dynamics)

1. Consider the dynamics of an assembly of N point vortices of strengths Γ_j $(j = 1, 2, .., N)$ in the plane. Show that the following 4 quantities are conserved by the dynamics,

$$H = -\frac{1}{4\pi} \sum_{\substack{j,k \\ j \neq k}}^{N} \Gamma_j \Gamma_k \log |z_j - z_k|$$

$$Q = \sum_{j=1}^{N} \Gamma_j x_j$$

$$P = \sum_{j=1}^{N} \Gamma_j y_j$$

$$I = \sum_{j=1}^{N} \Gamma_j (x_j^2 + y_j^2)$$

2. Consider the unbounded motion of N point vortices of strengh Γ_j $(j = 1, 2, .., N)$ each at positions $z_j(t) = x_j(t) + iy_j(t)$ in the plane. Let f and g denote any two functions depending of the point vortex positions. Define the *Poisson bracket* between f and g as follows:

$$[f, g] = \sum_{j=1}^{N} \frac{1}{\Gamma_j} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right).$$

Establish the following results

(a)

$$\dot{x}_j = [x_j, H], \qquad \dot{y}_j = [y_j, H],$$

(b)

$$[Q, H] = [P, H] = [I, H] = 0$$

where Q, P and I are as defined in question 1.

(c)

$$[I, P^2 + Q^2] = 0$$

3. Show that N equal-strength point vortices equally-spaced around the circumference of a circle of radius a centred at the origin rotates about the origin with a constant angular velocity Ω given as

$$\Omega = \frac{\Gamma(N-1)}{4\pi a^2}$$

4. Show that another steadily-rotating configuration of 3 identical point vortices of circulation Γ consists of 3 point vortices arranged on a line with one half-way between the other two. Determine the angular velocity of the configuration. Determine the linear stability properties of this rotating configuration.

5. (Foppl vortex pair behind a cylinder). Consider two point vortices of strengths $-\Gamma$ and Γ placed behind a cylinder of radius a in a uniform stream U at positions $z_0 = x + iy$ and $\overline{z_0}$ respectively (where $|z_0| > a$). Show that the configuration is in equilibrium provided that

$$r^2 - a^2 = 2ry, \qquad \Gamma = 4\pi U y(1 - a^4/r^4)$$

6. As a simpler model of the Foppl vortex pair behind a cylinder, suppose that the two vortices are sufficiently close to the stagnation point behind the cylinder that the cylinder can be modelled by an infinite impenetrable wall at $y = 0$. In $y \geq 0$, model the rear stagnation point flow by assuming that there exists the following irrotational strain flow

$$u = -\alpha x, \qquad v = \alpha y$$

where $\alpha > 0$ is constant. Suppose that one point vortex of strength $-\Gamma$ is at $z = z_1(t)$ and another of strength Γ is at $z = z_2(t)$. Write down the instantaneous complex potential for the whole flow by the method of images and, by letting the vortices move with the fluid according to the Helmholtz vortex theorems, show that

$$\frac{d\bar{z}_1(t)}{dt} = -\frac{i\Gamma}{2\pi}\left[\frac{1}{z_1 - z_2} - \frac{1}{z_1 - \bar{z}_2} + \frac{1}{z_1 - \bar{z}_1}\right] - \alpha z_1$$
$$\frac{d\bar{z}_2(t)}{dt} = \frac{i\Gamma}{2\pi}\left[\frac{1}{z_2 - z_1} - \frac{1}{z_2 - \bar{z}_1} + \frac{1}{z_2 - \bar{z}_2}\right] - \alpha z_2$$

where an overbar denotes complex conjugation. Verify that the vortices remain at rest if they are placed at the points

$$z_1 = d(-1 + i) \qquad z_2 = d(1 + i)$$

where $d^2 = \frac{\Gamma}{8\pi\alpha}$.

Perform the linear stability analysis of this configuration and show that it is linearly unstable.

7. The instability of Q.6 was used as an early explanation for the asymmetry seen in the von Karman vortex street behind a cylinder. The case of an asymmetric vortex street was considered in lectures. In this case, there are two rows of vortices at $z_n = na$ and $z_n = (n + \frac{1}{2})a + ib$. Consider instead the case of a *symmetric* vortex street in which the vortices in the two rows are at positions $z_n = na$ (all with strength Γ) and at $z_n = na + ib$ (all with strength $-\Gamma$). Show that the whole street may, in principle, maintain its form by moving to the left with speed

$$V = \frac{\Gamma}{2a} \coth\left(\frac{\pi b}{a}\right)$$

[Note: A linear stability analysis of this equilibrium shows that it is linearly unstable for *all* aspect ratios $\frac{b}{a}$ and, in contrast to the asymmetric vortex street, there is no special choice of aspect ratio for which it is linearly stable.]