M4A32: Vortex Dynamics

Problem Sheet 1

1. Fluid is barotropic which means $p = p(\rho)$. The Euler equation, in presence of a conservative body force, is

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\chi.$$

This can be written, on use of a vector identity,

$$\frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{1}{2}|\underline{u}|^2\right) - \underline{u}\wedge\underline{\omega} = -\frac{1}{\rho}\nabla p - \nabla\chi. \tag{1}$$

Take the curl:

$$\frac{\partial \underline{\omega}}{\partial t} - \nabla\wedge(\underline{u}\wedge\underline{\omega}) = -\nabla\left(\frac{1}{\rho}\right)\wedge\nabla p = \frac{p'(\rho)}{\rho^2}\nabla\rho\wedge\nabla\rho = 0. \tag{2}$$

On use of a vector identity we get

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{u}.\nabla\underline{\omega} - \underline{\omega}.\nabla\underline{u} - \underline{u}(\nabla.\underline{\omega}) + \underline{\omega}(\nabla.\underline{u}). \tag{3}$$

Now, $\nabla.\underline{\omega} = 0$ since div curl=0. Now use conservation of mass equation to substitute for $\nabla.\underline{u}$:

$$\nabla.\underline{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{4}$$

so

$$\frac{D\underline{\omega}}{Dt} - \underline{\omega}.\nabla\underline{u} - \frac{\underline{\omega}}{\rho}\frac{D\rho}{Dt} = 0. \tag{5}$$

Dividing by ρ gives the final result

$$\frac{D}{Dt}\left(\frac{\underline{\omega}}{\rho}\right) = \frac{\underline{\omega}}{\rho}.\nabla\underline{u}. \tag{6}$$

2. Assume a barotropic fluid in a conservative force field. Let

$$\Gamma = \oint_C \underline{u}.\underline{dl}. \tag{7}$$

Take time derivative

$$\frac{d\Gamma}{dt} = \oint_C \frac{D\underline{u}}{Dt}.\underline{dl} + \underline{u}.\frac{D\underline{dl}}{Dt} \tag{8}$$

But it is known that $D\underline{dl}/Dt = \underline{d}u$. Using this, together with the Euler equation,

$$\frac{d\Gamma}{dt} = \oint_C \left(-\frac{1}{\rho}\nabla p - \nabla\chi\right).\underline{dl} + d\left(\frac{|\underline{u}|^2}{2}\right) \tag{9}$$

But this can be written in the form

$$\frac{d\Gamma}{dt} = \oint_C \nabla \left(-\int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' - \chi + \frac{|\underline{u}|^2}{2} \right) .\underline{dl} \tag{10}$$

which is the integral, around a closed loop, of a total spatial differential of a single-valued function. It is therefore zero and yields Kelvin's circulation theorem for a barotropic fluid.

3. For a barotropic fluid, Euler's equation can be written in the form

$$\frac{\partial \underline{u}}{\partial t} + + \frac{1}{2}\nabla|\underline{u}|^2 + \underline{\omega} \wedge \underline{u} = -\nabla \int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' - \nabla\chi. \tag{11}$$

First form of Bernoulli: Suppose the flow is steady. Taking the dot product of the Euler equation with $\underline{u}$ yields

$$-\underline{u}.\nabla \left(\int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' - \nabla\chi - \frac{1}{2}|\underline{u}|^2 \right) \tag{12}$$

which means that the quantity

$$\int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' - \nabla\chi - \frac{1}{2}|\underline{u}|^2 \tag{13}$$

is constant on streamlines.

Second form of Bernoulli: Suppose that the flow is irrotational. The Euler equation then says that

$$\nabla \left(\int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' + \nabla\chi + \frac{1}{2}|\underline{u}|^2 \right) = 0 \tag{14}$$

from which we deduce that

$$\int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' + \nabla\chi + \frac{1}{2}|\underline{u}|^2 \tag{15}$$

is constant everywhere.

