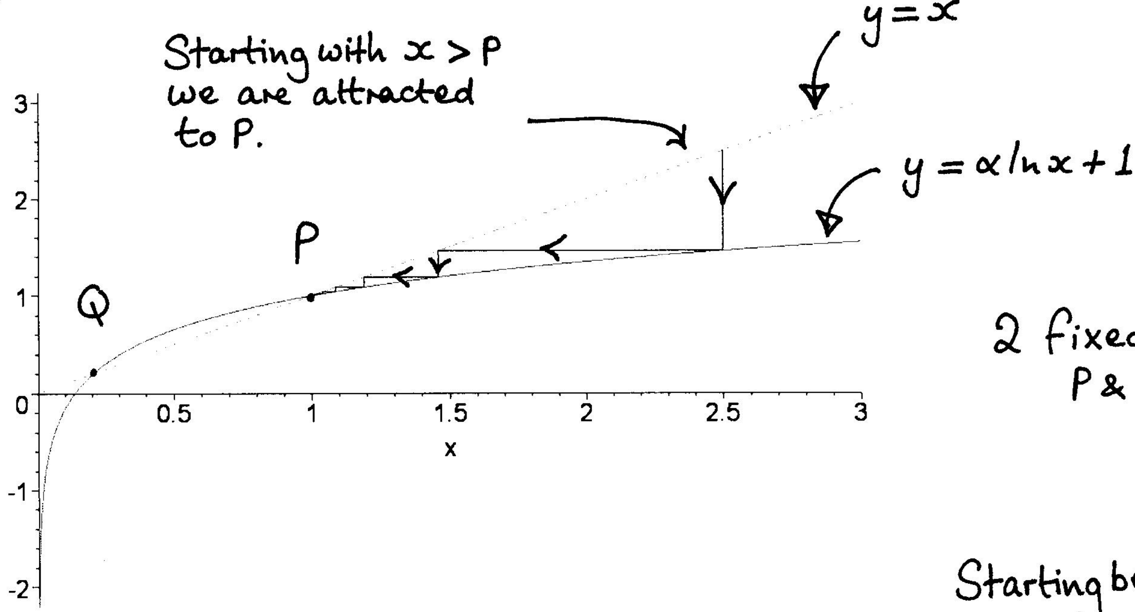
1/

ME3.6 Sheet 5

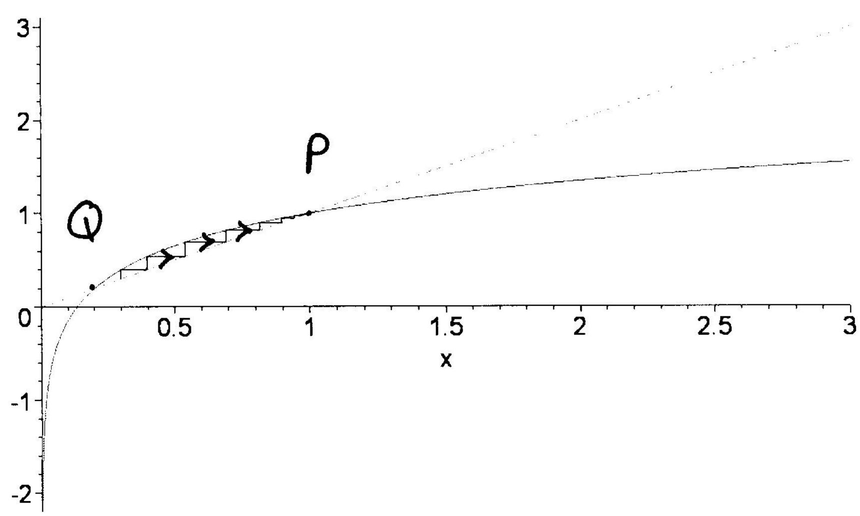
(i)

Starting with $x > P$ we are attracted to P.