Third form of Bernoulli: Suppose the flow is unsteady, but irrotational (note, by Q2, we still have the "persistence of irrotational flow" for a barotropic fluid so this is a consistent statement). Then $\underline{u} = \nabla\phi$ for some scalar ϕ. Then the Euler equation says that

$$\nabla \left(-\frac{\partial \phi}{\partial t} - \int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' - \nabla\chi - \frac{1}{2}|\underline{u}|^2 \right) = 0 \tag{16}$$

which means that

$$\frac{\partial \phi}{\partial t} + \int^{\rho} \frac{1}{\rho'} \frac{dp}{d\rho'} d\rho' + \nabla\chi + \frac{1}{2}|\underline{u}|^2 = H(t) \tag{17}$$

for some function of time $H(t)$.

4. Let u_i ρ and P be the velocity, density and pressure fields of any three-dimensional steady solution of the incompressible Euler equation, i.e.,

$$u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial P}{\partial x_i} = 0, \tag{18}$$

and

$$\frac{\partial \left(\rho u_i\right)}{\partial x_i} = 0. \tag{19}$$

Consider now the velocity, density and pressure fields $\hat{u}_i$, $\hat{\rho}$ and $\hat{P}$ given by

$$\hat{u}_i = \lambda u_i, \tag{20}$$

$$\hat{\rho} = \frac{\rho}{\lambda^2}, \tag{21}$$

$$\hat{P} = P \tag{22}$$

where λ is assumed to be such that

$$u_i \frac{\partial \lambda}{\partial x_i} = 0. \tag{23}$$

Note: this corresponds to the fact that λ is constant on streamlines. It must be shown that if (18), (19) and (23) hold then so do

$$\hat{u}_j \frac{\partial \hat{u}_i}{\partial x_j} + \frac{1}{\hat{\rho}} \frac{\partial \hat{P}}{\partial x_i} = 0, \tag{24}$$

and

$$\frac{\partial \left(\hat{\rho} \hat{u}_i\right)}{\partial x_i} = 0. \tag{25}$$

First, to show that (24) holds, note that by (20) and (22),

$$\begin{aligned} \hat{u}_j \frac{\partial \hat{u}_i}{\partial x_j} &+ \frac{1}{\hat{\rho}} \frac{\partial \hat{P}}{\partial x_i} \\ &= \lambda u_j \frac{\partial}{\partial x_j} \left(\lambda u_j\right) + \frac{\lambda^2}{\rho} \frac{\partial P}{\partial x_i} \\ &= \lambda^2 \left(u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} \right) + \lambda u_i u_j \frac{\partial \lambda}{\partial x_j} \\ &= 0 \end{aligned} \tag{26}$$

where the last equality follows by (18) and provided (23) holds.

To show that (25) holds, note that

$$\begin{aligned}\frac{\partial\left(\hat{\rho}\hat{u}_i\right)}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\rho u_i}{\lambda}\right) \\ &= \frac{1}{\lambda}\frac{\partial\left(\rho u_i\right)}{\partial x_i} + \rho u_i \frac{\partial}{\partial x_i}\left(\frac{1}{\lambda}\right) \\ &= -\frac{1}{\lambda^2}\rho u_i \frac{\partial \lambda}{\partial x_i} \\ &= 0\end{aligned} \tag{27}$$

where we have used both (19) and (23).

5. Assume an ideal fluid and a flow field of the form

$$\underline{u} = (u_r(r,z,t),\ 0,\ u_z(r,z,t)). \tag{28}$$

Taking the curl, in cylindrical polar coordinates,

$$\nabla \wedge \underline{\omega} = \begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u_r(r,z,r) & 0 & u_z(r,z,r) \end{vmatrix} = (0,\ \omega(r,z,t),\ 0) \tag{29}$$

where

$$\omega(r,z,t) = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}. \tag{30}$$

The vorticity equation is

$$\frac{\partial \underline{\omega}}{\partial t} = \nabla \wedge \left(\underline{u} \wedge \underline{\omega}\right). \tag{31}$$

Computation of the right hand side gives

$$(0, \quad -\frac{\partial}{\partial z}\left(\omega u_z\right) - \frac{\partial}{\partial r}\left(\omega u_r\right),\ 0) \tag{32}$$

so only the azimuthal term gives a non-trivial equation i.e.,

$$\frac{\partial \omega}{\partial t} + \frac{\partial(\omega u_r)}{\partial r} + \frac{\partial(\omega u_z)}{\partial z} = 0 \tag{33}$$

or, equivalently,

$$\frac{\partial \omega}{\partial t} + \omega\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right) + u_r\frac{\partial \omega}{\partial r} + u_z\frac{\partial \omega}{\partial z} = 0. \tag{34}$$