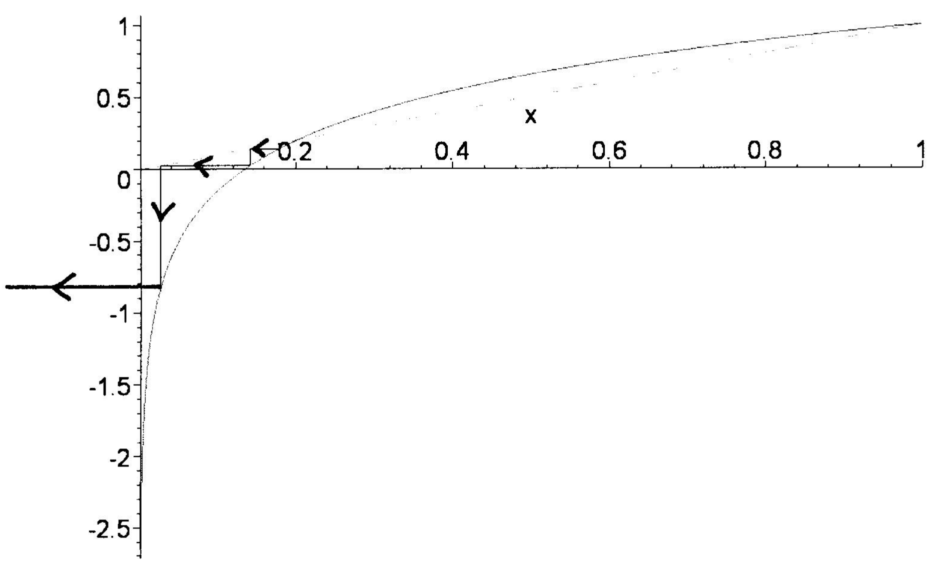


2 fixed points P & Q.

Starting between P & Q, we are attracted to P.



If we start with $x < Q$, then x becomes negative & then $\ln x$ is undefined.



P is <u>stable</u>

Q is <u>unstable</u>

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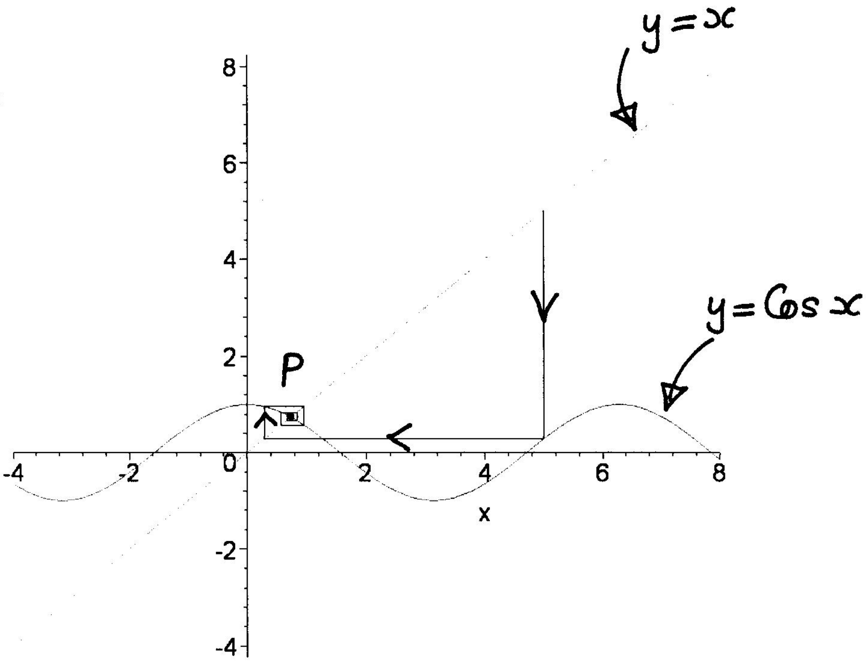
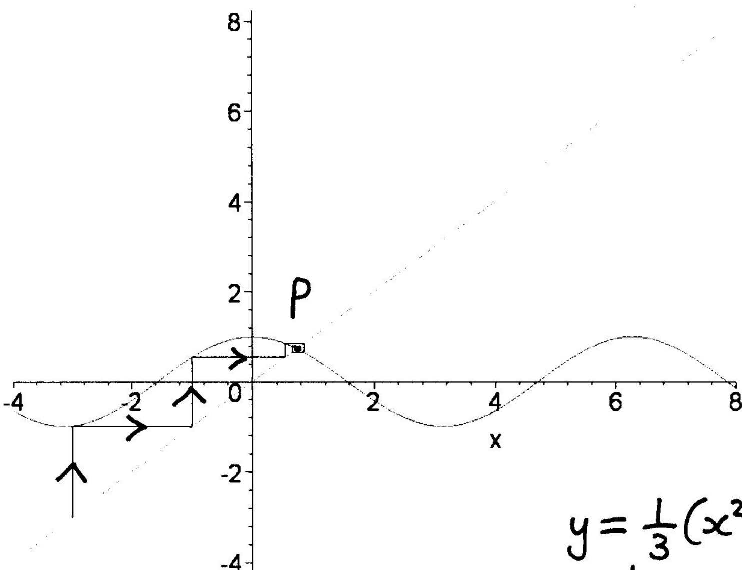
(ii)