But, the conservation of mass equation is $\nabla . \underline{u} = 0$ which, in cylindrical polar coordinates, takes the form

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \tag{35}$$

which can be used in the vorticity equation to reduce it to

$$\frac{\partial \omega}{\partial t} - \frac{\omega u_r}{r} + u_r \frac{\partial \omega}{\partial r} + u_z \frac{\partial \omega}{\partial z} = 0. \tag{36}$$

But, dividing this by r, it is simply

$$\left(\frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z} \right) \left(\frac{\omega}{r} \right) = 0 \tag{37}$$

which is the required result.

Note that if the radius of a vortex ring increases then the vorticity equation just derived shows that the dynamics is such that the vorticity ω changes linearly with the radius, thus as a ring is "stretched", the vorticity intensifies.

6. In spherical polars, the condition $\nabla . \underline{u} = 0$ takes the form

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \ u_\theta) = 0 \tag{38}$$

or, multiplying by $r^2 \sin \theta$,

$$\frac{\partial}{\partial r} (r^2 \sin \theta \ u_r) + \frac{\partial}{\partial \theta} (r \sin \theta \ u_\theta) = 0. \tag{39}$$

Therefore, introduce a streamfunction Ψ such that

$$r^2 \sin \theta \ u_r = \frac{\partial \Psi}{\partial \theta}, \quad r \sin \theta \ u_\theta = -\frac{\partial \Psi}{\partial r}. \tag{40}$$

Computing the vorticity field gives

$$\nabla \wedge \underline{\omega} = \begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & r \sin \theta \underline{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ u_r(r, \theta, t) & u_\theta(r, \theta, t) & 0 \end{vmatrix} = (0, 0, \ \omega(r, \theta, t)) \tag{41}$$

where

$$\omega(r, \theta, t) = \frac{1}{r} \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta}. \tag{42}$$

Substituting for u_r and u_θ in terms of Ψ then gives the final result

$$\omega = -\frac{1}{r \sin \theta} \left(\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial \Psi}{\partial \theta} \right). \tag{43}$$

For *irrotational* uniform flow past a sphere, we have $\omega = 0$ while $u_r \sim U \cos \theta$ and $u_\theta \sim -U \sin \theta$ as $r \to \infty$. Therefore, as $r \to \infty$, $\Psi \sim U r^2 \sin^2 \theta / 2$. This suggests trying a separable solution of the form

$$\Psi(r, \theta) = f(r) \sin^2 \theta. \tag{44}$$

Substituting this ansatz into the vorticity equation just derived with $\omega = 0$ yields the ordinary differential equation

$$0 = f''(r) - \frac{2f}{r^2} \tag{45}$$

which can be solved to yield

$$f(r) = Ar^2 + \frac{B}{r} \tag{46}$$

where A and B are constants. From the far-field conditions, we must pick $A = U/2$. On $r = a$ (the spherical boundary), we need Ψ to be constant. Take $\Psi = 0$ without loss of generality. This determines B and the final solution is

$$\Psi = \frac{U}{2}\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta. \tag{47}$$

Note: We will see this solution again when considering the "Hill's spherical vortex".

7. Seek the solution of

$$\nabla^2\psi = m\delta(\underline{x} - \underline{x}_0) \tag{48}$$

that decays at infinity. Without loss of generality, take $\underline{x}_0 = 0$. Multiply this equation by $e^{i\underline{k}.\underline{x}}$ and integrate over all space (i.e. take a Fourier transform):

$$\int_{\mathbb{R}^3} e^{i\underline{k}.\underline{x}}\nabla^2\psi = \int_{\mathbb{R}^3} e^{i\underline{k}.\underline{x}} m\delta(\underline{x} - \underline{x}_0) \tag{49}$$

Green's identity says that

$$\int_{\mathbb{R}^3}\left(u\nabla^2 v - v\nabla^2 u\right)dV = \lim_{R\to\infty}\int_{S_R}\left(u\nabla v - v\nabla u\right).\underline{dS} \tag{50}$$

where S_R is some radius-R spherical surface. The right side vanishes provided everything decays sufficiently fast at infinity. Letting $u = \psi$ and $v = e^{i\underline{k}.\underline{x}}$ gives

$$\int e^{i\underline{k}.\underline{x}}\nabla^2\psi dV = -|\underline{k}|^2\Psi(k) \tag{51}$$

where $\Psi(k)$ is the Fourier transform of ψ, that is

$$\Psi(\underline{k}) \equiv \int_{\mathbb{R}^3} e^{i\underline{k}.\underline{x}}\psi dV. \tag{52}$$