One fixed point P.

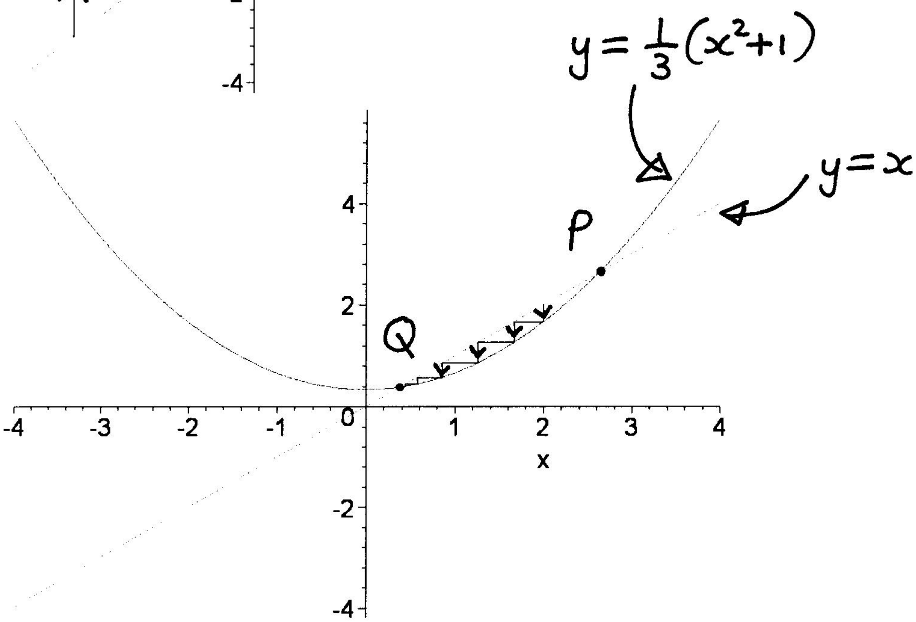
If we start with $x > P$, we are attracted to P.



If we start with $x < P$, we are also attracted to P.

<u>P is stable</u>.

(iii)



2 fixed points P & Q.

If we start between P & Q, we are attracted to Q.

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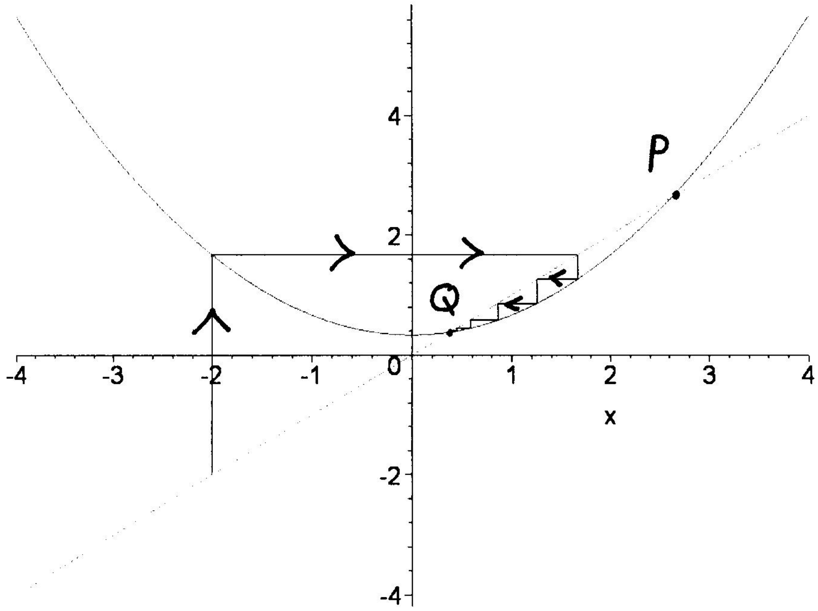
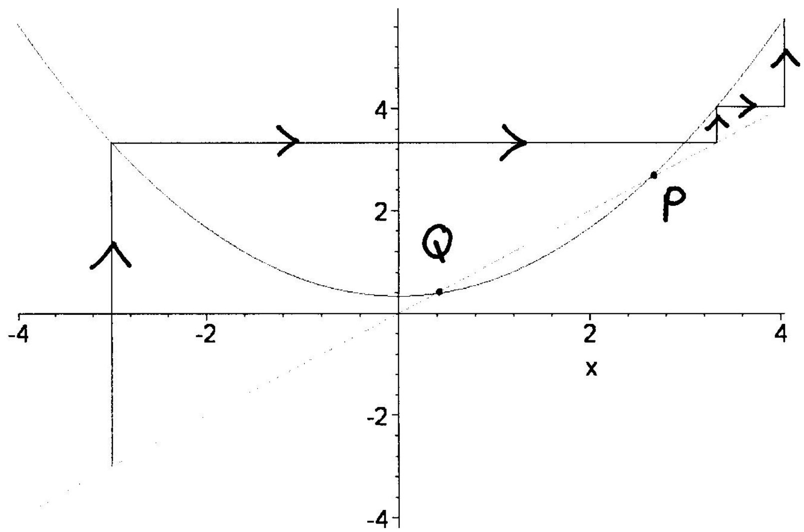
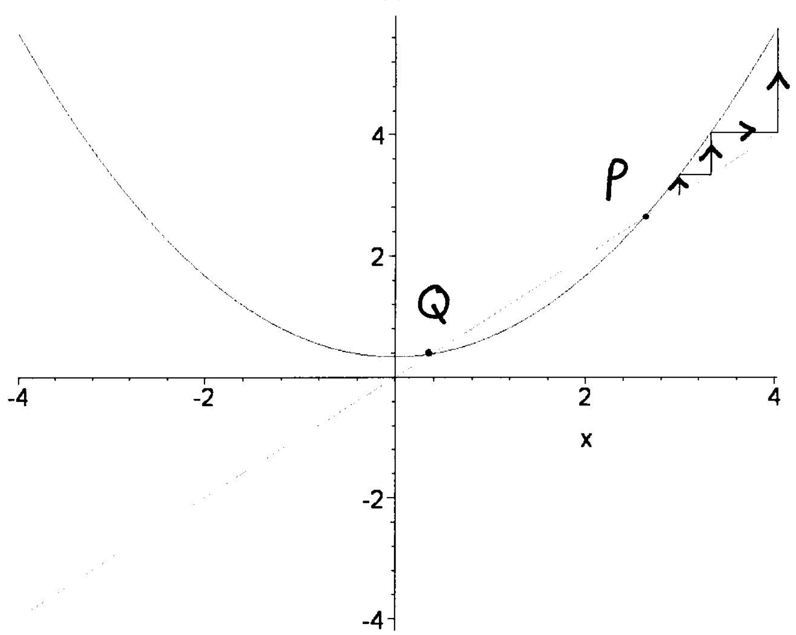
(iii) (ctd)



If we start with $x < Q$, we are attracted to Q.