Use of this in (48) gives the result

$$\Psi(\underline{k}) = -\frac{m}{|\underline{k}|^2} \tag{53}$$

Now, the easiest way to arrive at the result is to verify that the Fourier transform of $-m/(4\pi r)$ is $-m/|\underline{k}|^2$. But the Fourier transform of $-m/(4\pi r)$ is

$$\int_{\mathbb{R}^3} -\frac{1}{4\pi r} e^{i\underline{k}.\underline{x}} dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} -\frac{1}{4\pi r} e^{i|\underline{k}|r\cos\theta} r^2 \sin\theta d\theta d\phi dr \tag{54}$$

where we have adopted spherical polar coordinates to perform the integration. Carrying out the ϕ integration gives

$$-\frac{1}{4\pi}(2\pi) \int_0^{\infty} dr \int_0^{\pi} d\theta e^{i|\underline{k}|r\cos\theta} r \sin\theta d\theta \tag{55}$$

but this allows a further integration with respect to θ yielding

$$\int_0^{\infty} \frac{dr}{|\underline{k}|} \left[\frac{e^{-i|\underline{k}|r} - e^{i|\underline{k}|r}}{2i} \right] = -\int_0^{\infty} \frac{\sin|\underline{k}|r}{|\underline{k}|} dr = -\frac{1}{|\underline{k}|} \mathrm{Im} \int_C e^{i|\underline{k}|z} dz \tag{56}$$

where C is the contour consisting of the infinite ray along the positive real z-axis. But $e^{i|\underline{k}|z}$ is an analytic function of z in the first quadrant of the z-plane, moreover it decays exponentially on the contour C_R consisting of a large radius-R quarter-circle between the positive real and imaginary axes of the first quadrant. This means that Cauchy's theorem can be used to argue that the required integral is the same as

$$-\frac{1}{|\underline{k}|} \mathrm{Im} \int_{\hat{C}} e^{i|\underline{k}|z} dz \tag{57}$$

where $\hat{C}$ is the ray consisting of the positive imaginary axis of the z-plane. Parametrizing this contour as $z = iy$ for $0 \leq y < \infty$ the integral becomes

$$-\frac{1}{|\underline{k}|} \mathrm{Im} \int_0^{\infty} e^{-|\underline{k}|y} i dy = -\frac{1}{|\underline{k}|^2} \tag{58}$$

which verifies that the Fourier transform of $-1/(4\pi r)$ is $-1/|\underline{k}|^2$ as required.

8. From the Biot-Savart integral in 3d,

$$\underline{u} = -\frac{1}{4\pi} \int_{flow} \frac{1}{r^3} (\underline{x} - \underline{x}') \wedge \underline{\omega}(\underline{x}') dx' dy' dz' \tag{59}$$

Now assume $\underline{\omega} = 0$ everywhere off the plane $z = 0$ and assume that $\underline{x}$ lies in this plane. Then

$$\underline{u}(\underline{x}) = -\frac{1}{4\pi} \int_{flow} (\underline{x} - \underline{x}') \wedge \underline{\omega}(\underline{x}') \frac{dx' dy' dz'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \tag{60}$$

so that, performing the z integration (using the hint),

$$\begin{aligned} \underline{u}(\underline{x}) &= -\frac{1}{4\pi} \int_{flow} (\underline{x} - \underline{x}') \wedge \underline{\omega}(\underline{x}') \frac{2dx' dy'}{[(x - x')^2 + (y - y')^2]} \\ &= -\frac{1}{2\pi} \int_{flow} (\underline{x} - \underline{x}') \wedge \underline{\omega}(\underline{x}') \frac{dx' dy'}{\hat{r}^2} \end{aligned} \tag{61}$$

where $\hat{r}^2 = |\underline{x} - \underline{x}'|^2$ is the distance between two vectors $\underline{x}$ and $\underline{x}'$ in the plane $z = 0$. This is precisely the 2d Biot-Savart result.