However if we start too far below Q we move off to $+\infty$.



If we start with $x > P$ we also go off to $+\infty$.

P is <u>unstable</u>

Q is <u>stable</u>.

ME3.6 Sheet 5 Answers

1. See separate sheets.

2 (**i**) Fixed point X satisfies $X = (2a - X^3)/X^2 \Rightarrow 2X^3 = 2a \Rightarrow \underline{X = a^{1/3}}$
$f(x) = (2a - x^3)/x^2 \Rightarrow f'(x) = -(4a/x^3) - 1 \Rightarrow f'(X) = -4a/a - 1 = -5$
Thus $|f'(X)| = 5 > 1 \Rightarrow$ fixed point is $\underline{\text{unstable}}$.

(**ii**) Fixed point X satisfies $X = (a + X^3)/2X^2 \Rightarrow X^3 = a \Rightarrow \underline{X = a^{1/3}}$
$f(x) = (a + x^3)/2x^2 \Rightarrow f'(x) = -a/x^3 + 1/2 \Rightarrow f'(X) = -1 + 1/2 = -1/2$
Thus $|f'(X)| = 1/2 < 1 \Rightarrow$ fixed point is $\underline{\text{stable}}$.

(**iii**) $X = (a + 2X^3)/3X^2 \Rightarrow X^3 = a \Rightarrow \underline{X = a^{1/3}}$
$f(x) = a/3x^2 + 2x/3 \Rightarrow f'(x) = -2a/3x^3 + 2/3 \Rightarrow f'(X) = -2/3 + 2/3 = 0$
Thus $|f'(X)| = 0 \Rightarrow$ fixed point is $\underline{\text{stable}}$.

3 (**i**) Fixed point X satisfies $X = X^2 + 1/4 - a$.
Solving the quadratic: $X = 1/2 \pm \sqrt{a}$. Therefore we have 2 fixed points if $a > 0$.
Call them X_1, X_2.
Let $g(x) = x^2 + 1/4 - a \Rightarrow g'(x) = 2x \Rightarrow |g'(X_1)| = 2|1/2 + \sqrt{a}| > 1$
$\Rightarrow \underline{X_1 \text{ is unstable}}$.
$|g'(X_2)| = 2|1/2 - \sqrt{a}| < 1$ for stability, i.e. we need $-1/2 < 1/2 - \sqrt{a} < 1/2 \Rightarrow 0 < a < 1$
So the fixed point at X_2 is stable for $a < 1$ and unstable for $a > 1$. So $\underline{a_1 = 1}$.

(**ii**) Period 2 solutions satisfy $x = g(g(x))$.
Now $g(g(x)) = (g(x))^2 + 1/4 - a = x^4 + (1/2 - 2a)x^2 + (1/4 - a)^2 + 1/4 - a$.
Therefore $x = g(g(x))$ can be rewritten as $x^4 + (1/2 - 2a)x^2 - x + (1/4 - a)(5/4 - a) = 0$.
We know that X_1 and X_2 are also roots of this equation, so we can take out the factor $(x^2 - x + 1/4 - a)$. This leaves us with

$$(x^2 - x + 1/4 - a)(x^2 + x + 5/4 - a) = 0,$$

so that the period 2 solutions are the roots of $x^2 + x + 5/4 - a = 0$. $(*)$
Using quadratic equation formula: $X_{3,4} = -1/2 \pm \sqrt{a - 1}$.
These solutions are real provided $a \geq 1$, which shows that the period 2 solution comes into existence at $\underline{a = a_1 = 1}$.

The period 2 solution is stable provided

$$\left| \frac{d}{dx} g(g(x)) \right|_{x=X} < 1,$$

where X is either X_3 or X_4. This simplifies to

$$|g'(X)g'(g(X))|_{x=X} < 1.$$

Now, $X_3 = g(X_4)$ and $X_4 = g(X_3)$,
so the criterion for stability of the period 2 solution can be rewritten as

$$|g'(X_3)g'(X_4)| < 1.$$

In this case $g'(x) = 2x$ and so the condition for stability is $|4X_3X_4| < 1$.
From the quadratic equation $(*)$ we can see that $X_3X_4 = 5/4 - a$, so that stability is ensured provided $|5 - 4a| < 1 \Rightarrow 1 < a < 3/2$.
$\underline{\text{Thus the period 2 solution loses stability at } a = a_2 = 3/2.}$

4. (**i**) We have $b = f(a) \Rightarrow f(b) = f(f(a))$. But $f(b) = c$, so we have
$c = f(f(a)) \Rightarrow f(c) = f(f(f(a)))$.

But $a = f(c)$ and so we get $\underline{a = f^{(3)}(a)}$.
Similarly for b and c starting with $c = f(b)$and $a = f(c)$ respectively.
The 3-cycle is stable provided

$$\left| \frac{d}{dx} f(f(f(x))) \right|_{x=X} < 1,$$

where X is either a, b or c. The LHS simplifies to

$$\left| \frac{d}{dx} f(f(x)) f'(f(f(x))) \right|_{x=X}$$

and further, to

$$|f'(x) f'(f(x)) f'(f(f(x)))|_{x=X}$$

Setting $X = a$ (or b or c) we have

$$|f'(a) f'(b) f'(c)| < 1,$$

as required.
(**ii**) We have that a, b, c satisfy $X = f^{(3)}(X)$.
Fixed points of f (i.e. points X such that $X = f(X)$ also satisfy $X = f^{(3)}(X)$.
These roots can therefore be removed by dividing by $x - f(x)$.
(**iii**) It is not recommended to do this by hand (although it is possible).
In Mathematica we could use the following commands to do this:
$f[x\_, \lambda\_] := \lambda x(1-x)$
$Collect[Simplify[Expand[(x - f[f[f[x, \lambda]]])/(x - f[x, \lambda])]], x]$
(**iv**) If you plot the polynomial you will see that for $\lambda < 1 + 2\sqrt{2}$ it lies above the x –axis, i.e. there are no real roots and therefore no 3-cycle is possible.
At $\lambda = 1 + 2\sqrt{2}$ the polynomial just touches the axis in three places
$\Rightarrow$ three real roots are created. Thus the 3-cycle is born at this value of λ.
For graphs and more information see the Mathematica notebook logisticmap3cycle.nb on the web page http : //www.ma.ic.ac.uk/~agw/me.html.

5. (**i**) If we separate the variables in the ODE we get

$$\int \frac{dx}{x(3-4x)} = \int 100\, dt$$

Then if we use partial fractions the RHS becomes

$$\int \frac{1/3}{x} + \frac{4/3}{3-4x}\, dx = \frac{1}{3} \ln x - \frac{1}{3} \ln(3-4x) = \frac{1}{3} \ln\left(\frac{x}{3-4x} \right).$$

Taking exponentials of both sides:

$$\frac{x}{3-4x} = A \exp(300t)$$

where A is an arbitrary constant. Applying $x(0) = 1 \Rightarrow A = -1$.
Solving for x we have

$$x = \frac{3\exp(300t)}{4\exp(300t) - 1}.$$

It therefore follows that $\underline{x \to 3/4 \text{ as } t \to \infty}$.
(**ii**) Discretizing the ODE:

$$\frac{x_{i+1} - x_i}{\Delta t} = 100 x_i (3 - 4x_i)$$

$\Rightarrow x_{i+1} = 100 \Delta t x_i (3 - 4x_i) + x_i$. With $\Delta t = 0.01$ this becomes

$$x_{i+1} = 4x_i(1 - x_i),$$

as required. This is the logistic map with $\lambda = 4$.
At this value of λ the solutions are chaotic, so the student will get a very strange answer!
In particular the sequence does not tend to a limit as $i \to \infty$.